CLASSIFICATION OF SURFACES

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ABSTRACT. The sphere, Möbius strip, torus, real projective plane and Klein bottle are all important examples of surfaces (topological 2-manifolds). In fact, via the connected sum operation, all surfaces can be constructed out of these. After introducing these topological spaces, we define and study important properties and invariants of surfaces, such as orientability, Euler characteristic, and genus. Last, we give a proof of the classification theorem for surfaces, which states that all compact surfaces are homeomorphic to the sphere, the connected sum of tori, or the connected sum of projective planes.

Contents

1. Introduction

The main goal of this paper is to develop a geometric intuition for topological subjects. The structure of the paper is as follows. In §2, we make some basic definitions and describe the construction of basic surfaces such as the sphere, torus, real[1] projective plane, and Klein bottle.

In §3, we define some fundamental invariants of surfaces: orientability, genus, and Euler characteristic. Our definitions, in keeping with an intuitionistic spirit and a geometric preference, will occasionally be less-than-entirely rigorous, but we hope they will capture the imagination of the reader for whom this account may provide a first introduction.

In §4, we proceed to the proof of the classification theorem for surfaces. The classification theorem was proved in the 19th century, and is a beautiful example of geometric topology. Our proof for the classification theorem is elementary and follows [1]. It emphasizes geometric intuition over technical rigor. For this reason, we will assume the triangulation theorem for surfaces, a classical fact with a difficult proof. We also assume that Euler characteristic is a topological invariant. This fact is implicit in the definition of Euler characteristic as the alternating sum of Betti numbers, but for our naïve definition, it is not obvious.

2. Torus, projective plane and Klein bottle

In this section, we understand the basic definitions for the discussion of surfaces and turn to examine the geometric construction of different surfaces that bear importance in further discussion.

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[1]Henceforth 'real' shall be understood.

2.1. **First definitions.**

Definition 2.1. A topological space X is said to be a **Hausdorff space** if given any pair of distinct points $p_1, p_2 \in X$, there exist neighborhoods U_1 of p_1 and U_2 of p_2 with $U_1 \cap U_2 = \emptyset$

A Hausdorff space means for any two distinct points on the space, there exist two disjoint neighborhoods that contain the two points respectively, or, points can be separated by open sets.

Definition 2.2. A surface is a Hausdorff space that is locally homeomorphic to $\mathbf{R}^2$.

This definition is a rigorous mathematical expression of our intuitive understanding of surfaces, that is, if we place a small ant on a surface, the ant will think it is standing on a flat plane.

Definition 2.3. A surface is **connected** if every two distinct points on the surface are connected by a path in the surface.

Definition 2.4. A surface is said to be **closed** if it contains no boundary.

The sphere and torus are closed and connected. An example of an unconnected surface is a pair of linked tori. Examples of surfaces that are not closed are the disc, cylinder and Möbius strip.

Definition 2.5. Triangulation is the process of adding vertices, edges (non-self-intersecting arcs) and faces on the surface such that the surface is cut into triangles by the edges. Moreover, a triangulation of a closed, connected surface must satisfy the following two properties:

(1) Any edge must be shared by exactly two triangles.
(2) At least 3 edges meet at any vertex v and all triangles with v as a vertex fit round into a circle.

Definition 2.6. Two surfaces A, B are **homeomorphic** if there exists a continuous bijection $f : A \to B$ with continuous inverse $f^{-1} : B \to A$. If two surfaces are homeomorphic, we say they are **topologically equivalent**.

2.2. **Basic surfaces.** We now discuss the construction of the sphere, torus, real projective plane and Klein bottle, and their triangulations.

2.2.1. *Sphere.* The sphere can be defined as a set of points that are at the same distance from a given point in three dimensional space. Another geometric definition for the sphere is the result of a circle rotating along one of its diameter. However, we can construct the sphere from glueing and identifying the same points in a square. If we identify the points in A with the points in A' along with the direction pointed by the arrow, we can glue the edges together. Continue similar process for B and B' and we can get a structure of two connected cones, which is topologically equivalent to a sphere.

