THE CLASSIFICATION OF SURFACES

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ABSTRACT. The sphere, the torus, and the projective plane are all examples of surfaces, or topological 2-manifolds. An important result in topology, known as the classification theorem, is that any surface is a connected sum of the above examples. This paper will introduce these basic surfaces and provide two different proofs of the classification theorem. While concepts like triangulation will be fundamental to both, the first method relies on representing surfaces as the quotient space obtained by pasting edges of a polygon together, while the second builds surfaces by attaching handles and Mobius strips to a sphere.

CONTENTS

1. PRELIMINARIES

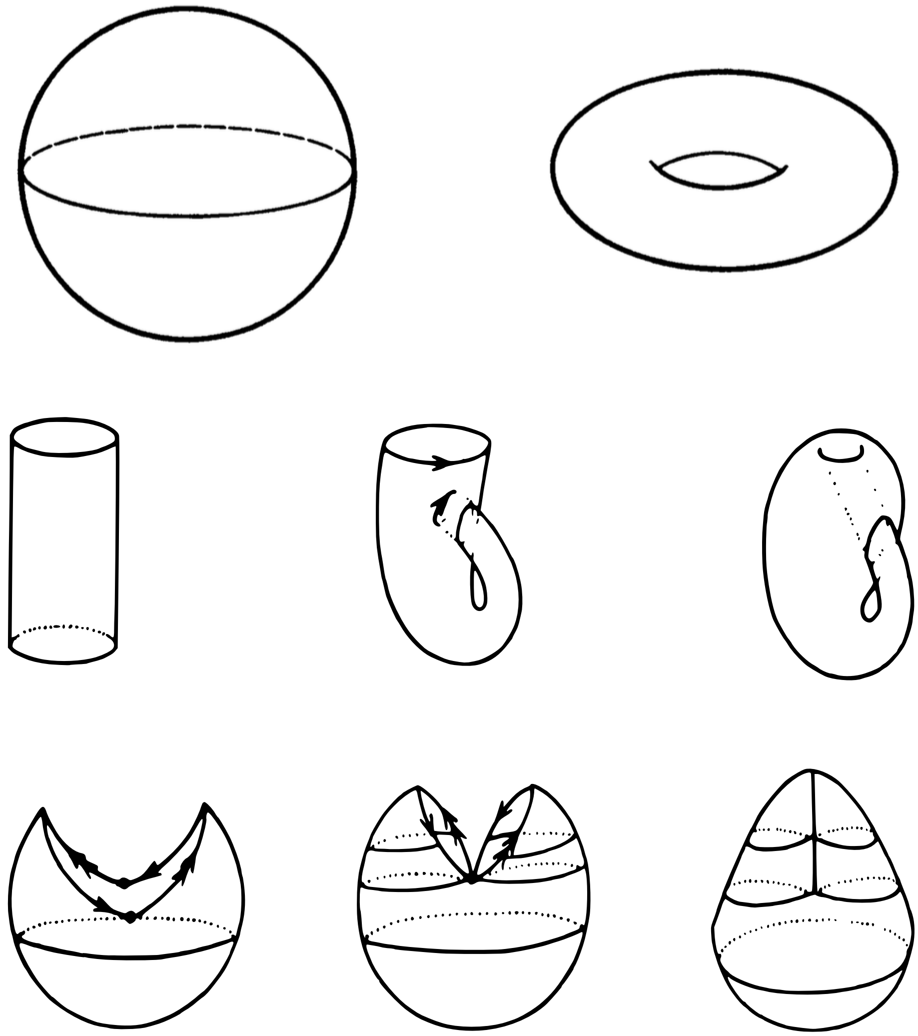
Definition 1.1. A topological space is ***Hausdorff*** if for all $x_1, x_2 \in X$, there exist disjoint neighborhoods $U_1 \ni x_1, U_2 \ni x_2$.

Definition 1.2. A ***basis***, B for a topology, τ on X is a collection of open sets in τ such that every open set in τ can be written as a union of elements in B.

Definition 1.3. A ***surface*** is a Hausdorff space with a countable basis, for which each point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^2$.

This paper will focus on compact connected surfaces, which we refer to simply as surfaces. Below are some examples of surfaces.[1] The first two are the sphere and torus, respectively. The subsequent sequences of images illustrate the construction of the Klein bottle and the projective plane.

[1]Images from Andrews, *The Classification of Surfaces*.

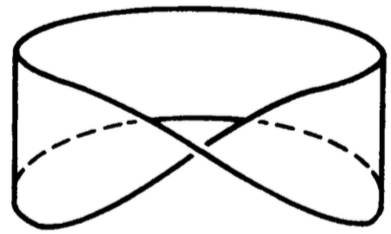


The above surfaces may be glued together by cutting a hole in each one and sewing the two surfaces together along the boundaries. This is also known as a *connected sum*. A connected sum of n tori is called the *n-fold torus*. A connected sum of m projective planes is called the *m-fold projective plane*.

The sphere and the torus are what we call *orientable* surfaces because they have a distinct inside and outside, while the Klein bottle and projective plane are *nonorientable*. The latter two examples seem to have self-intersections because they cannot be embedded in $\mathbb{R}^3$.

It is also worth mentioning another important surface, the Mobius strip.[2]

[2]Images from Armstrong, *Basic Topology*.



Although the Mobius strip is not a surface as we are considering in the word of this paper, because it has a boundary, it is fundamental in the classification of surfaces. We may see already an important observation: sewing two Mobius strips together along their boundaries yields a Klein bottle.

Definition 1.4. A surface is ***nonorientable*** if it contains a Mobius strip. Otherwise, a surface is ***orientable***.

Definition 1.5. A ***triangulation*** of a surface, X, is a collection of curved triangles $T_1, \ldots, T_n$ such that their union is X, and for any two T_i, T_j such that $i \neq j$, either $T_i \cap T_j$ is empty, $T_i \cap T_j$ is an edge of both, or $T_i \cap T_j$ is a vertex of both. If a surface has a triangulation it is said to be ***triangulable***.

