CLASSIFICATION OF SURFACES AND CHARACTERIZATION OF GRAPH EMBEDDINGS

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Abstract. Surfaces appear in many places in math so it is helpful to have some generic ways to interact with them. Helpfully, surfaces can be specialized to a small set of cases, the sphere, torus and projective plane through homeomorphism. This paper proves this. The method of proving this uses triangulations of surfaces. The paper then relates these triangulations to graph embeddings.

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1. Introduction

This paper is divided into two halves, both of which having to do with surfaces. The first half of the paper attempts to provide a straightforward view of the classification theorem for surfaces. This half will stay in simpler terms for most of the paper, attempting to be very explicit, possibly to the point of drudgery, in an attempt to convey additional explanation to other proofs. In particular, this half provides a pathway to reading Munkres’ proof of the classification theorem as well as Massey’s proof. This paper refers to these documents for supplemental information and more complex proofs that leverage deeper knowledge of topology.

The second half of the paper takes a tangent into graph theory and its relationship to topology. The proofs for that theorem take a slightly higher level approach and relate graphs to surfaces on which they can imbed without crossing lines. In the end, graphs are related to the process by which we classified surfaces in the terms of triangulations of surfaces.

2. Definitions

We begin with some basic definitions for topological theory.

Definition 2.1. Suppose $X$ and $Y$ are topological spaces. A homeomorphism $h$ is a continuous bijection from $X$ to $Y$ such that $h^{-1}$ is continuous.
Definition 2.2. A **Hausdorff Space** is a topological space $X$ such that for any distinct $x, y \in X$, there exists an open neighborhood of $x$ and another of $y$ which are disjoint.

Definition 2.3. An $m$-manifold is a Hausdorff space $M \subset \mathbb{R}^n$ for $n \geq m$ such that for all $x \in M$, there exists an open neighborhood $U$ of $x$ such that $U$ is homeomorphic to an open $m$-ball.

An $m$-manifold with boundary is a Hausdorff space $M \subset \mathbb{R}^n$ for $n \geq m$ such that for $x \in M$, there exists an open neighborhood $U$ of $x$ such that $U$ is homeomorphic to an open half-$m$-ball $\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid \sum x_i^2 < 1 \text{ and } x_m \geq 0\}$.

A surface is a compact 2-manifold.

Definition 2.4. Suppose $X$ and $A$ are topological spaces. A **quotient map** is a surjection $p : X \to A$ such that if $U \subset A$ is open in $A$, then $p^{-1}(U)$ is open in $X$.

Furthermore, any surjection $q : Y \to B$ where $Y$ is a topological space induces a topology on $B$ called a **quotient topology** where a set $U \subset B$ is open in $B$ if $q^{-1}(U)$ is open in $Y$. This space is said to be formed by identifying points in $Y$ which map to the same point in $B$ together.

Definition 2.5. A **closed arc** in a subspace $X$ is a set which is homeomorphic to the closed interval $[0, 1]$. An **open arc** is a set which is homeomorphic to the open interval $(0, 1)$.

Definition 2.6. A **continuum** is a compact, connected Hausdorff space.

Definition 2.7. A **graph** $G$ is an ordered pair $(V, E)$ where $V$ is a non-empty finite set of points called **vertices** and $E$ is a set of ordered pairs of distinct points of $V$ called **edges**. The number of vertices of a graph is given by $|V|$ and the number of edges by $|E|$.

For the sake of this paper, assume that all graphs are **connected**. That is for any two vertices $x$ and $y$ there exist a sequence of edges connecting those vertices.

Definition 2.8. Suppose $G$ is a graph on vertices $v_1, \ldots, v_n$ and edges $e_1, \ldots, e_m$ and $S$ is a surface. An embedding of $G$ in $S$ is a set $G(S) \subset S$ such that $G(S) = v_1(S) \cup \cdots \cup v_n(S) \cup e_1(S) \cup \cdots \cup e_m(S)$ where

1. $v_1(S), \ldots, v_n(S)$ are distinct points of $S$.
2. $e_1(S), \ldots, e_m(S)$ are open arcs of $S$ which are disjoint in pairs.
3. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$, $v_i(S)$ and $e_j(S)$ are disjoint.
4. If $e_k = (v_i, v_j)$, then $v_i(S) \cup e_k(S) \cup v_j(S)$ is a closed arc with endpoints $v_i(S)$ and $v_j(S)$.

Intuitively, this describes a drawing of the graph on the surface where no two edges intersect except at vertices.

The number of connected components of $S \setminus G(S)$ is given by $\|G(S)\|$.

$G(S)$ is a **2-cell** if every connected component of $S \setminus G(S)$ is a 2-cell. And $G(S)$ is maximum 2-cell if $\|G(S)\|$ is maximum among 2-cell embeddings of $G$.

3. The Classification Theorem

This paper begins with a treatment of the Classification Theorem of Surfaces.

**Theorem 3.1.** [The Classification Theorem]

If $S$ is a surface then $S$ is homeomorphic to exactly one of

1. The unit sphere in $\mathbb{R}^3$. 


(2) A connected sum of tori.
(3) A connected sum of projective planes.

This proof follows similarly to Munkres’ [1] proof. Some definitions follow which set up the mechanisms used to prove this theorem.

**Notation 3.2.** A polygon on \( n \) vertices will also be referred to as an \( n \)-gon. Additionally, we require that a polygon must have at least 3 vertices.