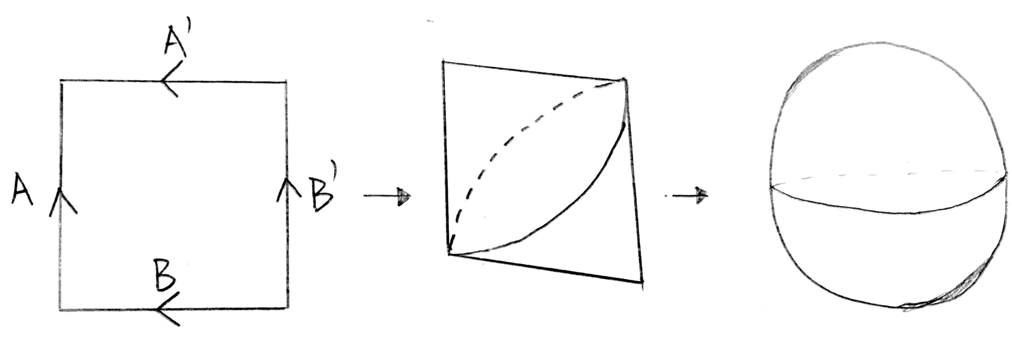

FIGURE 1. Construction of the sphere

2.2.2. *Torus.* The torus can be generated by the rotation of circle in a circular direction, but the similar square construction also applies. If we identify the opposite edges of a square with the same direction, we can get a torus.

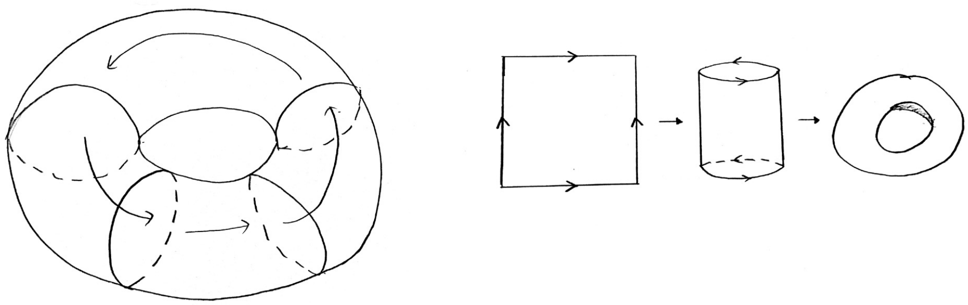

FIGURE 2. Construction of the torus

2.2.3. *Möbius strip.* If we identify a pair of opposite sides in a square oppositely, we can get a Möbius strip. Notice that Möbius strip has only one side: if we use a paper strip to make a Möbius strip, choose a path and continuously paint the paper red along the path until we paint the strip all red. According to this one-sideness, a Möbius strip has only one closed boundary.

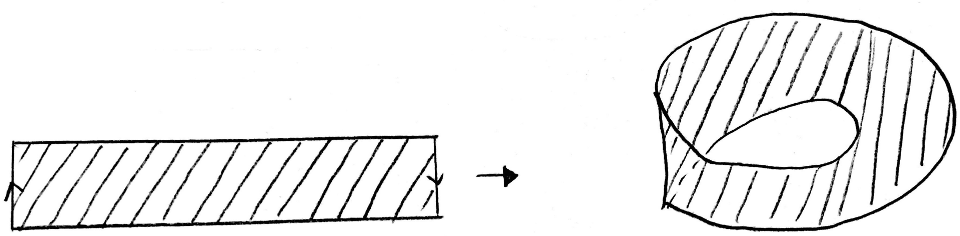

FIGURE 3. Construction of the Möbius strip

2.2.4. *Real Projective Plane.* The real projective plane ($\mathbf{R}P^2$) is classically defined as the space of lines through the origin in Euclidean 3-space. The geometric construction of $\mathbf{R}P^2$ is to identify both pair of sides in square oppositely.

