FUCHSIAN GROUPS AND FUNDAMENTAL REGIONS

BEN BEACH

Abstract. This paper provides an introduction to the theory of Fuchsian Groups and their fundamental regions.

Contents

Introduction	1
1. Hyperbolic Geometry	1
1.1. The Hyperbolic Metric	2
1.2. Isometries and The Special Linear Group	2
2. Fuchsian Groups	4
2.1. Basic Properties	4
2.2. Discrete and Properly Discontinuous Groups	4
2.3. Fixed Points and Abelian Groups	7
3. Fundamental Regions	8
3.1. The Dirichlet Region	9
Conclusions	10
Acknowledgments	10
References	10

Introduction

When considering the geometry of the hyperbolic plane, the most important group that arises is the group of orientation preserving isometries of the plane. This group is known as $PSL_2(\mathbb{R})$ and is formed by the operation $SL_2(\mathbb{R})/(\pm I)$ where $SL_2(\mathbb{R})$ is the group of 2×2 matricies with determinant 1 and I denotes the 2×2 identity matrix. For this reason, $PSL_2(\mathbb{R})$ is also known as the 2×2 projective special linear group over $\mathbb{R}$. It then follows that one may examine the set of discrete subgroups of $PSL_2(\mathbb{R})$, or Fuchsian Groups. Fuchsian Groups are particularly important to the areas of hyperbolic geometry of Riemann surfaces and algebraic curves.

1. Hyperbolic Geometry

The hyperbolic plane is obtained by maintaining that all of Euclid's axioms are true with the exception of the parallel postulate, which is negated. This is stated as follows:

Postulate 1.0.1. Suppose there exists a line $\mathscr{L}_0$ and fix a point $p \notin \mathscr{L}_0$. Then for a plane R such that $p \in R$ and $\mathscr{L}_0 \in R$, there exists at least 2 lines $p \in \mathscr{L}_1, \mathscr{L}_2$ such that $\mathscr{L}_1 \cap R = \emptyset$ and $\mathscr{L}_2 \cap R = \emptyset$.

The result of this change from standard Euclidean geometry is significant, however, for it is critical when developing the concept of a Fuchsian group. Furthermore, while a complete development of hyperbolic geometry is outside the scope of this paper, it is important to note several results when considering the derivation of Fuchsian groups.

1.1. **The Hyperbolic Metric.**

When considering $PSL_2(\mathbb{R})$, we deal with the upper half plane $\mathbb{H} \subset \mathbb{C}$ which is given by

$$\mathbb{H} = \{(x, y) \mid y > 0 \quad \text{and} \quad x, y \in \mathbb{R}\}$$

To realize the hyperbolic geometry, we first apply a metric to the space.

Definition 1.1.1. The Poincaré hyperbolic metric is given by

$$\mathrm{d}s = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y}$$

By applying Definition 1.1.1 it immediately follows that:

Definition 1.1.2. A geodesic is the shortest length between two points under the Poincaré hyperbolic metric.

Remark 1.1.3. Let $s\colon [0, 1] \to \mathbb{H}$ be a piece-wise differentiable curve given by $s(t) = x(t) + iy(t)$. It then follows that the hyperbolic length l of $s \in \mathbb{H}$ is given by

$$\int_0^1 \frac{\sqrt{\frac{\mathrm{d}x}{\mathrm{d}t}^2 + \frac{\mathrm{d}y}{\mathrm{d}t}^2}}{y(t)}\,\mathrm{d}t$$

but we know that

$$\sqrt{\frac{\mathrm{d}x}{\mathrm{d}t}^2 + \frac{\mathrm{d}y}{\mathrm{d}t}^2} = |\frac{\mathrm{d}z}{\mathrm{d}t}|$$

and so

$$l(s) = \int_0^1 \frac{|\frac{\mathrm{d}z}{\mathrm{d}t}|}{y(t)}\,\mathrm{d}t$$

Definition 1.1.4. The hyperbolic distance between two points a and b is given by

$$d(a, b) = \inf_{s \in S_{a,b}} (l(s))$$

where $S_{a,b}$ is the set of paths joining a and b.

