KAKUTANI'S FIXED POINT THEOREM AND THE MINIMAX THEOREM IN GAME THEORY

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ABSTRACT. The minimax theorem is one of the most important results in game theory. It was first introduced by John von Neumann in the paper *Zur Theorie Der Gesellschaftsspiele*. Later, John Forbes Nash Jr. provided an alternative proof of the minimax theorem using Brouwer's fixed point theorem. We describe in detail Kakutani's proof of the minimax theorem using Kakutani's fixed point theorem, and discuss applications of Kakutani's fixed point theorem to economics and the theory of zero-sum games.

CONTENTS

1. INTRODUCTION

Game theory predicts how rational individuals would behave under interdependence. It provides a great mathematical tool to infer outcomes when people have conflicts of interests, which often happens in economic situations.

The minimax theorem is one of the most important results in game theory. The theorem shows that there exists a strategy that both minimizes the maximum loss and maximizes the minimum gain. In other words, there is a strategy which rational people would take assuming the worst case scenario. Moreover, by working backwards from the outcome of the minimax theorem, it is possible to calculate the best strategy the players could take.

In 1928, John von Neumann proved the minimax theorem using a notion of integral in Euclidean spaces. John Nash later provided an alternative proof of the minimax theorem using Brouwer's fixed point theorem. This paper aims to introduce Kakutani's fixed point theorem, a generalized version of Brouwer's fixed point theorem, and use it to provide an alternative proof of the minimax theorem. Then, we will discuss applications of Kakutani's fixed point theorem to economics and the theory of zero-sum games.

Since this paper aims to be as self-contained as possible, the following are some basic topological concepts necessary for discussion and an introduction of Brouwer's fixed point theorem.

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Definition 1.1. A **convex hull** of a set X of points in the Euclidean plane or Euclidean space is the smallest convex set that contains X.

Definition 1.2. A **polytope** is a geometric object with flat sides, and may exist in any general number of dimensions n as an n-dimensional polytope.

Definition 1.3. A **simplex** is a k-dimensional space polytope which is the convex hull of its $k+1$ vertices. Suppose the $k+1$ points are $u_0, u_1, ..., u_k \in \mathbb{R}^k$ and are affinely independent, which means $u_1 - u_0, u_2 - u_0, ..., u_k - u_0$ are linearly independent. Then, the simplex determined by them is the set of points

$$C = \{\theta_0 u_0 + \theta_1 u_1 + ... + \theta_k u_k,\ \theta_i \geq 0\ (i = 0, 1, 2, ..., k),\ \sum_{i=0}^{k} \theta_i = 1\} \tag{1.4}$$

Definition 1.5. A **continuous mapping** is such a function that maps convergent sequences into convergent sequences: if $x_n \to x$ then $g(x_n) \to g(x)$.

Theorem 1.6 (Brouwer Fixed Point)**.** *If $x \to \varphi(x)$ is a continuous point-to-point mapping of an r-dimensional closed simplex S into itself, then there exists an $x_0 \in S$ such that $x_0 = \varphi(x_0)$.*

The proof of Brouwer's fixed point theorem uses the following

Proposition 1.7. *There exists no retraction $D^2 \to \partial D^2 = S^1$. Moreover, there exists no retraction $D^{n+1} \to /partial D^{n+1} = S^n$.*

The modern proof of this proposition is one line and uses the fundamental group. As familiarity with the fundamental group is not assumed, we take the proposition for granted and proceed to the proof of Brouwer's fixed point theorem. A reference for the proof is [4].

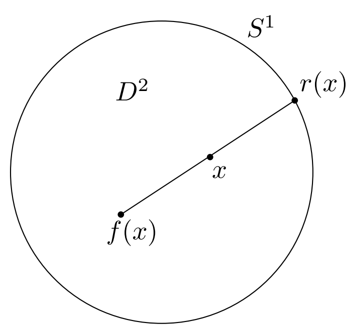

FIGURE 1.