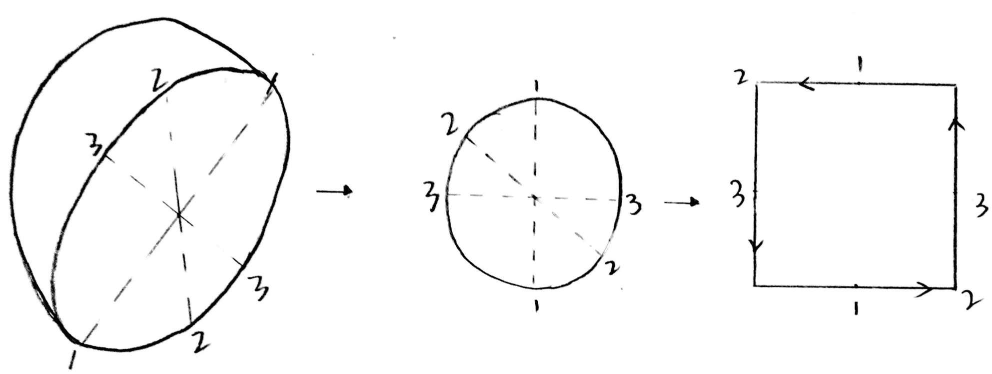

FIGURE 4. Reduction of $\mathbf{R}P^2$

The following argument will bring the classic definition and the geometric construction together. Since lines are determined by their directions, we can choose the line segments from the origin to the points on the unite sphere to represent the entire lines. Since the segments from the origin to opposite points on sphere are on the same line, we only need to consider the line segments in one hemisphere. If we discard the other hemisphere, it remains to consider the equator: the point on the equator will be identified with its opposite point. Since the hemisphere is topologically a disk, $\mathbf{R}P^2$ is in essence a disk with boundary sewed diametrically. If we stretch the sphere into a square, the effect is same as identifying both pairs of sides oppositely.

2.2.5. *Klein bottle.* Identifying a pair of sides in a square generates a cylinder. If we identity the remaining pair of sides with the same direction, we get a torus. If we identify them with opposite direction, we will get a Klein bottle.

The Klein bottle is challenging for imagination, because it is a 2-dimensional surface embedded in 4-dimensional space. In 3-dimensional space, no matter how we twist a cylinder, the two boundaries cannot be glued oppositely. However, this construction is possible in 4-dimensional space, since the self-intersection will disappear in a background of higher dimension. Think, for example, of a pair of intersecting lines at 2-dimensional space; this intersection will disappear if we gently lift one of the lines up, extending the structure to 3-dimensional. Similar method applies for the Klein bottle to resolve the problem of self-intersection.

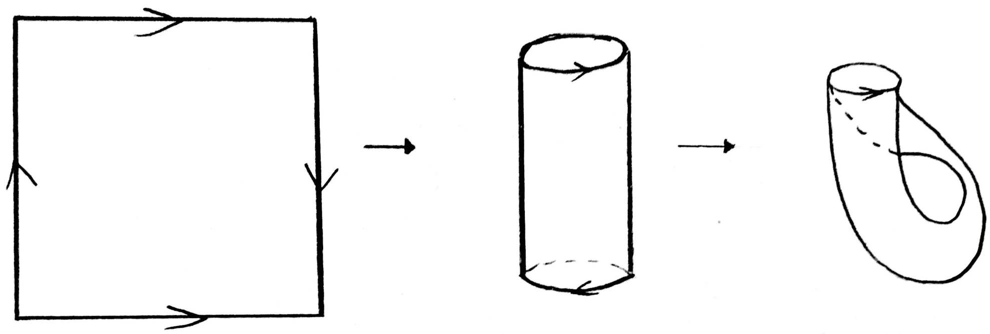

FIGURE 5. Construction of the Klein bottle

3. Orientability, Genus and Euler characteristic

Definition 3.1. A closed and connected surface is **orientable** if the direction does not change after one sails around the surface by any path. A surface is non-orientable if it is not orientable.

For example, a Möbius strip is non-orientable. If we place an ant and a gift on the Möbius strip, if the ant travels around the strip and goes back to the starting point on the strip, it cannot get the gift, because it is standing opposite to the gift. The ant and the gift face opposite direction, because if a point of the strip is on the left of the ant, then the same point is on the right of the gift.

Theorem 3.2. *A closed and connected surface is non-orientable if and only if it contains a Möbius strip.*

Proof. Since Möbius strip is non-orientable, there exists a closed path on the path that changes direction, which is also on the surface containing a Möbius strip. This is the proof for forward direction.