1.2. **Isometries and The Special Linear Group.**

Definition 1.2.1. The special linear group of 2×2 matrices $SL_2(\mathbb{R})$ is the group of matrices γ of the form

$$\gamma = \begin{bmatrix} a & b \\ c & b \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \quad \text{and} \quad ad - bc = 1$$

Definition 1.2.2. A möbius transformation $f\colon \mathbb{C} \to \mathbb{C}$ is given by the function

$$f(z) = \frac{az + b}{cz + d}$$

Remark 1.2.3. The special linear group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by möbius transformations given $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Definition 1.2.4. Each möbius transformation under the conditions of Remark 1.2.3 is represented by matrices $\pm A \in SL_2(\mathbb{R})$.

Definition 1.2.5. The group of all möbius transformations under the conditions of Remark 1.2.3 is given by $PSL_2(\mathbb{R})$ and is isomorphic to $SL_2(\mathbb{R})/\{\pm I_2\}$ where I_2 is given by the 2×2 identity matrix.

Next, the first critical theorem regarding $PSL_2(\mathbb{R})$ is addressed:

Theorem 1.2.6. *$PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by homeomorphisms.*

Proof. Let $f \in PSL_2(\mathbb{R})$ and $\kappa = f(z) = \frac{az+b}{cz+d}$. Then we know (given that $\overline{z}$ denotes the complex conjugate of z) that:

$$\kappa = \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}$$

And if we denoted the imaginary coefficient of κ as κ_y and similarly for z, then:

$$\kappa_y = \frac{\kappa - \overline{\kappa}}{2i} = \frac{z - \overline{z}}{2i|cz+d|^2} = \frac{z_y}{|cz+d|^2}$$

This means that $z_y > 0 \implies \kappa_y > 0$. Furthermore, since κ is continuous and bijective, it follows that $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by homeomorphisms. □

Definition 1.2.7. A möbius transformation from $\mathbb{H} \to \mathbb{H}$ is an isometry if hyperbolic distance is preserved; the set of all isometries form a group which we denote as $\mathcal{G}$.

Theorem 1.2.8. $PSL_2(\mathbb{R}) \subset \mathcal{G}$

Proof. From Theorem 1.2.6 we know that all möbius transformations κ of $PSL_2(\mathbb{R})$ are such that $\kappa \colon \mathbb{H} \to \mathbb{H}$. First we take $s \colon [0,1] \to \mathbb{H}$ to be $z(t) = (x(t), y(t))$ and give $\kappa(t) = f(z(t)) = v(t) + iw(t)$. We then have

$$\frac{\mathrm{d}\kappa}{\mathrm{d}z} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

and from the proof of Theorem 1.2.6 we know that $w = \dfrac{y}{|cz+d)|^2}$ and multiplying this by $\frac{1}{y}$ gives us $|\frac{\mathrm{d}\kappa}{\mathrm{d}z}|$. Applying the methods from Definition 1.1.3 then yields the following expression:

$$l(f(s)) = \int_0^1 \frac{|\frac{\mathrm{d}\kappa}{\mathrm{d}t}|}{w(t)}\,\mathrm{d}t = \int_0^1 \frac{|\frac{\mathrm{d}\kappa}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t}|}{w(t)}\,\mathrm{d}t = \int_0^1 \frac{|\frac{\mathrm{d}z}{\mathrm{d}t}|}{y(t)}\,\mathrm{d}t \tag{1.9}$$

and therefore by Definition 1.1.3 it follows that $l(f(s)) = l(s)$. Furthermore, we know that hyperbolic distance is invariant, so by Defintion 1.2.7 it follows that $PSL_2(\mathbb{R}) \subset \mathcal{G}$. □

The following remarks help develop the nature of isometries when considering $PSL_2(\mathbb{R})$:

Remark 1.2.10. The set of Möbius transformations in $PSL_2(\mathbb{R})$ given by the map $z \to -\overline{z}$ is denoted by $\mathcal{G}$, which is isomorphic to $PSL_2(\mathbb{R})$. Furthermore, $PSL_2(\mathbb{R})$ has index 2 with respect to $\mathcal{G}$.