We will take for granted that all surfaces are triangulable. This fact will be fundamental to both proofs of the classification theorem.

2. Method One: the Fundamental Polygon

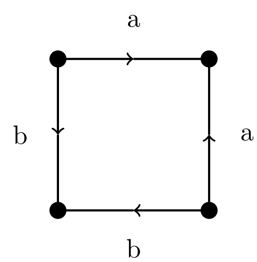
The first method of proof for the classification theorem is from Chapter 12 of Munkres's *Topology*. It relies on the fact that all surfaces may be constructed by pasting the edges of a polygon together in pairs, a fact that will be proved later in this section.

Definition 2.1. Let L be a line segment in $\mathbb{R}^2$ with endpoints a and b. An ***orientation*** of L is an ordering of its endpoints. If a is the ***initial point*** and b is the ***final point***, we say L is ***oriented from a to b***.

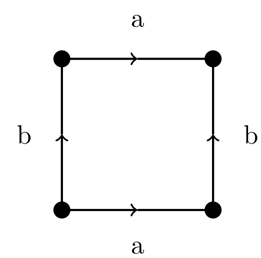
Definition 2.2. Let P be a polygonal region in the plane. A ***labelling*** of P is a map from its edges to a set of labels.

We may define an equivalence relation on P given a labelling and an orientation of each of its edges. We say that each point on the interior of P (not on an edge) is equivalent to only itself. Let e, e' be two edges with the same label, with e oriented from a to b and e' oriented from c to d. For $x \in e$, x is equivalent to $h(x) \in e'$ defined by the homeomorphism, h, which maps $x = (1-s)a + sb$ for some $s \in [0,1]$ to $(1-s)c + sd$. The quotient space obtained from this relation is said to be the result of pasting the edges of P together in the manner specified by the labelling and orientations. The polygon, equipped with a labelling and orientations of its edges, whose quotient space is a surface is called the *fundamental polygon* of that surface.

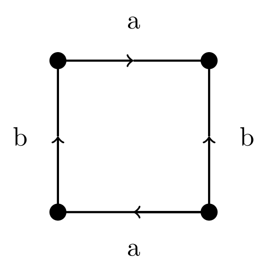
The following are the fundamental polygons of the surfaces given in the Preliminaries.



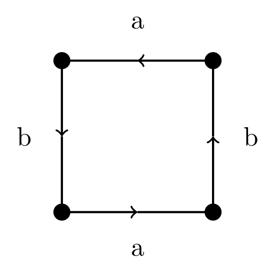
sphere



torus



Klein bottle



projective plane

Definition 2.3. Let P be a polygonal region with vertices $p_0, p_1, ..., p_n = p_0$ arranged in counterclockwise order. Let $\{a_1, ..., a_k\}$ be the set of labels for the edges of P. Let e_j be the edge connecting p_{j-1} and p_j and let a_{i_j} be the label of e_j. Then we may specify the ***labelling scheme*** of P by $w = (a_{i_1})^{\epsilon_1}(a_{i_2})^{\epsilon_2} \dots (a_{i_n})^{\epsilon_n}$ where

$$\epsilon_j = \begin{cases} +1 & \text{if } e_j \text{ is oriented from } p_{j-1} \text{ to } p_j \\ -1 & \text{if } e_j \text{ is oriented from } p_j \text{ to } p_{j-1} \end{cases}$$

Example 2.4. The fundamental polygons of the labelling schemes above are as follows

(1) sphere: $aa^{-1}bb^{-1}$
(2) torus: $aba^{-1}b^{-1}$
(3) Klein bottle: $ab^{-1}ab$
(4) projective plane: $abab$

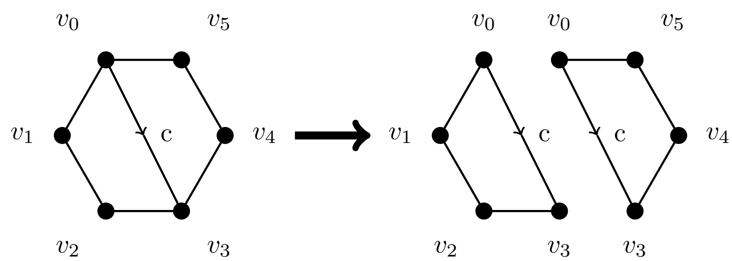
For the remainder of this section we will use w_0, y_0, y_1, etc. to denote what may be multiple adjacent labels.

Elementary Scheme Operations

The following operations may be performed on a scheme without affecting its resulting quotient space. They will be crucial in this proof of the classification theorem.

(1) Cut
If we have a polygon with vertices $v_0, ..., v_n$, and we separate it into two polygons, one having vertices $v_0, ..., v_k$ and one having vertices $v_0, v_k, ..., v_n$,

then we say that we *cut* along the line c, as illustrated below:



In this case, a scheme y_0y_1 will become two schemes of two regions y_0c and $c^{-1}y_1$.

(2) Paste
This is the reverse procedure of cutting. We may take two schemes y_0c and $c^{-1}y_1$ and paste along c to obtain y_0y_1.

(3) Relabel
We may change all occurences of a given label to a different letter that does not occur elsewhere in the scheme. We may also change the exponent of all occurences of a givel label.

(4) Permute
Cyclic permutation of the labels in a scheme does not affect the resulting space.

(5) Flip
We may reverse the order and switch the exponents of the labels in a given scheme. Specifically, the scheme $y_0y_1y_2$ becomes $y_2^{-1}y_1^{-1}y_0^{-1}$. This amounts to flipping the polygonal region over.

(6) Cancel
We may remove any adjacent terms of the same label but of opposite sign, provided that when those labels are removed, the resulting scheme has at least four terms. Specifically, the scheme $y_0aa^{-1}y_1$ is replaced by y_0y_1, so long as a does not appear elsewhere in the scheme.