If a closed connected surface is non-orientable, there exists a closed path that changes direction. If we cut this path out from the original surface, we can get a Möbius strip. □

This theorem tells us non-orientability has something to do with one-sidedness, since Möbius strip only had one side, and is closely related to non-orientable surfaces. The real projective plane and the klein bottle are two examples of non-orientable surfaces, the following graphs illustrate that they both contain Möbius strip.

Definition 3.3. **Genus** is the number of handles sewed on the surface.

Definition 3.4. Let M be a triangulable surface. Let v be the number of vertices, e the number of edges and t the number of triangles. Then the **Euler characteristic** $\chi(M) = v - e + t$.

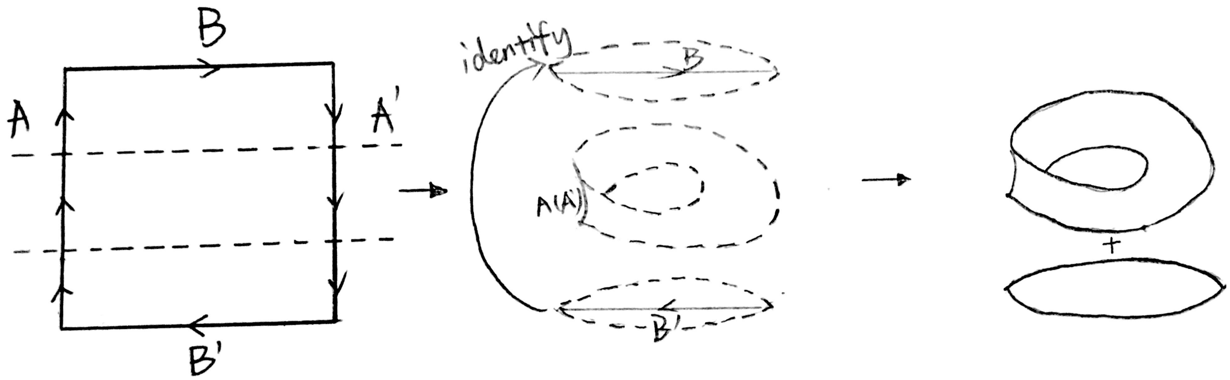

FIGURE 6. Real projective plane = Disk + Mobius strip

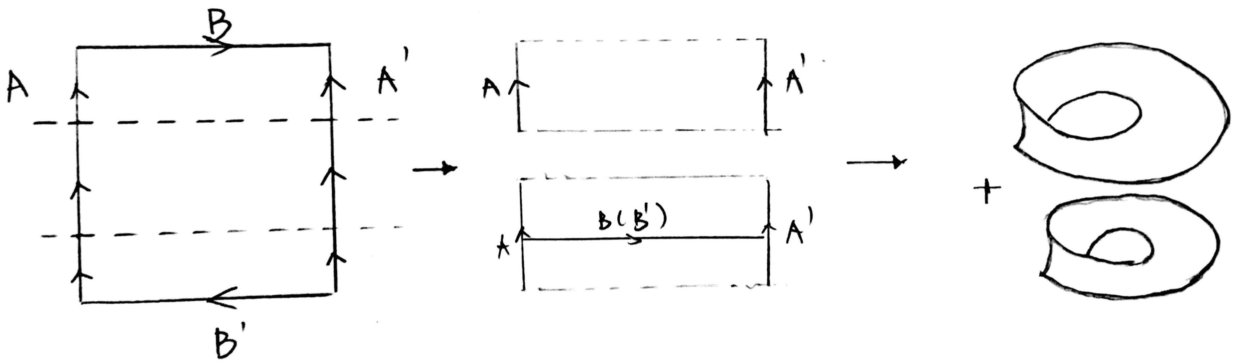

FIGURE 7. Klein bottle = 2 Mobius strip

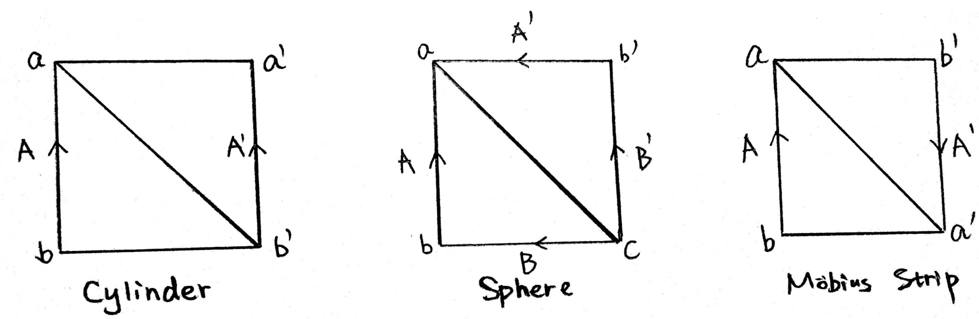

FIGURE 8. Triangulations

3.1. **Computations.** We will calculate the Euler characteristic of some surfaces to facilitate understanding. For the cylinder, since we identity A with A', there are two vertices $a(a')$ and $b(b')$, four edges and two triangles. So $\chi(Cylinder) = 2 - 4 + 2 = 0$. The calculation of the Euler characteristic of Möbius strip is similar to that of cylinder. The triangulation of sphere creates three vertices, three edges and two faces, so its Euler characteristic is 2.

3.2. **Connected Sum.**

Definition 3.5. $X*Y$ is the **connected sum** of two closed surfaces X, Y, a joint surface we get by removing a disk from each surface and gluing together along the boundary.

3.2.1. *the orientability of the connected sum.*

Theorem 3.6. *$X * Y$ is orientable if and only if both X and Y are orientable.*

Proof. This is a corollary from 3.2. If X and Y are orientable, then they don't contain Möbius strip, so their connected sum doesn't contain Möbius strip. Thus $X * Y$ is orientable. If X contains a Möbius strip, then X with a disk removed also contains the strip, so $X * Y$ is non-orientable. □

3.2.2. *the genus of the connected sum.*

Theorem 3.7. *If X is a closed and connected surface with genus n, and Y is a closed and connected surface with genus m, then $X * Y$ is a closed and connected surface with genus $n + m$.*