Remark 1.2.11. The topology resulting from the Poincaré metric applied to $\mathbb{H}$ is the same as for the Euclidean metric.

2. Fuchsian Groups

Recall that

$$PSL_2(\mathbb{R}) = \left\{ f(z) = \frac{az+b}{cz+d} \mid z \in \mathbb{C} \quad \text{and} \quad ad - bc = 1 \right\}$$

Within the group $PSL_2(\mathbb{R})$, there are three kinds of elements, which are each determined by the trace of the input matrix and given by $tr(f)$.

Definition 2.0.1.

(1) If the trace $|tr(f)| < 2$, then f is elliptic.
(2) If the trace $|tr(f)| = 2$, then f is parabolic.
(3) If the trace $|tr(f)| > 2$, then f is hyperbolic.

This definition is important when considering the fixed points of a given transformation, since to find them, one solves $z = \dfrac{az+b}{cz+d}$ given the standard limitations in $PSL_2(\mathbb{R})$ (that is, $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$). By computing the discriminant of the resulting quadratic, one can show that a hyperbolic transformation will have 2 fixed points $x, y \in \mathbb{R} \cup \{\infty\}$, a parabolic transformation will fix a single point, and an elliptic transformation has a fixed point $p \in \mathbb{H}$.

2.1. **Basic Properties.**

Now, we examine several basic properties of the group $PSL_2(\mathbb{R})$:

Definition 2.1.1. A geodesic contained in $\mathbb{H}$ which joins two fixed points of a hyperbolic transformation f is called the axis of f, denoted $a(f)$.

Theorem 2.1.2. *There exists a homeomorphism between $PSL_2(\mathbb{R})$ and the unit tangent bundle $S\mathbb{H}$ of $\mathbb{H}$ such that the action of $PSL_2(\mathbb{R})$ on itself by left multiplication is equivalent to the action of $PSL_2(\mathbb{R})$ on $S\mathbb{H}$ induced by its action on $\mathbb{H}$ by Möbius transformations.*

Proof. Let $\vec{v}$ be the unit vector at a point p on the imaginary axis tangent to the imaginary axis in the complex plane such that $(p, \vec{v})$ is a fixed element of $S\mathbb{H}$. Furthermore, suppose we have an element $(p_0, \vec{v}_0) \in S\mathbb{H}$. Then we know that there exists a transformation $T \in PSL_2(\mathbb{R})$ that takes the imaginary axis and transforms it to the geodesic through p_0 tangent to $\vec{v}_0$ and so $T(p) = p_0$. Therefore we know that since $DT(\vec{v}) = \vec{v}_0$, we have $T(p, \vec{v}) = (p_0, \vec{v}_0)$. Furthermore, it follows that $(p_0, \vec{v}_0) \longrightarrow T$ is a homeomorphism. If we take $S \in PSL_2(\mathbb{R})$ with $S(p_0, \vec{v}_0) = (p_0', \vec{v}_0')$ then from the above we know that $S(p_0, \vec{v}_0) = ST(p, \vec{v})$ and $S(p_0, \vec{v}_0)$ maps to ST. Thus, he action of $PSL_2(\mathbb{R})$ on itself by left multiplication is equivalent to the action of $PSL_2(\mathbb{R})$ on $S\mathbb{H}$ induced by its action on $\mathbb{H}$ by Möbius transformations. □

Definition 2.1.3. A subgroup given as $\Gamma \subset \mathcal{G}$ is discrete if the subspace topology on Γ is the discrete topology.

2.2. **Discrete and Properly Discontinuous Groups.**

Definition 2.2.1. A discrete subgroup of $\mathcal{G}$ is a Fuchsian Group if it is a discrete subgroup of $PSL_2(\mathbb{R})$.

Remark 2.2.2. The action of $PSL_2(\mathbb{R})$ on the upper half plane $\mathbb{H}$ by isometries lifts to the action on $S\mathbb{H}$.