**Definition 3.3.** Suppose \( L \) is a line segment in \( \mathbb{R}^2 \). An orientation of \( L \) labels one endpoint of \( L \) the initial point and the other the final point. Suppose these points are \( a \) and \( b \) respectively. Then \( L \) is from \( a \) to \( b \). Suppose \( L' \) is another line segment from \( c \) to \( d \). Then a positive linear map of \( L \) onto \( L' \) is the homeomorphism \( h: L \rightarrow L' \) such that if \( x = (1 - s)a + sb \in L \) then \( h(x) = (1 - s)c + sd \).

**Definition 3.4.** A labeling of a polygon \( P \) is a mapping of edges of \( P \) to a set of labels. A labeling is proper if each label has two edges which map to it. Given an orientation for each edge, we then define an equivalence relation \( \sim \) on \( P \). If \( x \) is in \( \text{Int} \, P \), \( x \sim x \) only. If \( x, y \) are elements of \( \text{Bd} \, P \), then \( x \sim y \) if \( x \) is on an edge with the same label as an edge which \( y \) is on and the positive linear map under the provided orientation of those two edges maps \( x \) to \( y \). The quotient space \( X \) obtained from this relation is said to be obtained from pasting the edges of \( P \) together.

**Definition 3.5.** Suppose \( P \) is a polygon with successive vertices \( p_0, \ldots, p_n \) where \( p_0 = p_n \). Given orientations and labels for \( P \), let \( a_1, \ldots, a_m \) be the distinct labels of edges. We let \( a_i \) denote the label of the \( k \)-th edge of \( P \). We let \( \epsilon_k = \begin{cases} +1 & \text{if edge } k \text{ is from } p_k \text{ to } p_{k+1} \\ -1 & \text{otherwise} \end{cases} \) denote the orientation of each edge. Then we can complete specify this polygon by writing and drawing as below.

\[ w = (a_{i_1})^{\epsilon_1}(a_{i_2})^{\epsilon_2} \cdots (a_{i_n})^{\epsilon_n}. \]

We call this symbol a labeling scheme length \( n \). It is not difficult to show that the same labeling scheme applied to two different \( n \)-gons give rise to the same space up to homeomorphism. An example scheme would be \( a^{-1}b^{-1}ab \) (note we ignore exponents of \( +1 \) when writing). When drawing schemes, unknown sections will be represented with snaked lines as above and if orientations of certain edges are known they will be described with arrows in the appropriate direction as opposite to exponents.
We can extend this definition further to allow any finite number of polygons under a collection of labeling schemes \( w_1, \ldots, w_n \). A quotient space can then be generated in a similar manner as before identifying any edge in any scheme together.

Now we define exactly what connected sums of tori or projective planes are.

**Definition 3.6.** A torus is the quotient space \( T \) which arises from the labeling scheme of a 4-gon \( aba^{-1}b^{-1} \). An \( n \)-fold torus is the quotient space of a \( 4n \)-gon under the labeling scheme \( (a_1b_1a_1^{-1}b_1^{-1})(a_2b_2a_2^{-1}b_2^{-1}) \cdots (a_nb_n^{-1}b_n^{-1}) \). A way in which the 1-fold torus matches the typical idea of a torus is presented below.

**Definition 3.7.** The projective plane is the quotient space \( X \) which arises from labeling a 4-gon with the scheme \( aabb^{-1} \). It is most often represented as a quotient space on the 2-sphere with diametrically opposite points being identified together.

For \( m > 1 \) the \( m \)-fold projective plane is the quotient space of a \( 2m \)-gon under the labeling scheme \( (a_1a_1)(a_2a_2) \cdots (a_ma_m) \).

These \( n \)-fold tori and \( m \)-fold projective planes are often also referred to as the connected sum of \( n \) tori and \( m \) projective planes respectively and are constructed by cutting holes on these surfaces and then identifying those holes together. The idea is summarized in pictures below with the 2-fold torus.

**Elementary Scheme Operations**

From here, some methods of modifying labeling schemes and their underlying polygons will be developed for the final proof of the classification theorem. The following operations affect a labeling scheme \( w_1, w_2, \ldots, w_n \) (and its underlying polygons) without changing the resulting quotient space up to homeomorphism.
(i) **Cutting.** Suppose \( w_1 = y_0y_1 \) where \( y_0 \) and \( y_1 \) are sub-labeling schemes with at least 2 elements. Suppose \( c \) is a label which does not appear anywhere in the scheme. Then \( y_0c, c^{-1}y_1, w_2, \ldots, w_n \) represents the same space.

(ii) **Pasting.** Suppose \( w_1 = y_0c \) and \( w_2 = c^{-1}y_1 \) and \( c \) appears in only these two locations in the scheme. Then the scheme \( y_0y_1, w_3, w_4, \ldots, w_n \) represents the same space.

(iii) **Relabel.** The occurrence of any label \( a \) can be replaced with a free label (occurs nowhere in the scheme) \( b \) without affecting the resulting space. Similarly, all exponents of a particular label can be flipped (-1 to 1 and 1 to -1).

(iv) **Permute.** Any subscheme \( w_i \) can be replaced with a cyclic permutation of itself without affecting the resulting scheme.