Proof. Since X is homeomorphic to a ball sewed with n handles, we can locate the disk on the ball for convenience. Similarly, Y is homeomorphic to a ball sewed with m handles, so we locate the removable disk on the ball, too. Glueing together X and Y along the boundary means glueing together two balls along the boundary, since the boundaries of removed disks only appear on the balls, which results in a new ball with all the handles on the previous two balls preserved. Thus, the genus of $X * Y$ is $m + n$. □

3.2.3. *the Euler characteristic of the connected sum.*

Lemma 3.8. *If X, Y are surfaces with closed boundaries, and Z is the result of glueing X and Y along the boundary, then $\chi(Z) = \chi(X) + \chi(Y)$.*

Proof. Let M_X, M_Y be triangulations on X and Y such that M_X, M_Y have the same number of vertices and edges on the boundary.[2] Let M_Z be formed by adding up the vertices,edges and triangles in M_X,M_Y, and subtract the difference between the number of vertices and the number of edges on the boundary. M_Z is a triangulation of Z, since the glueing process preserves all the triangulation patterns of X and Y except merging the two boundaries. Since the boundaries on X and Y are closed, the number of vertices equals to the number of edges. Thus the number of vertices/edges/triangles in M_Z equals to the sum of number of vertices/edges/triangles of X and Y. Therefore, $\chi(Z) = \chi(X) + \chi(Y)$. □

Theorem 3.9. $\chi(X * Y) = \chi(X) + \chi(Y) - 2$

Proof. First, let's compute the Euler characteristic of the disk. If we pick three vertices along the boundary and one vertex in the center, they form a triangulation with four vertices, six edges and three triangles. Thus, $\chi(disk) = 1$. Denote X', Y' as the surface we get by removing a disk from X and Y, then the previous lemma tells us that

$$\chi(X) = \chi(X') + \chi(disk)$$

and

$$\chi(Y) = \chi(Y') + \chi(disk).$$

Applying the lemma again, since we glue X' and Y' along their closed boundaries to get $X * Y$,

$$\chi(X * Y) = \chi(X') + \chi(Y') = \chi(X) + \chi(Y) - 2.$$

□

[2]We assume that all surfaces can be triangulated, since the complete proof of this statement is beyond the scope of this paper. Please refer the complete proof to P.J.Hilton and S. Wiley:*Homology Theory* (Cambridge) 1960

4. The Classification Theorem

In this section, we will prove the classification theorem, which states that any closed and connected surface is homeomorphic to the sphere, the connected sum of tori, or the connected sum of real projective planes. To prove this theorem, we need to derive two important lemmas from the graph theory.