From this definition, however, it is useful to develop a notion of discontinuity that helps when considering Fuchsian Groups. Thus, the following definitions are presented:

Definition 2.2.3. A family $\{F_p \mid p \in P\}$ of subsets of a metric space M indexed by elements of P is locally finite if and only if $F_p \cap X \neq \emptyset$ for only finitely many $p \in P$ where $X \subset M$ is compact.

Definition 2.2.4. Let G be a group of homeomorphisms of the metric space M. Then given a family $G(x) = \{g(x) \mid g \in G\}$ where $x \in M$, we refer to this family as the G-orbit of a point x. Furthermore, we denote the stabilizer of x in G as G_x, the order of which is the multiplicity of the points of $G(x)$.

Definition 2.2.5. A group G acts properly discontinuously on M if the family of G-orbits of a point $p \in M$ is locally finite.

These three definitions then allow us to consider the nature of properly discontinuous group actions.

Theorem 2.2.6. *A group G acts properly discontinuously on M if and only if each $p \in M$ has a neighborhood U such that for only finite $T \in G$:*

$$T(U) \cap U \neq \emptyset$$

Proof. This proof is broken into two segments, which follow:

(1) First, suppose that G acts properly discontinuously on M. Then we know each orbit $G(p)$ is discrete and G_p is finite for each point p. Thus, it follows that for some ball $B(p, \epsilon)$ centered at a point p with radius ϵ, $G_x \cap B(p, \epsilon) = x$. If we let $U \subset B(p, \frac{\epsilon}{2})$ be a neighborhood of p, it also follows that $T(X) \cap U \neq \emptyset$, which means that $T \in G_p$ and this holds for only finitely many $T \in G$. Therefore, if G acts properly discontinuously on M, then each $p \in M$ has a neighborhood U such that $T(U) \cap U \neq \emptyset$ for only finite $T \in G$.

(2) Now, suppose that each $p \in M$ has a neighborhood U such that $T(U) \cap U \neq \emptyset$ for only finitely many $T \in G$. If G is indiscrete, then there exists a limit point σ_0 such that any neighborhood B_{σ_0} will have infinite images in G, which contradicts the assumption. Furthermore, suppose that $T(\sigma) = \sigma$ for infinite $T \in G$. Then we know that any neighborhood U of σ has a non-finite number of images under G. Thus, the stabilizer of each point G_σ is locally finite and so by Definition 2.2.5 we know that G acts properly discontinuously on M as desired.

From the above it follows that a group G acts properly discontinuously on M if and only if each $p \in M$ has a neighborhood U such that for only finite $T \in G$, $T(U) \cap U \neq \emptyset$ is true. □

Example 2.2.7. One simple example of a Fuchsian group is the modular group $PSL_2(\mathbb{Z})$. This is described by the relation

$$PSL_2(\mathbb{Z}) = \left\{ f(z) = \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1 \right\}$$

.

The next step is to examine the interrelations between discrete subgroups of $PSL_2(\mathbb{R})$ and properly discontinuous action on $\mathbb{H}$. For this reason, the two following lemmas are introduced:

Lemma 2.2.8. *Suppose that $p \in X$ and that $X \subset \mathbb{H}$ is compact. Then $\Lambda = \{T \in PSL_2(\mathbb{R}) \mid p \in X\}$ is compact.*

Proof. We know that there exists a continuous map $\varphi\colon SL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R})$ given by $\varphi(A) = T$ and $T(z) = \dfrac{az+b}{cz+d}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In order to show that $\{T \in PSL_2(\mathbb{R}) \mid T(z_0) \in X\}$ is compact, we show that