(v) **Flip.** One can replace the scheme \( w_i = (a_{i_1})^{e_1}(a_{i_2})^{e_2}\cdots(a_{i_k})^{e_k} \) with its inverse \( w_i^{-1} = (a_{i_1})^{-e_1}\cdots(a_{i_k})^{-e_k} \). This is equivalent to visiting each edge in a clockwise order or flipping the polygon “over” on the page.

(vi) **Joining.** Suppose the labels \( a \) and \( b \) always appear sequentially in the labeling scheme and the orientations of these subsequent edges are the same. Then all occurrences can be replaced by a new label \( c \) with the orientation of \( a \). The reverse is also possible.

(vii) **Cancel.** Suppose that \( y_0aa^{-1}y_1 \) is a subscheme and that \( a \) does not appear elsewhere. Then this scheme may be replaced with \( y_0y_1 \).

(viii) **Uncancel.** This is the reverse of cancel. Suppose \( y_0y_1 \) is a subscheme. This may be replaced with \( y_0aa^{-1}y_1 \) provided that \( a \) is a free label.

**Proof of (i).** Suppose \( y_0y_1 \) is a labeling scheme of a single polygon \( P \) where \( y_0, y_1 \) are subschemes of length at least 2. Let \( p_0, p_1, \ldots, p_{n-1} \) be the vertices of \( P \). Let \( k \) be the length of \( y_0 \). Let \( c \) be a free label. We will cut \( P \) into two separate polygons \( Q_1 \) and \( Q_2 \) and translate \( Q_1 \) so that \( Q_1 \) does not intersect \( Q_2 \). The vertices of these new polygons are \( p'_0, p'_1, \ldots, p'_{k} \) which are the points \( p_i \) translated and \( p_k, p_{k+1}, \ldots, p_n = p_0 \). We then identify the two new edges from \( p'_k \) to \( p'_0 \) and from \( p_k \) to \( p_0 \) created in this process by a postive linear map. It should be clear that these are homeomorphic and are two polygons with the schemes \( y_0c \) and \( c^{-1}y_1 \).

**Proof of (ii).** Suppose \( Q'_1 \) and \( Q_2 \) are two polygons described by \( y_0c \) and \( c^{-1}y_1 \) respectively with vertices \( q_0, q_1, \ldots, q_k = q_0 \) and \( Q_2 \) as vertices \( p_0, p_1, p_{k+1}, \ldots, p_n = p_0 \). Because choice of polygon does not matter, we assume that the vertices \( Q_2 \) are inscribed on a circle. Let \( p_1, \ldots, p_{k-1} \) be a sequence of counterclockwise points on that circle between the points \( p_0 \) and \( p_{k-1} \). Thus, \( p_0, p_1, \ldots, p_k \) form a \( k \) length
polygon, $Q_1$, inscribed in the same circle as $Q_2$. Thus, $Q'_1$ is homeomorphic to $Q_1$. Additionally, the line segment from $q_k$ to $q_0$ is mapped linearly to the line segment from $p_k$ to $p_0$ so the fact that these edges are identified together is preserved. Thus there is a homeomorphism from the quotient space of $Q'_1$ and $Q_2$ to the union of $Q_1$ and $Q_2$ under the scheme $y_0y_1$. We say that the polygon $P$ arising from this is obtained by pasting $Q'_1$ and $Q_2$ together. 

We will omit proofs for (iii), (iv) and (v) as they are not hard to verify.

**Proof of (vi).** Suppose $w$ is a labeling scheme of length $n$ such that $a$ and $b$ are labels in the scheme that always appear as $ab$ or $b^{-1}a^{-1}$ and there are at least 2 other elements of the scheme that are not $a$ or $b$. Let $c$ be a free label of $w$. Consider the scheme $w'$ created by replacing occurrences of $ab$ with $c$ and $b^{-1}a^{-1}$ with $c^{-1}$. Let $Q$ be the underlying polygon of $w$ and $P$ the underlying polygon of $w'$. Consider the homeomorphism $h : \text{Bd}Q \to \text{Bd}P$ which does the following.

1. non-$a$ or $b$ labelled edges of $Q$ map to their corresponding edge of $P$.
2. For $ab$-labeled edges, $h$ maps the segment from $p_i$ to $p_{i+1}$ to the segment from $q_k$ to $q_{\text{mid}}$. And the segment from $p_{i+1}$ to $p_{i+2}$ to the segment from $q_{\text{mid}}$ to $q_{k+1}$. And similarly for $b^{-1}a^{-1}$-labelled edges.

Then extend this into a homeomorphism of $Q$ and $P$ in their entirety. It should be clear by construction that for $x$ and $y \in Q$ such that $x \sim_w y$ ($x$ and $y$ are identified by the scheme $w$), we have $h(x) \sim_w h(y)$. Following from the existence of this homeomorphism, it should not be hard to see that the quotient space arising from $w$ is homeomorphic to that arising from $w'$. The fact that the reverse operation also induces an equivalent labeling scheme follows similarly to this.

**Proof of (vii).** We will now show that if $w_i = y_0aa^{-1}y_1$ then we can replace $w_i$ with $y_0y_1$ without affecting the quotient space up to homeomorphism. We accomplish this by cutting $w_i$ between $a$ and $a^{-1}$ to receive $y_0ab, b^{-1}a^{-1}y_1$. We then apply joining using a free label $c$ to receive $y_0c, c^{-1}y_1$. Then we paste to receive $y_0y_1$. Thus we may replace $y_0aa^{-1}y_1$ with $y_0y_1$.