(7) Uncancel
This is the reverse operation of (6). The scheme y_0y_1 may be replaced by $y_0aa^{-1}y_1$, provided a does not appear elsewhere in the scheme.

Definition 2.5. Because the above operations do not affect the quotient space of the original scheme, we may define two labelling schemes to be ***equivalent*** if one may be obtained by performing elementary scheme operations on the other.

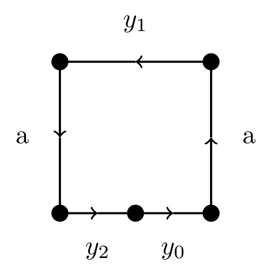
Definition 2.6. We say that a labelling scheme w is a ***proper scheme*** if every label appears twice in the scheme. A proper scheme is ***torus type*** if for every label, a, both a and a^{-1} appear in the scheme. A proper scheme is ***projective type*** if it is not torus type.

Lemma 2.7. *Let w be a scheme of projective type. Then w can be written in the form $w = y_0ay_1ay_2$ and $w \sim aay_0y_1^{-1}y_2$.*

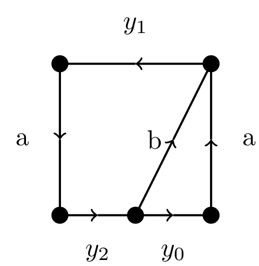
Proof. w is of projective type, so there must be some label, a such that a appears twice in the scheme, both times having the same sign. We may assume this sign is positive. Then it follows that $w = y_0ay_1ay_2$ where some of the y_i may be empty. We consider several cases:

(1) y_0 is empty:
   (a) y_1 is empty: In this case, we have $w = aay_2$, which is already in the desired form.
   (b) y_2 is empty: Then $w = ay_1a$. After flipping, $w = a^{-1}y_1^{-1}a^{-1}$. We may relabel to obtain $w = ay_1^{-1}a$. After permuting, $w = aay_1^{-1}$.
   (c) y_1, y_2 are nonempty: Then we may cut along the line b to obtain two disjoint regions y_2ba and $y_1b^{-1}a \sim a^{-1}by_1^{-1}$. Gluing along a, we have $w \sim y_2bby_1^{-1}$. Permuting and relabelling, we obtain $w \sim aay_1^{-1}y_2$.

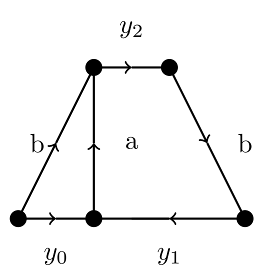
(2) y_0 is nonempty: The cases when y_1 and y_2 are empty follow similarly from the first case. We now consider the case when all y_i are nonempty. This procedure requires two cuts. We start with $w \sim y_0ay_1ay_2$.



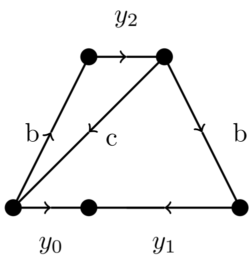
Cutting along b, we obtain $b^{-1}y_0a$ and y_1ay_2b

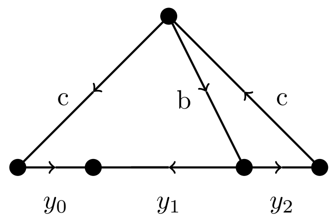


Aftering flipping one of the regions and gluing along a we have that $w \sim b^{-1}y_0y_1^{-1}b^{-1}c^{-1}cy_2^{-1}$.



We may follow a similar procedure by cutting along c and gluing along b to obtain $w \sim ccy_0y_1^{-1}y_2$, where we may relabel c to a, completing the proof. This step is illustrated below:





□

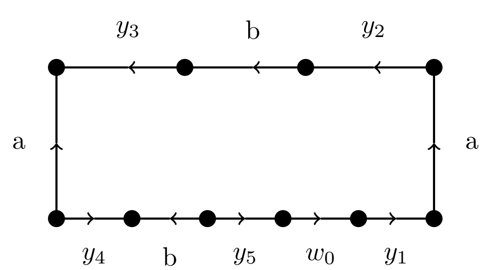
Corollary 2.8. *If w is a scheme of projective type then w is equivalent to a scheme of the same length taking the form $(a_1a_1)(a_2a_2)\dots(a_ka_k)w_1$ where w_1 is either empty or torus type.*

The proof follows by repeatedly applying the previous lemma.

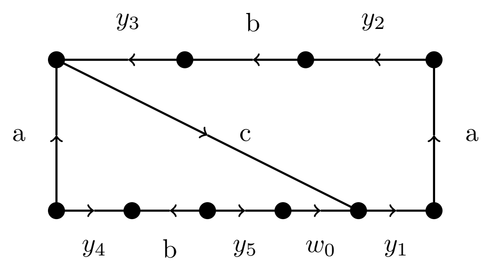
Lemma 2.9. *Let w be a proper scheme of the form w_0w_1 where w_1 is a scheme of torus type that does not contain two adjacent terms having the same label but opposite sign. Then w is equivalent to a scheme of the form w_0w_2 with w_2 having the same length as w_1 and having the form $w_2 = aba^{-1}b^{-1}w_3$ where w_3 is either torus type or empty.*

Proof. Let w be a proper scheme of the form w_0w_1 where w_1 is torus type and does not contain any adjacent terms having the same label but opposite sign. Let a be the label in w_1 such that the number of labels between a and a^{-1} is minimum. We may assume, without loss of generality, that a comes first in the scheme. Because w_1 does not have any adjacent terms with opposite label, there must be some term, b, between a and a^{-1}. Then $w_1 = y_1ay_2by_3a^{-1}y_4b^{-1}y_5$ where some of the y_i may be empty. We may now begin cutting and pasting. The proof of this lemma requires three steps.