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \mid \frac{az_0+b}{cz_0+d} \in X \right\}$$

is compact. First, examine the matrix A. We can identify the elements $\{a, b, c, d\}$ with $\mathbb{R}^4$ such that they are mapped to the coordinates (a, b, c, d). Thus, we have a continuous map $\psi\colon SL_2(\mathbb{R}) \longrightarrow \mathbb{H}$ given by $\psi(A) = \varphi(A)z_0$. Furthermore,

$$\psi^{-1}(X) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \mid \frac{az_0+b}{cz_0+d} \in X \right\}$$

and from this it follows that the pre-image of the closed set is closed. We also know that the set is bounded since X is bounded and so there exists $\mu > 0$ such that $|\dfrac{az_0+b}{cz_0+d}| < \mu$ for all $A \in \psi^{-1}(X)$. Furthermore, since X is compact in $\mathbb{H}$, we know that there exists some $\epsilon_0 > 0$ such that the imaginary component of $\frac{az_0+b}{cz_0+d}$ is greater than or equal to ϵ_0 and so by Theorem 1.2.6 we know that $\dfrac{az_0+b}{cz_0+d} = \dfrac{z_{0y}}{|cz_0+d|^2}$. We can therefore conclude that

$$|cz_0 + d| \leq \sqrt{\frac{z_{0y}}{\epsilon_0}}$$

and finally

$$|az_0 + b| \leq \epsilon \cdot \sqrt{\frac{z_{0y}}{\epsilon_0}}$$

so it follows that $\psi^{-1}(X)$ is bounded with respect to $\mathbb{R}^4$. Therefore, it is closed and bounded and by Heine-Borel, compact. Since $\psi^{-1}(X)$ is compact and $\Lambda = \varphi(\psi^{-1}(X))$ we know that $\Lambda = \{T \in PSL_2(\mathbb{R}) \mid T(z_0) \in X\}$ is compact. $\square$

Lemma 2.2.9. *Suppose $\Gamma \subset PSL_2(\mathbb{R})$ acts properly discontinuously on $\mathbb{H}$. If an element of Γ fixes $p \in \mathbb{H}$, then there exists a neighborhood X containing p such that the only element of X fixed by a non-identity element of Γ is p.*

Proof. Let $T(p) = p$ where T is not the identity and $T \in \Gamma$. Suppose there exists a sequence $\{p_1, \ldots, p_n, p\} \in \mathbb{H}$ such that for $\Gamma_n \in \Gamma$ we have $T_n(p_n) = p_n$. Then if we examine the closed ball $\overline{B(p, 3\epsilon)}$, since the topology given by the hyperbolic metric is equivalent to the Euclidean (Remark 1.2.11), we can conclude that $\overline{B(p, 3\epsilon)}$ is compact. Based on the fact that we know Γ acts properly discontinuously, we know that $\{T \in \Gamma \mid T(p) \in \overline{B(p, 3\epsilon)}\}$ is finite. It also follows for some $n > N \in \mathbb{N}$ that $d_{\mathbb{H}}(T_n(p), p) > 3\epsilon$. Hence, $d_{\mathbb{H}}(p_n, p) < \epsilon$ which means since T_N preserves the hyperbolic metric that: $d_{\mathbb{H}}(T_n(p), p) \leq d_{\mathbb{H}}(T_n(p), T_n(p_n)) + d_{\mathbb{H}}(T_n(p_n), p) = d_{\mathbb{H}}(p, p_n) + d_{\mathbb{H}}(p_n, p) < 2\epsilon$ by the triangle inequality. However, this contradicts the above statement that $d_{\mathbb{H}}(T_n(p), p) > 3\epsilon$ and so if $\Gamma \subset PSL_2(\mathbb{R})$ acts properly discontinuously on $\mathbb{H}$ and an element of Γ fixes $p \in \mathbb{H}$, we assert that there exists a neighborhood X containing p such that only p is fixed by Γ. $\square$

From these two lemmas the desired result now follows:

Theorem 2.2.10. *Suppose that $\Gamma \subset PSL_2(\mathbb{R})$. Then Γ is a Fuchsian group if and only if it acts properly discontinuously on $\mathbb{H}$.*

Proof. This proof may be broken into two segments, which follow:

(1) Suppose that Γ acts properly discontinuously on $\mathbb{H}$. Then if we let $p \in \mathbb{H}$ with $X \subset \mathbb{H}$ compact, we can take the set of transformations give by $\{T \in \Gamma \mid T(p) \in X\}$, which is just $\{T \in PSL_2(\mathbb{R}) \mid T(p) \in X\} \cap \Gamma$, to be finite by the compactness of the set Lemma 2.2.8 and its intersection with the discrete group Γ. Therefore, by Definition 2.2.5, it follows that if Γ acts properly discontinuously on $\mathbb{H}$, then it is a Fuchsian group.