**Proof of (viii).** The proof of this inverse operation follows by cutting, flipping the joining process (which we will not prove), and then pasting.

**Definition 3.8.** We define two labeling schemes $w$ and $w'$ to be equivalent if a sequence of elementary operations can go from $w$ to $w'$. Because these operations all have inverses this properly defines an equivalence class. Furthermore, as we have
shown above, any two equivalent labeling schemes give rise to the same space up to homeomorphism.

We will now show that any surface can be obtained by such a labeling scheme up to homeomorphism. And more specifically, every surface can be represented by a labeling scheme where each label appears precisely twice. To do this, we require a rather tedious concept called triangulation.

**Definition 3.9.** Suppose \( X \) is a compact Hausdorff space. A curved triangle is a subspace \( A \subset X \) such that there exists a homeomorphism \( h : T \to A \) where \( T \) is a closed triangle. If \( e \subset T \) is an edge of \( T \), then \( h(e) \) is an edge of \( A \). Similarly with vertices.

**Definition 3.10.** A triangulation of a surface \( S \) is a finite set \( A_1, A_2, \ldots, A_n \) of curved triangles in \( S \) such that for any \( i \neq j \) we have that \( A_i \cap A_j \) is empty, contains only one common vertex, or is an edge of both and \( \bigcup_{i \in [n]} A_i = S \).

The fact that every surface has a triangulations is, while not too difficult to prove, tedious, so simply refer to [2] for proof. Using this fact, it can be proved that any surface is homeomorphic to a collection of triangles whose edges are identified in pairs. The following theorem uses this fact in its proof. For the sake of brevity, we refer the reader to [1, pgs 471-6] or [3, pg 14].

**Theorem 3.11.** If \( S \) is a surface then there exists a polygon \( P \) and labeling scheme \( w \) such that the quotient space arising from these is homeomorphic to \( S \) and every label in \( w \) appears twice.

Due to this theorem we know that the classification theorem boils down to showing

**Theorem 3.12.** ([Classification Theorem])

If \( w \) is a labeling scheme of a single polygon where each label appears exactly twice, then it is equivalent to exactly one of

1. A Sphere \((ab^{-1}a^{-1})\)
2. An \( n \)-fold torus
3. An \( m \)-fold projective plane

To this aim, we will write a few lemmas which show allow us to quickly simplify labeling schemes to a few easy to handle cases.

**Definition 3.13.** We say that a labeling scheme is proper if every label appears exactly twice. A labeling scheme is of torus type if every label occurs once with positive exponent and once with negative. If in a torus type scheme a label appears twice in a row, then the cancel operation will shorten the scheme and remove those two edges. We assume this is the case from now on. Thus a torus type scheme has no label which is adjacent to itself. Every other scheme is of projective type.

**Lemma 3.14.** Suppose \( w = [y_0]a[y_1]a[y_2] \) is a proper scheme. Then \( w \sim aa[y_0y_1^{-1}y_2] \)

**Proof.** Suppose \( y_0 \) is empty. If \( y_2 \) is empty then the proof is trivial by permuting. So we assume that \( y_2 \) has at least one label. The proof is also trivial if \( y_1 \) is empty, so we assume \( y_1 \) has one label.
\begin{align*}
a[y_1]a[y_2] & \sim a[y_1]b, b^{-1}a[y_2] \quad \text{by cutting} \\
& \sim b^{-1}[y_1^{-1}]a^{-1}, a[y_2]b^{-1} \quad \text{by flipping the first, and permuting the second.} \\
& \sim b^{-1}[y_1^{-1}y_2]b^{-1} \quad \text{through pasting} \\
& \sim aa[y_1^{-1}y_2] \sim aa[y_1^{-1}y_2] \quad \text{by permuting and relabeling.}
\end{align*}

Now suppose \( y_0 \) is not empty.

\[ [y_0]a[y_1]a[y_2] \sim [y_0]ab, b^{-1}[y_1]a[y_2] \quad \text{through cutting.} \\
& \sim b^{-1}a^{-1}[y_0^{-1}], b^{-1}[y_1]a[y_2] \quad \text{by flipping.} \\
& \sim [y_0^{-1}]b^{-1}a^{-1}, a[y_2]b^{-1}[y_1] \quad \text{by permuting.} \\
& \sim [y_0^{-1}]b^{-1}[y_2]b^{-1}[y_1] \quad \text{by pasting.} \\
& \sim b^{-1}[y_2]b^{-1}[y_1y_0^{-1}] \quad \text{through permuting.} \\
& \sim b^{-1}b^{-1}[y_2^{-1}y_1y_0^{-1}] \quad \text{by the part above.} \\
& \sim [y_0y_1^{-1}y_2]bb \quad \text{by flipping.} \\
& \sim aa[y_0y_1^{-1}y_2] \quad \text{through permuting and relabeling.}
\]

}\]

\textbf{Lemma 3.15.} If \( w \) is a proper scheme then it is similar to a scheme \( a_1a_1a_2a_2 \cdots a_ka_kw' \) where \( w' \) is of torus type or empty.

\textbf{Proof.} Suppose \( w \) is any proper labeling scheme. If \( w \) is of torus type then we are finished.