Definition 4.1. An **edge** is an unordered pair of vertices. A graph is a pair $G = (V, E)$ where V is the set of vertices and E is the set of edges.

Definition 4.2. A graph G is **connected** if there is a path between each pair of vertices.

Definition 4.3. A **tree** is a connected graph without cycles.

Lemma 4.4. *A tree contain at least one end point, which is a vertex only on one edge.*

Proof. Let $G = (V, E)$ be a tree. Suppose otherwise, then each vertex is on at least two edges. If we choose a vertex v and an edge incident to it, we can reach another vertex v', and we can find another edge incident to v'. This path can proceed infinitely, since every vertex we encounter provides an additional edge. Since the number of vertices is finite, by pigeon hole principal, we will encounter a vertex twice, which creates a loop. Since the tree does not contain any loop, the contradiction arises.

□

Definition 4.5. The **Euler characteristic** for a graph is the number of vertices minus the number of the edges. $\chi(G) = v - e$.

Lemma 4.6. *If T is a tree, then $\chi(T) = 1$.*

Proof. We will prove by induction on the number of edges. For the base case, when $e = 0$, we have a graph of one vertex, then $\chi(T_0) = 1 - 0 = 1$. Suppose that a tree with k edges has the Euler characteristic 1. Then let T_{k+1} denote a tree with $k+1$ edges, by the last theorem, we can find an end point and remove the end point as well as the edge incident to it. The result is a tree T_k with k edges. Then $\chi(T_{k+1}) = \chi(T_k) + 1 - 1 = 1$. □

Lemma 4.7. *If G is connected and contains a loop, then $\chi(G) < 1$.*

Proof. Let G be a graph containing loop, we can at least remove an edge on the loop without lost of connectedness. Continue tho remove the edges until we get a loop-free connected graph G', which is a tree. Let r be the number of edges we removed, since the total number of edges is finite, r is finite and $r \geq 1$. Therefore, $\chi(G) = \chi(G') - r < \chi(G') = 1$. □

Proposition 4.8. *If G is connected, then $\chi(G) \leq 1$.*

Definition 4.9. Let M be a surface, and choose a triangulation G for M. Within each triangle X in G, choose a point x inside the triangle as the **dual-vertex**. If X, Y are triangles in G that share an edge, then join the dual-vertex x, y to form a **dual-edge** xy. A **dual triangulation** is a graph consists of dual-vertices and dual-edges.

Definition 4.10. The **complement** K of a dual-tree T is the set of all vertices and edges in the triangulation of M that does not intersect with the edges in T.

Lemma 4.11. *The complement K of a dual-tree T is connected.*

Proof. We will prove by induction on the number of the edges in dual-tree. For the base case, when $e = 0$, the complement is the vertices and edges on M, which is connected. Suppose the complement of the dual-tree with n edges is connected, then let K_{n+1} be the complement of a dual tree T_{n+1} with $n+1$ edges. Let the end point in dual tree be x, and let the vertex connected to x be Y. Let the corresponding triangle to x, y be X, Y. Label the vertices of X as a, b, c as shown in the graph. Any two vertices in K_n is linked by a path. If the path does not contain ab, then the path is preserved in K_{n+1}. If the path contains ab, then ac and cb can replace ab in the path, so any two vertices in K_{n+1} is connected. □

Definition 4.12. A dual-tree is **maximal** is it is not contained in any larger dual-tree