(2) For the next part, let Γ act properly discontinuously on on $\mathbb{H}$. Suppose Γ is not a Fuchsian group. Let $p \in \mathbb{H}$ be a point that is fixed only by the identity element of Γ in accordance with Lemma 2.2.10. Since Γ is not Fuchsian, it is not a discrete subgroup and so we know that there exists some sequence of distinct transformations $\{T_i\}$ such that $\lim_{i\to\infty} T_i(p) = p$ and furthermore p is not fixed so there exists a sequence with $p \notin \{T_i(p)\}$. Therefore, it follows that for $\epsilon > 0$ every $\overline{B(s,\epsilon)}$ contains infinitely many points composing the Γ-orbit of p. By Definition 2.2.5 this means Γ does not act properly discontinuously on $\mathbb{H}$.

From these two parts, we know that if Γ acts properly discontinuously on $\mathbb{H}$ it is a Fuchsian group and conversely if Γ is not a Fuchsian group, it does not act properly discontinuously. Thus, for $\Gamma \subset PSL_2(\mathbb{R})$, Γ is a Fuchsian group if and only if it acts properly discontinuously on $\mathbb{H}$. $\square$

As illustrated above, Theorem 2.2.10 provides the critical connection between discrete subgroups of $PSL_2(\mathbb{R})$ and properly discontinuous group actions on $\mathbb{H}$, which characterize Fuchsian groups as subgroups of $PSL_2(\mathbb{R})$.

2.3. **Fixed Points and Abelian Groups.**

We now turn our attention to some of the properties of Fuchsian groups, specifically how the group's algebraic properties are developed. In order to consider these properties, first we consider the fixed point set of a transformation as given in the following lemma:

Lemma 2.3.1. *Suppose $\Lambda T = T\Lambda$. If we let the fixed point set of T be denoted by κ, then $\Lambda\colon \kappa \to \kappa$.*

Proof. Suppose $p \in \kappa$. Then we know that $\Lambda(p) = \Lambda T(p) = T\Lambda(p)$ and therefore $\Lambda(p)$ is fixed and $\Lambda\colon \kappa \to \kappa$ as desired. $\square$

Remark 2.3.2. Commutativity of elements of $PSL_2(\mathbb{R})$ is equivalent to the elements maintaining the same sets of fixed points. (See Katok, pg. 35)

Now we turn our attention to cyclic Fuchsian groups, which produces the following lemma:

Lemma 2.3.3.

(1) *If $\Gamma \subset S^1$ where Γ is discrete, then it is finite cyclic.*

Proof. Suppose that $\Gamma \subset \{x \in \mathbb{C} \mid x = e^{i\theta}\}$ is discrete. Then we know that there exists some minimal valued θ_0 for which $x \in \Gamma$ and there exists a $y \in \mathbb{Z}$ such that $y\theta_0 = 2\pi$. If this were not the case, θ_0 would contradict. Therefore, since $y \in \mathbb{Z}$ is finite, we know that if $\Gamma \subset S^1$ where Γ is discrete, then it is finite cyclic. □

(2) *If $\Gamma \subset \mathbb{R}$ is discrete, then it is infinite cyclic.*

Proof. Suppose that $\Gamma \subset \mathbb{R}$ is discrete. then we know that $0 \in \Gamma$ and there exists some $p \in \mathbb{R}$ such that $p > 0$ and so $\{n \cdot p \mid n \in \mathbb{Z}\} \subset \Gamma$ Now, let $q \in \Gamma$ with $q \neq n \cdot p$. Then $q > 0$ ($q < 0$ would be in Γ) and so we know there exists some $m \in \mathbb{Z}$ such that $m \geq 0$ and $mp < q < (m+1)p$. Furthermore, $q - mp < p$ and it follows that $(q - mp) \in \Gamma$. This is a contradiction from the assumption that $p > 0$ and so a subgroup cannot be finite cyclic. Therefore, for a discrete $\Gamma \subset \mathbb{R}$, Γ is infinite cyclic. □