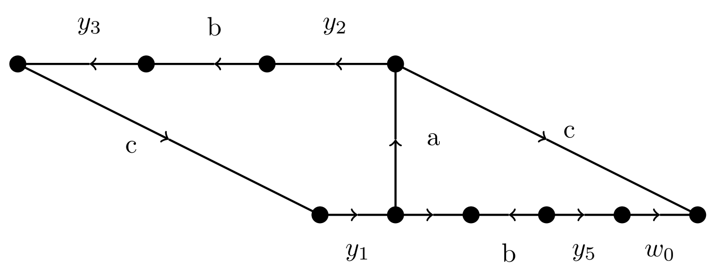
(1) $w_0y_1ay_2by_3a^{-1}y_4b^{-1}y_5 \sim w_0c^{-1}y_2by_3cy_1y_4b^{-1}y_5$.



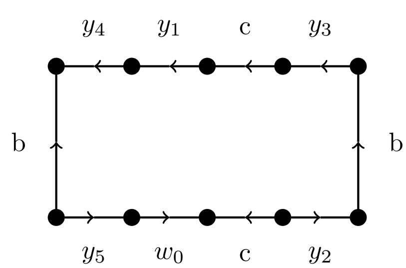
Cut along the line c to obtain two disjoint polygonal regions: $a^{-1}y_4b^{-1}y_5w_0c^{-1}$ and $cy_1ay_2by_3$.



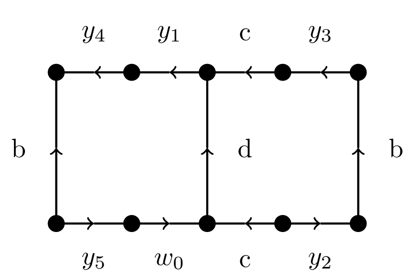
Glue along a to obtain the region $w_0c^{-1}y_2by_3cy_1y_4b^{-1}y_5$.



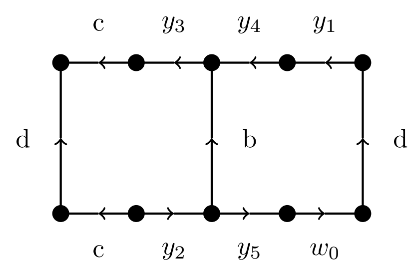
(2) $w_0c^{-1}y_2by_3cy_1y_4b^{-1}y_5 \sim w_0dy_1y_4y_3cd^{-1}c^{-1}y_2y_5$.



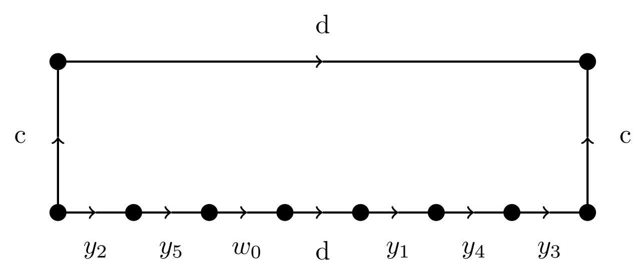
Cut along the line d to obtain two disjoint polygonal regions: $y_1y_4b^{-1}y_5w_0d$ and $d^{-1}c^{-1}y_2by_3c$.



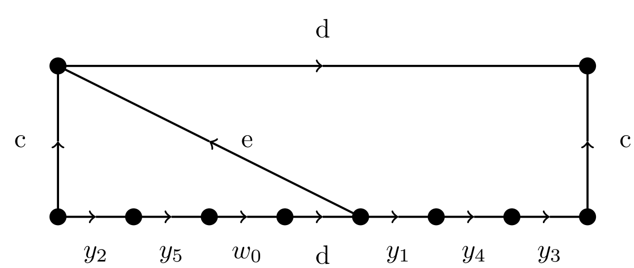
Glue along b the obtain the region $w_0dy_1y_4y_3cd^{-1}c^{-1}y_2y_5$.



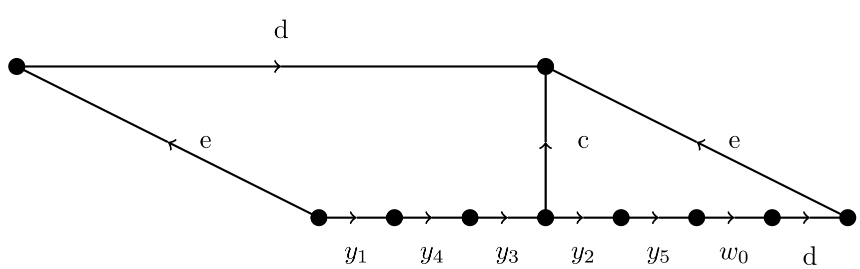
(3) $w_0 d y_1 y_4 y_3 c d^{-1} c^{-1} y_2 y_5 \sim w_0 d e d^{-1} d^{-1} y_1 y_4 y_3 y_2 y_1$.



Cut along the line e to obtain two disjoint polygonal regions: $c^{-1}y_2y_5w_0de$ and $d^{-1}e^{-1}y_1y_4y_3c$.



Glue along c to obtain the region $w_0ded^{-1}e^{-1}y_1y_4y_3y_2y_5$.



We may relabel d to a and e to b. Because w_1 is torus type, $y_1y_4y_3y_2y_5$ is either torus type, or empty. Let $w_3 = y_1y_4y_3y_2y_5$. Then we have shown that $w \sim w_0aba^{-1}b^{-1}w_3$, where w_3 is either torus type or empty. $\square$

Lemma 2.10. *Let w be a proper scheme of the form $w = w_0 ccaba^{-1}b^{-1}w_1$. Then $w \sim w_0 aabbccw_1$.*

Proof. By permuting, $w \sim ccaba^{-1}b^{-1}w_1w_0$. Appling Lemma 2.7, we have that $w \sim abcbacw_1w_0$. We may apply this lemma again to b, then a, to obtain $w \sim aabbccw_1w_0$. By permuting we obtain the desired result. □

Theorem 2.11. *Let X be the quotient space obtained from a proper labelling scheme. Then X is homeomorphic to one of the following:*