Lemma 4.13. *A maximal dual-tree contains all the dual-vertices.*

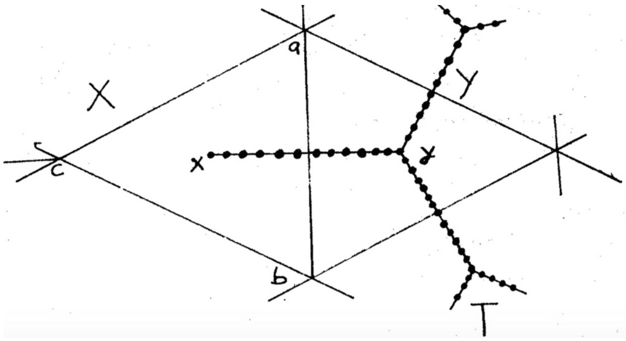

FIGURE 9. Illustration for Lemma 3.13

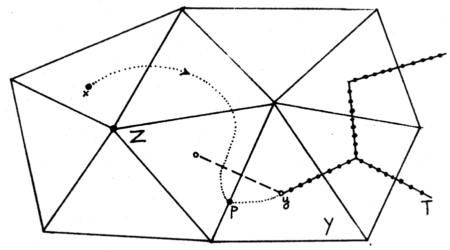

FIGURE 10. Illustration for Lemma 3.14

Proof. Suppose for contradiction that there exists a dual vertex x not contained in the dual-tree T. Let v be a dual-vertex in T, we can draw a directed path P from x to v such that P does not path through vertices in M, so P only intersects with triangles at their edges. Let Y be the first triangle on P, whose corresponding dual-vertex y is contained in T. Let p be the first point in P that lies in the triangle Y, then p is on an edge of Y, and let the dual-edge corresponding to that edge be yz, then z is the dual-vertex of a triangle adjacent to Y. Since y is the first dual-vertex in T on the path, z is not contained in T. Let T' be the dual-tree obtained by adding vertex z and edge yz, then T is not maximal.

□

Theorem 4.14. *If M is a closed and connected surface, then $\chi(M) \leq 2$*

Proof. Let T be a maximal dual-tree, and let G be the complement of T. Since T consists of all the dual-vertices, then G consists of no triangles, then G is a connected graph. By definition, the vertices of M are the vertices of G, and the edges of M are either in G or have a corresponding intersecting dual-edge in T. The triangles in M corresponds to dual-vertices in T. Thus, $\chi(M) = \chi(T) + \chi(G) \leq 2$. □

Definition 4.15. Let M be a surface and choose a triangulation. M is **spherelike** if every closed curve consists of vertices and edges of the triangulation separates M into two disconnected parts.

Theorem 4.16. *If M is a closed and connected surface, then the following statements are equivalent:*

(1) M is spherelike.
(2) $\chi(M) = 2$.
(3) M is homeomorphic to a sphere.

Proof. First, we will prove that (*1*) implies (*2*) by contradiction. Suppose that $\chi(M) \neq 2$, then by 4.14, $\chi(M) < 2$, and $\chi(G) = \chi(M) - \chi(T) < 1$, if we define T, G similarly. Thus, G is not a tree, and consequently contains a loop C. By the definition of spherelike, C is a closed curve and will separate M. However, the complement of T is connected, which is a contradiction.

Next, assume that $\chi(M) = 2$, and we need to show that M is homeomorphic to a sphere. We want to find a method to divide M into to two disks with same boundary. Let T be a maximal dual tree, and K be its complement, then K is also a tree, since $\chi(K) = \chi(M) - \chi(T) = 1$. Define $N(T)$ and $N(K)$ as neighbors of T and K, such that for any point x on the surface M, let $t(x)$ and $k(x)$ denote the distance from x to $T \cap X$ and $K \cap X$, where X is the triangle that contains x: if $t(x) \leq k(x)$, then $x \in N(T)$; if $k(x) \leq t(x)$, then $x \in K(T)$. Thus, $x \in N(T) \cap N(K)$ if and only if $t(x) = g(x)$, which defines the shared boundary of two neighbors. Then, we want to show that $N(T)$ and $N(K)$ are disks. Since the tree has an end point, and if we remove an end point and its incident edge from the tree, the remaining graph is still a tree, we can continue this process inductively until we shrink the tree into a point. Let the neighbor of the tree shrink and wrap around the point, we will reverse the shrinking process by adding an edge and a point each time to the graph. Each time the neighbor stretches to fit along the new antenna, but it never does more than homeomorphism. Thus, the neighbor is equivalent to the disk surrounding the point when we reduced the tree. Therefore, $N(T)$ and $N(G)$ are disks that share a boundary, so M is the product of sewing two disks along the same boundary, which is a sphere.