Theorem 2.3.4. *If $\Gamma \subset PSL_2(\mathbb{R})$ is a Fuchsian group with non-identity elements sharing a fixed point set κ, then it is cyclic.*

Proof. From Definition 2.0.1 we know that an element of $PSL_2(\mathbb{R})$ is categorized by its fixed point set and all elements of a subset $\Gamma \subset PSL_2(\mathbb{R})$ are of the same type. Therefore, this proof can be broken down into three parts.

(1) Suppose that Γ is composed of elliptic elements. We know that Γ is discrete and a subgroup of the orientation preserving isometries of the unit disk. Thus, the conjugate group must have the fixed point set containing 0 and so Γ is isomorphic to a subgroup of S^1. Thus, from Lemma 2.3.3 it follows that Γ is cyclic.
(2) Suppose Γ is composed of hyperbolic elements. Then if we choose p such that $p\Gamma p^{-1}$ is a conjugate group such that it fixes 0 and ∞, we know that $\Gamma \subset K$ is discrete where $K = \{z \longrightarrow cz \mid c > 0\}$. Furthermore, since this is isomorphic to $\mathbb{R}$ through the multiplicative group of real numbers $x \in \mathbb{R}$ with $x > 0$. Thus, by Lemma 2.3.3, we know that Γ is infinite cyclic.
(3) Suppose that Γ is composed of parabolic elements. Then it is infinite cyclic by similar logic as for the hyperbolic case.

From the above it therefore follows that if $\Gamma \subset PSL_2(\mathbb{R})$ is a Fuchsian group with non-identity elements sharing the fixed point set κ, it is cyclic. □

Theorem 2.3.5. *Abelian Fuchsian groups are cyclic.*

Proof. From remark 2.3.2 we know that all elements (with the exception of the identity elements) in a commutative Fuchsian group have a common fixed point set. Thus, from Theorem 2.3.4 we know that Abelian Fuchsian groups are cyclic. □

3. Fundamental Regions

We know that for a group action on a topological space, if we take a single point in the topological space, the images of the given point under the group action form the orbit of the action. Thus, we define a fundamental domain of the group as the subset of the topological space that contains one point from each of these orbits.

3.1. **The Dirichlet Region.**

When considering the fundamental regions of Fuchsian groups, the most critical concept is that of a Dirichlet region of a Fuchsian group. These are defined as follows:

Definition 3.1.1. If $D_p(\Gamma)$ is a Dirichlet region centered at a point p, then it is given by the equation $D_p(\Gamma) = \{x \in \mathbb{H} \mid d_{\mathbb{H}}(x,p) \leq d_{\mathbb{H}}(x, T(p))\, \forall T \in \Gamma\}$

Remark 3.1.2. From Definition 3.1.1 we then know that for each fixed $T_0 \in PSL_2(\mathbb{R})$, we have $\{x \in \mathbb{H} \mid d_{\mathbb{H}}(x,p) \leq d_{\mathbb{H}}(x, T_0(p))\}$ which is described by the set of points $x \in \mathbb{H}$. Furthermore, under the hyperbolic metric, each $x \in \mathbb{H}$ is closer to p than to any of its images.

Definition 3.1.3. We define the perpendicular bisector of a geodesic $\omega = [p_0, p_*]$ as the unique geodesic passing through the midpoint ψ of ω that is also orthogonal to ω.