(1) $aa^{-1}bb^{-1}$, the sphere
(2) $abab$, the projective plane
(3) $(a_1a_1)\dots(a_na_n)$ for $n \geq 2$, the n-fold projective plane
(4) $(a_1b_1a_1^{-1}b_1^{-a})\dots(a_mb_ba_m^{-1}b_m^{-1})$ for $m \geq 1$, the m-fold torus

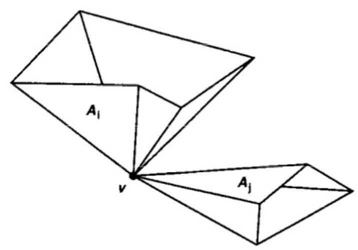
Proof. Let w be the labelling scheme. If w is torus type, proceed inductively as follows. We begin with the base case, with the length of w being 4. w is of torus type, so either $w = aba^{-1}b^{-1}$, which is type (4), or $w = aa^{-1}bb^{-1}$, which is type (1). Assume that if w is a torus type scheme of length $n-1$, w is one of the four types outlined in the statement of the theorem. Let w be a torus type scheme of length n. If w has any adjacent terms with the same label but opposite sign, we may cancel them and reduce to a case covered in the inductive hypothesis. If w does not have any adjacent terms of the same label but opposite sign, then we may apply Lemma 2.9 with w_0 empty. Then $w \sim aba^{-1}b^{-1}w_1$ where w_1 is torus type (w_1 may not be empty because the length of w is greater than 4.) We then apply the lemma again with $w_0 = aba^{-1}b^{-1}$. We may continue similarly until we end up with a scheme of type (4).

If w is projective type, then we proceed again by induction. For the base case, we consider a projective scheme of length 4. By Corollary 2.8, either $w = aabb$, which is type (3), or $w = aab^{-1}b$. If $w = aab^{-1}b$, we may apply Lemma 2.7 to obtain $w \sim abab$, which is type (2). Now assume that if w is a projective type scheme of length $n-1$, w is one of the four types outlined in the statement of the theorem. Let w be a projective type scheme of length n. As in the first case, we may reduce to cases that have no adjacent terms with opposite signs. By Corollary 2.8, $w \sim (a_1a_1)\dots(a_ka_k)w_1$, where w_1 is either torus type or empty. If w_1 is empty, then w is type 3. If w_1 is torus type, then it follows from Lemma 2.9 that $w \sim (a_1a_1)\dots(a_ka_k)aba^{-1}b^{-1}w_2$, where w_2 is either torus type or empty. By Lemma 2.10, $w \sim (aa)(bb)(a_1a_1)\dots(a_ka_k)w_2$. We proceed similarly to conclude that w is of type (3). □

We have now shown that the quotient space optained from any proper labelling scheme is one of the standard surfaces. What we have yet to show is that all surfaces can be represented as the quotient space of a proper labelling scheme. This is the subject of the final part of this section.

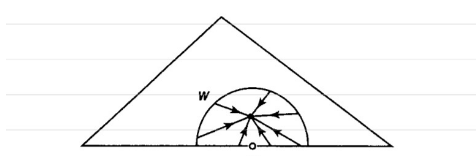
Theorem 2.12. *If X is a compact triangulable surface, then X is homeomorphic to the quotient space obtained from a collection of disjoint triangular regions in the plane by pasting their edges together in pairs.*

Proof. In order to prove this theorem, we must show two things: (1) that for each edge, e, of each triangle, T_i, there exists exactly one different triangle, T_j, such that $T_i \cap T_j = e$; (2) that if the intersection of two triangles T_i and T_j is a vertex, v, then there is a sequence of triangles having v as a vertex, beginning with T_i and ending with T_j such that each triangle's intersection with its successor is an edge. (1) ensures that the edges are pasted together in pairs. (2) ensures that a surface X is uniquely determined by pasting the edges of triangles together. Otherwise, a situation like that illustrated below may arise, in which indentification of vertices must also be specified.[3]



The proofs of both of these statements rely on a fact stated in Definition 1.3: all points in X must have a neighborhood that is locally homeomorphic to an open subset of $\mathbb{R}^2$.
To prove (1) is satisfied we will show the following:

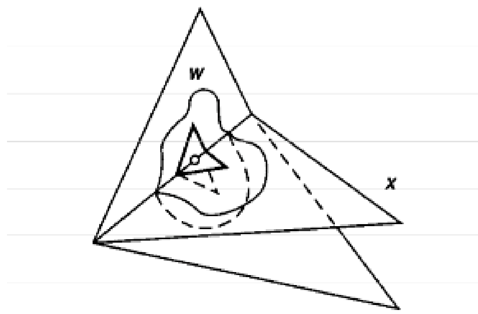
(a) For any edge, e of any triangle, T_i, there is at least one other triangle, T_j having e as an edge.
(b) There is no more than one other triangle having e as an edge.
(a) follows from the fact that if X is a triangular region in the plane, and x is a point interior to one of the edges of X, then x does not have a neighborhood which is homeomorphic to an open two-ball. This statement is made clear by the image below.[4]



(b) follows from the fact that if three or more triangles in $\mathbb{R}^3$ intersect in the same edge and x is a point on the interior of that edge, then x does not have a neighborhood that is locally homeomorphic to an open two-ball. This statement

[3]Image from Munkres, *Topology*.
[4]Ibid.

is illustrated below.[5]



Proving that (2) is satisfied is less complicated. Define two triangles A_i and A_j to be equivalent if there exists a sequence of triangles having v as a vertex, beginning with T_i and ending with T_j such that each triangle's intersection with its successor is an edge. Assume, for the sake of contradiction, that there exist more than one equivalence class. Call the union of all triangles in one equivalence class A, and the union of all triangles in the remaining equivalence classes B. By construction, the intersection of A and B is v. It follows that for every sufficiently small neighborhood, W, of v, $W - v$ is nonconnected. This is a contradiction because v has a neighborhood which is homeomorphic to an open two ball, so v must have arbitrarily small neighborhoods, W, such that $w - v$ is connected.