Otherwise we proceed by induction. We assume that \( w \sim a_1a_1 \cdots a_ka_kw' \) where \( w' \) is of projective type and \( k \) may be 0. Because \( w' \) is of projective type, then \( w' = [y_0]a_{k+1}[y_1]a_{k+1}[y_2] \). Applying Lemma 3.14 yields \( w = a_1a_1 \cdots a_ka_k[y_0]a_{k+1}[y_1][a_{k+1}][y_2] \sim a_1a_1a_1 \cdots a_ka_k[y_0y_1^{-1}y_2] \). Relabeling shows this is equivalent to \( a_1a_1 \cdots a_ka_k[0][y_0y_1^{-1}y_2] \). If \( [y_0y_1^{-1}y_2] \) is of torus type, then this process is completed, otherwise we continue the process. We note that \( [y_0y_1^{-1}y_2] \) is shorter than \( w \) and \( w \) is finite so this process must complete. Therefore, we have that \( w \sim a_1a_1 \cdots a_ka_kw' \) where \( w' \) is of torus type or empty.

\textbf{Lemma 3.16.} Suppose \( w \) is proper scheme and \( w = w_0w_1 \) where \( w_1 \) is of torus type.

Then \( w \sim w_0w_2 \) where \( w_2 = aba^{-1}b^{-1}w_3 \) where \( w_3 \) is torus type or empty and \( w_2 \) has the same length as \( w_1 \). Furthermore, \( w \) can be expressed as \( w_0a_1b_1a_1^{-1}b_1^{-1} \cdots a_n b_n a_n^{-1}b_n^{-1} \).

\textbf{Proof.} Let \( a \) be a label in \( w_1 \) such that it is closest to its second occurrence. Let \( b \) be any label between them. There exists at least one such label because \( a \) cannot be adjacent to itself as given in our definition of torus type. Because \( a \) is closest to its second occurrence, the other occurrence of \( b \) must come before the first of \( a \) or after the second. Because \( a, a^{-1}, b, b^{-1} \) must appear interspersed, we use relabeling to have them appear in that order. That is, \( w_1 = [y_0]a[y_1]b[y_2]a^{-1}[y_3]b^{-1}[y_4] \) where \( y_0, \ldots, y_4 \) are possibly empty labeling schemes.

Thus \( w = w_0[y_0]a[y_1]b[y_2]a^{-1}[y_3]b^{-1}[y_4] \). We will show that this is equivalent to \( w_0a[y_1]b[y_2]a^{-1}[y_0][y_3]b^{-1}[y_4] \).
Then time will result in ing this lemma multiple times using the resulting y operations above did not change the number of postive and negative edges. Apply-

The following shows that $w_0a[y_1]b[y_2]a^{-1}[y_0y_3]b^{-1}[y_1y_4] \sim w_0a[y_0y_3y_2]ba^{-1}b^{-1}[y_1y_4]$.

Finally, we show that this last scheme is equivalent to $w_0aba^{-1}b^{-1}[y_0y_3y_2y_1y_4]$. If $w_0, y_1$, and $y_4$ are empty, then the proof follows as so.

Otherwise we apply the following.

So we have that $w \sim w_0aba^{-1}b^{-1}[y_0y_3y_2y_1y_4]$. We also note that $aba^{-1}b^{-1}[y_0y_3y_2y_1y_4]$ must have the same number of elements as $w_1$ and also be of torus type as all the operations above did not change the number of positive and negative edges. Applying this lemma multiple times using the resulting $y_0y_3y_2y_1y_4$ as the torus type each time will result in $w \sim w_0a[y_0]c[aba^{-1}b^{-1}[w_1]$.

\[ w_0[y_0]a[y_1]b[y_2]a^{-1}[y_3]b^{-1}[y_4] \sim [y_0]a[y_1]b[y_2]a^{-1}[y_3]b^{-1}[y_4]w_0 \quad \text{permuting.} \]
\[ \sim [y_0]a[y_1]b[y_2]c, c^{-1}a^{-1}[y_3]b^{-1}[y_4]w_0 \quad \text{cutting} \]
\[ \sim [y_1]b[y_2]c[y_0]a, a^{-1}[y_3]b^{-1}[y_4]w_0c^{-1} \quad \text{permuting.} \]
\[ \sim [y_1]b[y_2]c[y_0][y_3]b^{-1}[y_4]w_0c^{-1} \quad \text{pasting on } a. \]
\[ \sim w_0a[y_1]b[y_2]a^{-1}[y_0y_3]b^{-1}[y_4] \quad \text{permuting and relabling } c \text{ to } a. \]

Theorem 3.17. Suppose $w$ is a proper scheme such that $w = [w_0]ccaba^{-1}b^{-1}[w_1]$. Then $w \sim [w_0]aabbecc[w_1]$. 

\[ a[y_0y_3y_2]ba^{-1}b^{-1} \sim ba^{-1}b^{-1}a[y_0y_3y_2] \quad \text{permuting.} \]
\[ \sim aba^{-1}b^{-1}[y_0y_3y_2] \quad \text{relabling } a \text{ to } b^{-1} \text{ and changing } b \text{ to } a. \]
Proof:

\[(w_0)cc(ababa^{-1})[w_1] \sim cc(ababa^{-1})w_1w_0\] by permuting.

\[= cc(ab)[ba]^{-1}w_1w_0\]

\[\sim [ab][ba]c[w_1w_0]\] by reading lemma 3.14 in reverse.

\[= [a][b][c][acw_1w_0]\]

\[\sim bb[ac^{-1}acw_1w_0]\] by lemma 3.14.

\[= [bb][ac^{-1}]a[cw_1w_0]\]

\[\sim aabbcww_0\] by lemma 3.14.

\[\sim [w_0]aabcw_1w_1\] by permuting.