Lemma 3.1.4. *A line given by $d_{\mathbb{H}}(p, p_0) = d_{\mathbb{H}}(p, p_1)$ is the perpendicular bisector of the segment $\omega = [p_0, p_1]$.*

Proof. Suppose without loss of generality that $p_0 = i$ and $p_1 = ir^2$ where $r > 0$. Then we know that $\psi = ir$ and so the bisector is given by the equation $|p| = r$ where $pin\mathbb{H}$ and it is orthogonal. Furthermore, $d_{\mathbb{H}}(p, p_0) = d_{\mathbb{H}}(p, p_1)$, so applying the fact that $z, w \in \mathbb{H}$, we know that $\cosh(p(z,w)) = 1 + \frac{|z-w|^2}{2z_y w_y}$ (See Katok, pg. 13) This is then equivalent to

$$\frac{|p - p_0|^2}{1} = \frac{|p - p_1|^2}{r^2}$$

which is

$$|p - p_0| = \frac{|p - p_1|}{r}$$

and this simplifies to $|p| = r$. Hence, $d_{\mathbb{H}}(p, p_0) = d_{\mathbb{H}}(p, p_1)$ is the perpendicular bisector of the segment $\omega = [p_0, p_1]$. $\square$

Theorem 3.1.5. *If p is not fixed by any non-identity element of $\Gamma PSL_2(\mathbb{R})$, then $D_p(\Gamma)$ is a fundamental region of $\Gamma \subset PSL_2(\mathbb{R})$. Furthermore, $D_p(\Gamma)$ is connected.*

Proof. Suppose that $q \in \mathbb{H}$ and Γ_q is its orbit in the discrete subgroup Γ. Since Γ_q is discrete, there exists a q_0 with minimal $d_{\mathbb{H}}(p, q_0)$. It then follows that for all $T \in \Gamma$ we have $d_{\mathbb{H}}(p, q_0) \leq d_{\mathbb{H}}(p, T(q_0))$ and so from Definition 3.1.1 we know that $q_0 \in D_p(\Gamma)$. Hence, $D_p(\Gamma)$ has at minimum one point from each of the Γ-orbits. Now, if we take $d_{\mathbb{H}}(q,p) = d_{\mathbb{H}}(T(q), p)$ with $T \in \Gamma$ and T not being the identity, we know that $d_{\mathbb{H}}(q,p) = d_{\mathbb{H}}(q, T^{-1}(p))$. From the above, we can conclude that $q \notin D_p(\Gamma)$ or $q \in \overline{D_p(\Gamma)} \setminus (D_p(\Gamma))$. If $q \in D_p(\Gamma)$, then $d_{\mathbb{H}}(q,p) < d_{\mathbb{H}}(T(q), p)$ with $T \in \Gamma$. Suppose two points q_0, q_1 are in the same Γ-orbit. Then we know that both $d_{\mathbb{H}}(q_0, p) < d_{\mathbb{H}}(q_1, p)$ and $d_{\mathbb{H}}(q_1, p) < d_{\mathbb{H}}(q_0, p)$. This is an obvious contradiction, however, and so the interior of the Dirichlet region given by $D_p(\Gamma)$ has no more than a single point per Γ-orbit. Hence, if p is not fixed by any non-identity element of $\Gamma PSL_2(\mathbb{R})$, then $D_p(\Gamma)$ is a fundamental region of $\Gamma \subset PSL_2(\mathbb{R})$. Furthermore, we know that $D_p(\Gamma)$ is connected since it is the intersection of closed half-planes, which gives a closed convex set and so connectedness follows. $\square$

Thus, from the Theorem 3.1.5, we are able to observe for any discrete $\Gamma \subset PSL_2(\mathbb{R})$ the Dirichlet region as a connected fundamental region. Furthermore, from this critical theorem, we are able to deduce further properties of Fuchsian groups, including those regarding the Ford fundamental region and properties of Riemann surfaces.

Conclusions

From the previous development of the properties of hyperbolic geometry and its applications to the consideration of $PSL_2(\mathbb{R})$ and its discrete subgroups, one may derive a base level of knowledge on some of the geometric and algebraic properties that compose Fuchsian groups. Furthermore, the discussion of fundamental regions and Dirichlet Regions provides some exposure to the powerful applications of Fuchsian groups to other areas of mathematics, such as the hyperbolic geometry of Riemann surfaces.

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References

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