□

Corollary 2.13. *If X is a compact connected triangulable surface, then X is homeomorphic to a space obtained from a polygonal region in the plane by pasting its edges together in pairs.*

This corollary follows by applying Theorem 2.12 to paste the edges of the triangles together in pairs. Eventually one will obtain a polygonal region whose edges may be pasted together in pairs to obtain X.

Thus, we have completed the first proof of the classification theorem. This proof provides an algorithm for classifying a surface, given its labelling scheme, using elementary scheme operations and the results used to prove Theorem 2.11:

(a) $y_0ay_1ay_2 \sim aay_0y_1^{-1}y_2$
(b) $w_0y_1ay_2by_3a^{-1}y_4b^{-1}y_5 \sim w_0aba^{-1}b^{-1}y_1y_4y_3y_2y_5$
(c) $w_0ccaba^{-1}b^{-1}w_1 \sim w_0aabbccw_1$

The following examples will illustrate this fact.

Example 2.14. Reduce the following schemes to one of the four basic forms. For clarity, let $\sim_{(i)}$ denote equivalence by the *ith* result. Note that letters denote the results outlined above and numbers refer to elementary scheme operations.

(1) $abca^{-1}cb$

$$abca^{-1}cb \sim_{(a)} ccabab \sim_{(a)} aaccb^{-1}b \sim_{(6)} aacc$$

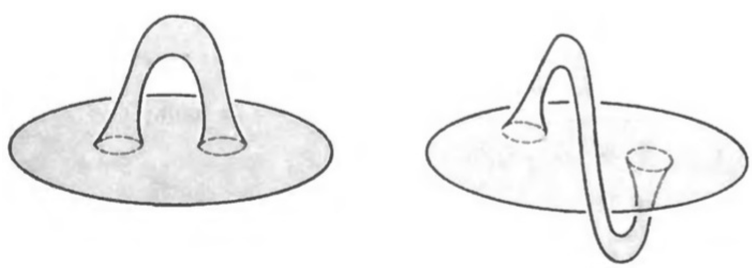
[5]Image from Munkres, *Topology.*

(2) $abcda^{-1}c^{-1}b^{-1}d^{-1}$

$$abcda^{-1}c^{-1}b^{-1}d^{-1} \sim_{(b)} aba^{-1}b^{-1}c^{-1}cdd^{-1} \sim_{(6)} aba^{-1}b^{-1}$$

3. Method Two: Sewing Handles and Mobius Strips

This section presents an alternate proof of the classification theorem, credited to Zeeman. In this proof, the n-fold torus is referred to as a sphere with n handles sewn on, and the m-fold projective plane is referred to as a sphere with m disks removed and replaced with Mobius strips. Thus, we will begin by understanding what it means to sew a handle to a sphere, or replace a disk with a Mobius strip. We begin with sewing on a handle. In order to do so, we must remove two disjoint disks and glue their edges together. We denote which way to glue the edges together with an arrow pointing either clockwise or counterclockwise around the boundary of the disk. Thus, there are two scenarios: (a), if the arrows are in opposite directions; (b), if the arrows are in the same direction.[6]

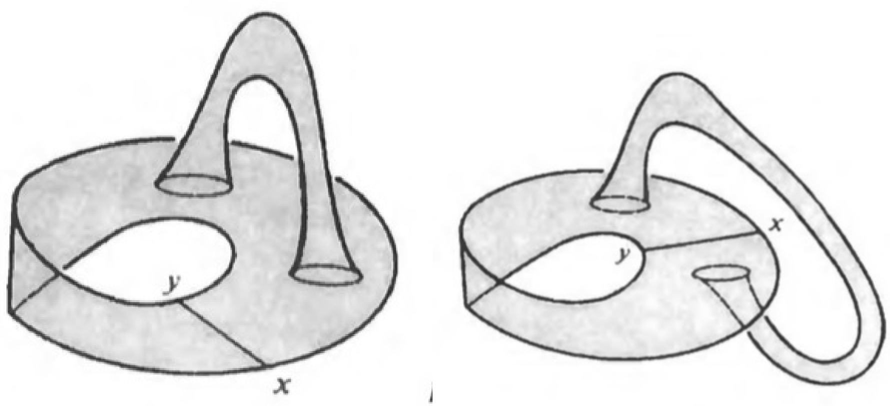


(a) (b)

(a) is homeomorphic to a punctured torus, while (b) is homeomorphic to a punctured Klein bottle. We know that a Klein bottle is two Mobius strips swen together, so sewing a punctured Klein bottle onto a sphere is equivalent to sewing on two Mobius strips.

The above images show handles being sewn onto two disks, but we may also sew them onto Mobius strips, as illustrated below.[7]

[6]Images from Armstrong, *Basic Topology.*
[7]Ibid.



(*a*) (*b*)

It is clear that (*a*) is homeomorphic to (*b*), as we may move one of the disks around the Mobius strip until it is on the other side. It follows that for any surface containing a Mobius strip, sewing a handle on is equivalent to sewing on two Mobius strips.

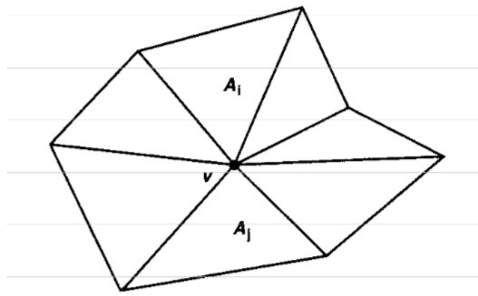
We now discuss what it means to replace a disk with a Mobius strip. Given a sphere, remove a disk and identify its boundary with the boundary of a Mobius strip. Sew the boundaries together. If we cut along the center line of the Mobius strip and untwist it, then sew it onto the sphere as specified before, then we can see that sewing a Mobisus strip on a sphere is equivalent to removing a disk of the sphere and sewing the boundary diametrically, i.e. sewing all pairs of diametrically opposite points on the boundary together. It is then easy to see that sewing diametrically is the same as sewing a cross-cap on, as pictured in the Preliminaries of this paper, when the projective plane was first introduced.