Finally, the proof of the classification theorem follows.

**Theorem 3.18.** [Classification Theorem]

If \(w\) is a labeling scheme of a single polygon where each label appears exactly twice, then it is equivalent to exactly one of

1. A Sphere \((abb^{-1}a^{-1})\)
2. An \(n\)-fold torus
3. An \(m\)-fold projective plane

**Proof.** Suppose \(w\) is a proper scheme.

Observe that \(w\) has at least length 4 to be a proper scheme as there must be an even number of labels in the scheme and by definition the length at least 3.

Applying the cancel operation as many times as possible yields \(w \sim w'\) where \(w'\) has no label which appears following itself with the opposite orientation. Similar the length of \(w'\) is at least 4.

If \(w'\) has length 4 then through relabeling there are three cases, \(w' \sim abb^{-1}a\) which is the sphere, \(w' \sim ababa^{-1}\) which is a 1-fold torus, \(w' \sim abab\) the projective plane or \(w' \sim aabb\) the 2-fold projective plane. Thus \(w\) is similar to (1), (2) or (3).

Now we assume \(w'\) is longer than 4.

By lemma 3.15 \(w' \sim a_1a_1a_2a_2 \cdots a_m a_m w''\) where \(m\) may be zero and the beginning segment empty and \(w''\) is torus type.

If \(w''\) is empty then we have that \(w\) is equivalent to an \(m\)-fold projective plane (3).

Otherwise we proceed as follows. Applying lemma 3.16 to \(a_1a_1a_2a_2 \cdots a_m a_m w''\) yields \(a_1a_1a_2a_2 \cdots a_m a_m w'' \sim (a_1a_1a_2a_2 \cdots a_m a_m)(b_1c_1c_1^{-1} \cdots b_n c_n b_n^{-1} c_n^{-1})\).

If \(m = 0\) then we have that \(w\) is equivalent to an \(n\)-fold torus (2).

If \(m \neq 0\), then apply theorem 3.17 multiple times to find that 
\[(a_1a_1a_2a_2 \cdots a_k a_k)(b_1c_1c_1^{-1} \cdots b_n c_n b_n^{-1} c_n^{-1}) \sim (a_1a_1a_2a_2 \cdots a_{k-1} a_{k-1}) (a_{k-1}b_1c_1 \cdots b_n c_n b_n c_n a_m a_m).\] This of course is a \((m+2n)\)-fold projective plane. Thus \(w\) is equivalent to (3).

The fact that these categories are disjoint is handled in [1, pgs. 454-7].

So in conclusion, as any surface \(S\) is homeomorphic to the quotient space arising from a polygon pasting edges in pairs, we have shown that any surface is homeomorphic to a sphere, an \(n\)-fold torus or an \(m\)-fold projective plane.

\[\square\]
From this classification arise two distinct groups.

**Definition 3.19.** An *orientable* surface, is one which is homeomorphic to a sphere or *n*-fold torus. If a surface is homeomorphic to an *m*-fold projective plane then it is *non-orientable*.

These characteristics have more precise definitions with more meaning, however that is not the focus of this paper. Given a surface *S*, there is a characteristic ascribed to it.

**Definition 3.20.** Suppose *S* is homeomorphic to the sphere. Then the *genus* of *S* is 0. If *S* is homeomorphic to an *n*-fold torus then it has genus *n*. If *S* is homeomorphic to the *m*-fold projective plane, then the genus of *S* is *m*. As these are mutually exclusive, we will refer to them simply as the *genus* of the surface in most cases, and with the notation $\gamma(S)$.

The *Euler characteristic* of the surface, is $\chi(S) = 2 - 2\gamma(S)$ for orientable surfaces and $\chi(S) = 2 - \gamma(S)$ for non-orientable surfaces.

The Euler characteristic has another, more general definition. $\chi(S) = \alpha_0 - \alpha_1 + \alpha_2$ where $\alpha_i$ is the number of *i*-complexes in a cellular decomposition of *S*.

It is useful to note that $\chi(S) \leq 2$ and $\chi(S) = 2$ if and only if *S* is homeomorphic to the sphere.

### 4. Graph Embeddings

From here, the paper segues into a discussion of graph embeddings. It should be clear that a finite graph can be drawn on some surface *S* without having lines cross. However, it is difficult to determine the “simplest” surface on which a graph can imbed. That is, the surface with lowest genus. This paper will show the following:

**Theorem 4.1.** Suppose *G* is a graph and *X* is a surface. Let $G(X)$ be an embedding of *G* in *X*. If $G(X)$ is the one skeleton of a triangulation of *X* then $G(X)$ is simplest. And furthermore, if $G(Y)$ is an embedding of *G* on a surface *Y* with the same Euler characteristics as *X*, then $G(X)$ is the one skeleton of a triangulation of *Y*.

This connection to triangulations of surfaces suggests a connection between graphs and 2-manifolds which may prove interesting further results. This section will start with a rigorous definition for minimal embeddings and define exactly what the “one skeleton” of a triangulation is.