At last, it remains to show that *(3)* implies *(1)*, which is equivalent to prove that a sphere is spherelike. Let C be polygonal consisting of a finite number of great-circle arcs. Let x be a point on the sphere, such that x is not on C or on any great circles that contain arc in C. Regard x as north pole, and if we pick up any point y on sphere except at the two poles, draw a meridian arc from x to y, and we can label y as even or odd depending on the parity of the numbers of intersection between arc xy and C. Note that if arc xy intersects C at a vertex, then we can count the intersection twice. Label x itself as even, and the south pole depending on number of intersection on one of the meridian arc from north pole from south. Thus we have labeled all points on the sphere and separate the sphere into groups of odd points and even points. Any path connected an odd point and an even point will necessarily cross through C. Therefore C separates the sphere.

□

Theorem 4.17 (the Classification Theorem). *Every closed and connected surface is homeomorphic to one of the following:*

(1) A sphere
(2) A connected sum of tori
(3) A connected sum of projective planes.

Proof. Given that M is spherelike if and only if $\chi(M) = 2$, and that $\chi(M) \leq 2$ for all closed and connected surface, it remains to discuss the situation where $\chi(M) < 2$, for M that is not a spherelike surface. By definition, there exists a closed curve C on M such that C does not separate M. We will perform a process of surgery on C, which can be discussed in two cases:

(1) C is orientation-preserving when C is contained in a cylinder. Cut along C, and insert two disks along the two new boundary curve we just created. Leave the direction on the new curves, because later we need to sew them together according to the direction.
(2) C is orientation-reversing when C is contained in a Möbius strip, so cutting along C will create only one new boundary. Cut along C and insert only one disk along the boundary curve.

Let M_1 be the resulting surface after surgery. We claim that

$$\chi(M_1) = \begin{cases} \chi(M) + 2, & \text{if } C \text{ is orientation-preserving} \\ \chi(M) + 1, & \text{if } C \text{ is orientation-reversing} \end{cases}$$

Since for the curve C, the number of vertices equals the numbers of edges, $\chi(C) = 0$, so the the removal of C does not change $\chi(M)$. Consider the triangulation of a disk, by adding a new vertex in the center, the number of new edges equals the number of new triangles. Thus, adding a disk increases the Euler characteristic by 1.

Proceed surgery inductively, then we can surge M_k into M_{k+1}, if M_k is not spherelike. Since the Euler characteristic is less than 2 for a closed connected surface, the surgery will only occur finitely until we hit a spherelike surface M_r and end the procedure.

We then will engage in a process of desurgery, which means sewing the disks together according to the directions. There are three cases for the curves:

a. If two curves have opposite direction, then the effect is to remove the two disks and sew a handle on them.
b. If two curves have same direction, then the effect is to remove the two disks and sew on a Klein bottle, which is equivalent to sewing on two Möbius strips.
c. If there is only one disk, remove the disk and sew the boundary of the disk diametrically, then the effect is to sew on a Möbius strip.

After performing the surgery we obtain a surface M^*, which is homeomorphic to our original surface M.

(1) If M is orientable, then M^* is orientable, so M^* does not contain any Möbius strip. Thus, only (a) type desurgery can happen. As a result, M^* is a sphere with n handles sewed on, where n is the genius of the connected sum of tori. We can compute n by the Euler characteristic:

$$n = 1 - \frac{\chi(M)}{2}$$

(2) If M is non-orientable, then not only type (a) occurs. If so, perform the desurgeries of type (b) and (c) first, so we get a surface with at least one Möbius strip sewed on. We can now convert type (a) into type (b), by pulling one of the disks along a Möbius strip. Just as an ant changes direction after walking around the strip, the disk also changes direction. Then the genus is twice the number of type (b) plus the number of (c), so $n = 2 - \chi(M)$.

$\square$

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References

[1] E.C.Zeeman. An introduction to topology: the classification theorem of surfaces. http://www.maths.ed.ac.uk/ aar/surgery/zeeman.pdf