Having introduced the basic ideas of this section, we now lay the foundations for a formal proof. First, we look back to the concept of triangulation.

Definition 3.1. Let X be a surface. Triangulate X. The ***Euler characteristic*** of X is defined to be $\chi(X) = v - e + t$, where v is the number of vertices of the triangulation, e the number of edges, and t the number of triangles.

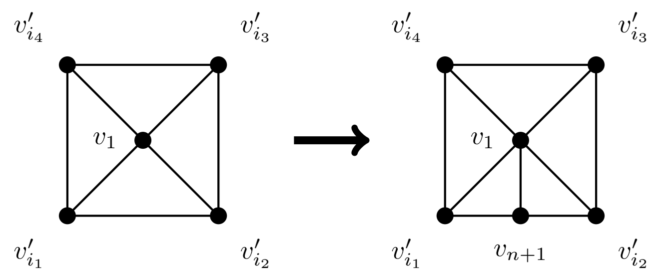
Theorem 3.2. *The Euler characteristic is invariant under choice of triangulation.*

Proof. First, an observation. Notice that in a given triangulation, each vertex, v must be surrounded by a sequence of triangles, all sharing v as a common vertex, and each one sharing an edge with its successor, as illustrated below:[8]



Now, some nomenclature. When two vertices are connected by an edge, call them adjacent. Let $deg(v)$ be the number of edges one vertex has.

We may now proceed with the proof. Let X be a surface. Let T and T' be two triangulations of X. Let n be the number of vertices of T and m be the number of vertices of T'. Without loss of generality, let $n \leq m$. Label the vertices of T, $v_1, ..., v_n$ and of T', $v'_1, ..., v'_m$. We know that v_1 must be adjacent to some other vertices, $v_l, ..., v_k$, and v'_1 must be adjacent to some other vertices, $v'_i, ..., v'_j$. If $\deg(v'_1) = \deg(v_1)$, then relabel $v_l, ..., v_k$ to $v'_i, ..., v'_j$. If $deg(v'_1) > \deg(v_1)$, then relabel $v_l, ..., v_k$ to $v'_i, ..., v'_z$ for some $z < j$ such that all vertices adjacent to v_1 are relabelled. Then add a new vertex, v_{n+1} on the interior of an edge of a triangle surrounding v_1 and connect it to v_1 by an edge, as illustrated below:



Connect v_{n+1} to all other possible vertices without crossing an existing edge to avoid adding additional vertices in the future. Relabel v_{n+1} to one of the vertices adjacent to v'_1 whose label has not already been taken. Continue as such until all vertices adjacent to v'_1 are present. Clearly, this operation does not change the Euler characteristic because we have added a new vertex, two new edges and one new face. Thus, $\Delta\chi(X) = 1 - 2 + 1 = 0$. If $deg(v'_1) < \deg(v_1)$, we may reverse the steps we made in the previous case. It is clear that this operation also does not change the Euler characteristic. We may continue this process on a vertex that was relabeled until we have every vertex of T' with its appropriate degree and have thus moved from one triangulation to another without affecting the Euler characteristic. $\square$

Definition 3.3. A ***graph***, $G(V, E)$ is a set of vertices, V, and edges, E, with each edge associated with two vertices. A graph is connected if there is a path between any two vertices.

[8]Image from Munkres, *Topology.*

Definition 3.4. A ***tree*** is a connected graph with no cycles. A ***spanning tree***, of a graph, G, is a tree that contains all vertices of G.

The following two lemmas will be left unproven in this paper, as they are standard results in graph theory.

Lemma 3.5. *Every connected graph has a spanning tree.*

Lemma 3.6. *Every tree has at least one vertex with degree one.*

Lemma 3.7. *All trees have neighborhoods that are homeomorphic to disks.*

Proof. We prove the lemma inductively. For the base case, T is a tree with one vertex. We may add a disk around the vertex. Assume that a tree with $n-1$ vertices has a neighborhood that is homeomorphic to a disk. Let T be a tree with n vertices. By Lemma 3.6, T has a vertex, v, with degree one. Let T' be the subgraph of T which includes all vertices except v. Then T' is connected. Because T has no cycles, T' has no cycles and is therefore a tree. By the inductive hypothesis, T' has a neighborhood that is homeomorphic to a disk. We can use this same neighborhood for T and extend it to include the missing vertex by adding an arm-like appendiage. Thus, all trees have neighborhoods that are homeomorphic to disks. $\square$

Lemma 3.8. $\chi(G) \leq 1$ *for all connected graphs. Specifically,* $\chi(G) = 1$ *if and only if* G *is a tree.*

Proof. Note that we define the Euler characteristic for a graph to be $\chi(G) = v - e$, where v is the number of vertices of the graph, and e is the number of edges. Let G be a tree. We can prove that $\chi(G) = 1$ by induction. The base case is trivial. Assume $\chi(G) = 1$ for any tree with $n-1$ vertices. Let G be a tree with n vertices. By Lemma 3.6, G has at least one vertex of degree one, so we may remove that vertex to obtain a tree, G', with $n-1$ vertices. $\chi(G') = 1$ by the inductive hypothesis. Clearly, $\chi(G) = \chi(G') + 1 - 1 = 1$.