**Definition 4.2.** Suppose *X* is a surface with an embeddings $G(X)$. $G(X)$ is *simplest* if given a surface *Y* with embedding $G(Y)$, $\gamma(X) \leq \gamma(Y)$. Recall based on our definition of Euler characteristic that $\chi(X) \geq \chi(Y)$.

Additionally, if *X* and *Y* are surfaces, *G* is a graph and $G(X), G(Y)$ are embeddings of *G* in *X* and *Y* respectively, then $G(X)$ is simpler than $G(Y)$ if $\chi(X) \geq \chi(Y)$.

**Definition 4.3.** Suppose $\{A_1, \ldots, A_n\}$ is a triangulation of a surface *S*. Then the *one skeleton* of that triangulation is $\bigcup_{i=1}^n \text{Bd} A_i$. Which can also be described as the union of edges and vertices of the triangulation.

We now introduce a method called the capping operation for taking an embedding $G(X)$ and creating a surface *X’* which has lower genus than *X* for which $G(X)$ is an embedding of *G* on *X’*. The capping operation takes the connected component
of the complement of \( G(X) \) and replaces them with collections of two-cell, thus reducing the genus of the surface. To define this process, a lemma referenced in [4, pg 306] and shown in [5, pg 851] is used.

**Lemma 4.4.** If \( G(X) \) is an embedding of a graph \( G \) on a surface \( X \). Let \( S \) be a component of \( X \setminus G(X) \). Then \( S \) contains a subcomplex \( T \) which is a 2-manifold with boundary. More specifically, if \( J_1 , \ldots , J_s \) where \( s \geq 1 \) are the Jordan-curves which constitute the boundary of \( T \), then

1. The components of \( (S \setminus T) \) are open cylinders \( L_1 , \ldots , L_s \)
2. \( \partial(L_i) \) has two components, \( J_i \) and a subset of \( G(X) \).

**Theorem 4.5.** Suppose \( G(X) \) is an embedding of a graph \( G \) on a surface \( X \). Let \( S \) be a connected component of \( X \setminus G(X) \). Let \( T, J_1 , \ldots , J_s \) and \( L_1 , \ldots , L_s \) be as in the lemma above. Let \( X_S \) be the surface obtained from \( X \setminus IntT \) by connecting the boundary curves \( J_i \) with open 2-cells \( C_i \). Then the following properties hold.

1. \( G(X) \) is also an embedding \( G(X_S) \) of \( G \) in \( X_S \).
2. \( \chi(X_S) \geq \chi(X) \) if and only if \( S \) is an open two-cell. And in this case we may 
   regard \( T \) as \( S \) and simply let \( X_S \) be \( X \). All properties will hold properly.
3. \( \|X_S\| \geq \|X\| \) and the components of \( X_S \setminus G(X_S) \) are open two cells and the components of \( X \setminus G(X) \) removing \( S \).

**Proof.** First, note that \( IntT \subset S \). As \( G(X) \subset X \setminus S \), \( G(X) \subset X \setminus IntT \subset X_S \) so \( G(X) \) has property 1.

Now, let \( T_S \) be the surface \( T \cup \bigcup_{i=1}^s C_i \). By computations on complexes, \( \chi(X) = \chi(X_S) + \chi(T_S) - 2s \). Thus \( \chi(X_S) = \chi(X) - \chi(T_S) + 2s \). We note that as for any surface, \( \chi(T_S) \leq 2 \). Thus \( 2s - \chi(T_S) \geq 0 \) as \( s \geq 1 \). Thus \( \chi(X_S) \geq \chi(X) \) and \( G(X_S) \) is simpler than \( G(X) \). So property 2 holds.

Furthermore, equality holds if and only if \( s = 1 \) and \( T_S \) is homeomorphic to the 2-sphere. Thus only if \( T \) and therefore \( S \) are open 2-cells. Thus property 3 holds.

Property 4 should be easy to see based on the operation. Connected components of \( X \setminus G(X) \) are carried over in the construction of \( X_S \) aside from \( S \) and \( S \) is replaced by \( s \) open two cells by connecting the curves \( J_i \).

\( \square \)

**Process 4.6** (The Capping Operation). Let \( G(X^0) \) be an embedding of a graph \( G \) on a surface \( X^0 \). Let \( S_1 , \ldots , S_n \) be the connected components of \( X^0 \setminus G(X^0) \). For \( i = 1 , \ldots , n \), let \( X^i \) be the surface \( X^0 \setminus S_i \) as given in the previous theorem. Then \( S_{i+1} , \ldots , S_n \) are connected components of \( X^i \) as well as a collection of 2-cells. Let \( X^* = X^0 \). We note that \( X^* \) has the following properties due to the previous theorem.

1. \( G(X^0) \) is an embedding \( G(X^*) \) of \( G \) in \( X^* \).
2. The components of \( X^* \setminus G(X^*) \) are open 2-cells.
3. \( \|G(X^*)\| \geq \|G(X^0)\| \).
4. \( \chi(X^*) \geq \chi(X^0) \) and furthermore, \( \chi(X^*) = \chi(X^0) \) if and only if \( X^0 \) is 2-cell.

It should be clear that the order of “capping” is irrelevant up to homeomorphism as no operation affects the component modified in the next.

The following is a useful intermediary characterization of graph embeddings and follows quickly from the capping operation.
Theorem 4.7. An embedding $G(X)$ of a graph $G$ in a surface $X$ is simplest if and only if it is maximum 2-cell.

Proof. Suppose $G(X)$ is simplest and $G(Y)$ is a maximal 2-cell embedding. By property (4) of the capping operation, $G(X)$ is 2-cell. As $G(X)$ and $G(Y)$ provide 2-cell decompositions of $X$ and $Y$ respectively, $\chi(X) = ||G_v|| - ||G_E|| + ||G(Y)||$ and $\chi(Y) = ||G_v|| - ||G_E|| + ||G(Y)||$. Thus $0 \leq \chi(X) - \chi(Y) = ||G(X)|| - ||G(Y)|| \leq 0$ by the fact that $X$ is simplest and $G(Y)$ is maximum. Thus $\chi(X) = \chi(Y)$ and $||G(X)|| = ||G(Y)||$. So $G(Y)$ is simplest as well and $G(X)$ is a maximum 2-cell embedding.

With this, a question arises. Suppose $T$ is the one skeleton of a triangulation of a surface $S$. This triangulation induces a graph where vertices of the triangulation are vertices of the graph and edges between vertices of the triangulation are edges of the graph. It should be clear that $T$ is an embedding of that graph in $S$. The question then is $T$ a simplest embedding? The answer, as will be proven, is yes.

Observation 4.8. Suppose $G(X)$ is a 2-cell embedding of a graph $G = \langle \{v_1, \ldots, v_n\}, \{e_1, \ldots, e_m\}\rangle$ in a surface $X$. Let $\mathcal{T}$ be the collection of components in $X \setminus G(X)$.

Let $i \in \{1, \ldots, m\}$ and $S \in \mathcal{T}$. If $e_i(X) \cap \text{Bd}S \neq \emptyset$ then $e_i(X) \subseteq \text{Bd}S$ as $e_i(X)$ will bound some 2-cell. Furthermore, because $X$ is a 2-manifold, there exists at most one $S' \in \mathcal{T} \setminus \{S\}$ such that $e_i(X) \subseteq \text{Bd}S'$. So for $i \in \{1, \ldots, m\}$ and $S$, one of the following occurs:

Case 0 $e_i(X) \cap \text{Bd}S = \emptyset$.
Case 1 $e_i(X) \subset \text{Bd}S$ and there exists some $S' \in \mathcal{T} \setminus \{S\}$ such that $e_i(X) \subset \text{Bd}S'$.
Case 2 $e_i(X) \subset \text{Bd}S$ and there exists no $S' \in \mathcal{T} \setminus \{S\}$ such that $e_i(X) \subset \text{Bd}S'$.

Let $f_S(i) = k$ where the $k$ is the case above. Thus $\sum_{S \in \mathcal{T}} f(i) = 2$ for a fixed $i$.

Note that if for a fixed $S \in \mathcal{T}$, $\sum_{i=1}^{m} f_S(i) = k$ then $S \cup \text{Bd}S$ is homeomorphic to a $k$-gon where each edge of the $k$-gon maps to an edge of the embedding. For each $k \in \mathbb{N}$, let $\mathcal{J}_k$ be the set of $S \in \mathcal{T}$ homeomorphic to a $k$-gon in this manner. Let $N_k = ||\mathcal{J}_k||$, $p = \min_{k \in \mathbb{N}}\{k \mid N_k \neq 0\}$ and $q = \max_{k \in \mathbb{N}}\{k \mid N_k \neq 0\}$.

As each edge is counted twice in summing the edges attached to each face, $2||G_E|| = \sum_{k \geq p} kN_k$. Thus $p \sum_{k \geq p} N_k \leq 2||G_E|| \leq q \sum_{k \geq p} N_k$. As the number of connected components is this sum, $p||G(X)|| \leq 2||G_E|| \leq q||G(X)||$.

Theorem 4.9. Suppose $G$ is a graph and $X$ is a surface. Let $G(X)$ be an embedding of $G$ in $X$. If $G(X)$ is the one skeleton of a triangulation of $X$ then $G(Y)$ is simplest. And furthermore, if $G(Y)$ is an embedding of $G$ on a surface $Y$ with the same Euler characteristic as $X$, then $G(X)$ is the one skeleton of a triangulation of $Y$.

Proof. $G(X)$ is the one skeleton of a triangulation of $X$, so $X \setminus G(X)$ contains the interiors of curved triangles as components which must be 2-cells. Therefore $G(X)$ is 2-cell. As all of the components are 3-gons by the above observation, $3||G(X)|| = 2||G_E||$. Suppose $G(Y)$ is a 2-cell embedding of $G$ in a surface $Y$. As $G(X)$ provides a triangulation, $G$ must have at least 3 vertices. Now consider the polygons of $Y \setminus G(Y)$. The degenerate cases of polygons, 1 and 2-gons, cannot occur in $Y \setminus G(Y)$ as a graph cannot have self-loops and the graph has 3 vertices so no edge will touch only two edges as that only occurs when the embedding has one
arc and its end points. Thus $p \geq 3$. Hence, $\|G(Y)\| = \frac{3}{p} \|G(X)\| \leq \|G(X)\|$. Thus $G(X)$ is a maximum 2-cell embedding on $X$ and therefore simplest by theorem 4.7.

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References