Let G be a connected graph that is not a tree. Then G has a spanning tree, G', and $\chi(G') = 1$. G has a cycle, so G must have at least one more edge than G', so $\chi(G) < \chi(G') = 1$. Thus, in proving that $\chi(G) = 1$ if and only if G is a tree, we have shown that $\chi(G) < 1$ for all connected graphs that are not trees, and hence, $\chi(G) \leq 1$ for all connected graphs. $\square$

Lemma 3.9. $\chi(X) \leq 2$ *for all surfaces,* X*.*

Proof. Let X be a surface. Choose a triangulation of X. Then there exist triangles $T_1, ..., T_n$ whose union is X. This triangulation forms a connected graph, G. Fix vertices $v_1, ..., v_n$ such that each v_i is located in the interior of T_i. Connect v_i and v_j by an edge if T_i and T_j share an edge. Call the resulting graph G'. G' is clearly connected. Let T be a spanning tree of G'. Let K be the dual graph of T, i.e. K is a subgraph of G such that if T does not cross through an edge of G, then that edge and the vertices at its endpoints are in K. Recall that $\chi(X) = v - e + t$. Clearly, T has t vertices and $e_0 < e$ edges. It is also easy to see that K has $e_1 < e$ edges where $e_0 + e_1 = e$. Now assume, for the sake of contradiction, that K has $v_0 < v$ vertices. Then there is some vertex, v_j of G such that $v_j \notin K$. This implies that for all other vertices, v_k, which are connected to v_j by an edge, e_k, there is an edge of T which crosses through e_k. Then T must have a cycle, which is a contradiction. Thus, K

must have v vertices. This implies that $\chi(X) = \chi(T) + \chi(K) = 1 + \chi(K) \leq 2$ by Lemma 3.8. □

Lemma 3.10. *If $\chi(X) = 2$, then X is homeomorphic to a sphere.*

Proof. Triangulate X. Let K and T be defined the same way as in the previous lemma. If $\chi(X) = 2$, then $\chi(K) = 1$, so by Lemma 3.8, K is a tree. K and G are clearly disjoint, so we can find disjoint neighborhoods of the two trees which are homeomorphic to disks. We may expand these neighborhoods until they cover X and meet along their edges. Thus, we have that X is the union of two disks, so X must be a sphere. □

Theorem 3.11. *Every surface is homeomorphic to one of the following*

(1) a sphere
(2) a sphere with n handles sewn on
(3) a sphere with m disks replaced by Mobius strips

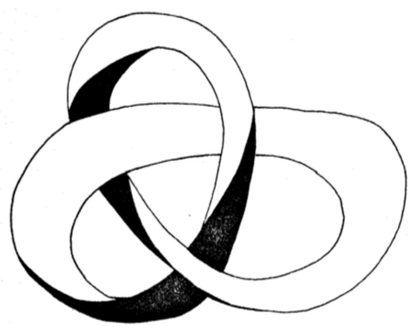
Proof. Let X be a surface. By Lemma 3.9, $\chi(X) \leq 2$. If $\chi(X) = 2$, then X is homeomorphic to a sphere by Lemma 3.10. We consider the case when $\chi(X) < 2$. Triangulate X. Define K and T as in the previous two lemmas. Because $\chi(X) < 2$, K is not a tree, so it has a cycle, C. T has a vertex in each triangle and does not cross K. Thus, any two points in $X - C$ are connected by a path in $X - C$, so C does not separate X. A neighborhood of C is either a cylinder or a Mobius strip. Consider the case when it is a cylinder, then when we cut along C we will obtain two disks. Cut along C and fill the resulting disks. Call this new surface X'. Because C is a curve of K, the edges of the disks are edges of the original triangulation of X. Assume C is made up of k edges. Then we may triangulate the resulting disks by adding a vertex in the middle and connecting all vertices in C to the central vertex. It follows that the Euler characteristic of each disk is $k + 1 - 2k + k = 1$. It follows that $\chi(X') = \chi(X) + 2$. Now consider the case when the neighborhood of C is a Mobius strip, then when we cut along C, we will obtain one disk. In this case, $\chi(X') = \chi(X) + 1$.

In both cases, we may continue similarly to obtain a surface whose Euler characteristic is 2 and is thus homeomorphic to a sphere. Cutting along the curve C and filling in disks is equivalent to removing either handles, Mobius strips, or Klein bottles. Recall that a Klein bottle is two Mobius strips sewn together along their boundaries, so removing a Klein bottle is equivalent to removing two Mobius strips. If a surface is orientable, then it does not contain any Mobius strips, so we could have only removed handles. It follows that if X is orientable, and it required n surgeries to reach a surface that is homeomorphic to a sphere, then X is a sphere with n handles sewn on, an n-fold torus. If X is nonorientable, then during surgery we may have removed handles and Mobius strips. If we have just removed Mobius strips, then we know our original surface is an m-fold projective plane. If we have removed Mobius strips and handles, then we know from the beginning of this section that the handles are equivalent to Klein bottles, and thus X is a $(m+2n)$-fold projective plane where m is the number of Mobius strips removed and n is the number of handles removed. □

The second proof of the classification theorem provides a different algorithm for classifying a surface, which will be demonstrated in the following example.

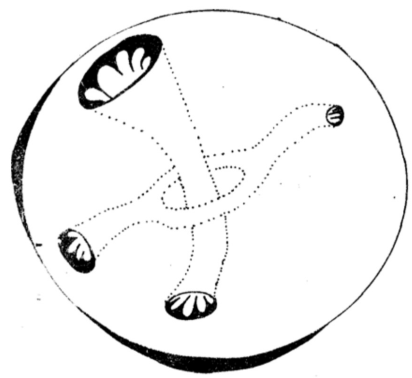
Example 3.12. Determine the classification of the following surfaces by cutting along curves that do not separate the surface, as outlined in the proof of Theorem 3.11.[9]

(*a*)



knotted torus

(*b*)



sphere with holes bored in

Note that both examples are orientable, and thus any cuts will only remove tori.

(a) Cut around the thin part of the torus and fill in the two resulting circles. Unknot the torus. The resulting surface is clearly homeomporphic to a sphere. Thus, the knotted torus is homeomorphic to the torus.
(b) Cut around the thin part of the vertical hole and fill in the two resulting disks. Repeat this procedure with the two sections of the other hole which surround the first hole. The resulting surface will be homeomorphic to a sphere. Thus, the original surface is homeomorphic to a 3-fold torus.

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[9]Images from Zeeman, *An Introduction to Topology.*