Proof of Brouwer's fixed point theorem (Theorem 1.6). Start with two-dimensional case. Let $x \to \varphi(x)$ be a continuous mapping such that $\varphi : D^2 \to D^2$, where D^2 is an unit disk. Let S^1 be the boundary of D^2. Assume $\varphi(x) \neq x$ for all x. Define $r(x) \in S^1$ be the intersection of S^1 with the ray that starts from $f(x)$ and goes through x (c.f. Figure 1). The map $r(x)$ is continuous mapping from D^2 to S^1, and if $x \in S^1$ then $r(x) = x$. However, note that if $j : S^1 \to D^2$, then there exists no continuous map $i : D^2 \to S^1$ such that $i \circ j = id$. If we only consider $x \in S^1$, then $\varphi(x)$ can be considered as a mapping from S^1 to D^2. However, $\varphi \circ r(x) = id$ for $x \in S^1$; i.e. φ is a retraction $D^2 \to S^1$. Since this contradicts Proposition 1.7, it must be that $\varphi(x) = x$ for some $x \in D^2$. The proof for arbitrary dimension n is analogous, with D^2 changed to D^n and S^1 to S^{n-1}. (The proof that there exists no retraction $D^n \to \partial D^n = S^{n-1}$ uses the higher homotopy group π_{n-1} in lieu of π_1.) □

2. Kakutani's Fixed Point Theorem

Kakutani's fixed point theorem generalizes Brouwer's fixed point theorem in two aspects. A point-to-point mapping is generalized to point-to-set mapping, and continuous mapping is generalized to upper semi-continuous mapping.

Definition 2.1. A **point-to-set map** is a relation where every input is associated with at least one output in which at least one input is associated with two or more outputs.

Definition 2.2. A point-to-set mapping $x \to \Phi(x) \in \Omega(S)$ of S into $\Omega(S)$ is called **upper semi-continuous** if $x_n \to x_0$, $y_n \in \Phi(x_n)$ and $y_n \to y_0$ imply $y_0 \in \Phi(x_0)$.

Definition 2.3. Barycentric simplicial subdivision is a canonical algorithm to divide an arbitrary convex polytope into simplices with the same dimension, by connecting the barycenters of their faces.

Lemma 2.1. *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let (x_n) be a sequence of real numbers bounded by $[0, 1]$. For sequences that are bounded by $[a, b]$ where $a, b \in \mathbb{R}$, subtract a and divide by $(b-a)/2$ for each element of the sequence to obtain a sequence (x_n), which converges if and only if the original sequence converges. Divide $[0, 1]$ into two intervals $[0, 1/2]$, $[1/2, 1]$. At least one of two intervals must contain infinitely many elements of (x_n), since if both intervals contain finite number of elements then the union of both sets cannot be infinite. Repeatedly split the intervals into halves and let I_n $(n = 1, 2, 3, ...)$ be the intervals that contain infinitely many elements in the each step. Then, $I_1 \subset I_2 \subset I_3 \subset ...$, where the length of $I_n = (1/2)^n$. Now, let the sequence (z_n) be a subsequence of (x_n) made up of one element from each interval. Then (z_n) converges since for any $N \in \mathbf{N}$, there exists m and n such that $|(z_m - z_n)| < (1/2)^N$ since any interval I_n for $n > N$ would have interval smaller than $(1/2)^N$, and $(1/2)^N$ can get arbitrarily small so (z_n) is a Cauchy sequence. Therefore, the subsequence (z_n) converges. □

Lemma 2.2 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof. Let (x_m) be a bounded sequence in $\mathbb{R}^n$. Now consider the sequence $(\pi_1(x_m))$. This sequence is a bounded sequence of real numbers, so it has a convergent subsequence $(\pi_1(x_{m_k}))$. Similarly, the sequences $(\pi_2(x_m))$, $(\pi_3(x_m)), ..., (\pi_n(x_m))$ all have convergent subsequence $(\pi_2(x_{m_k}))$, $(\pi_3(x_{m_k})), ..., (\pi_n(x_{m_k}))$. Let (z_m) be a subsequence of (x_m) such that $(z_m) = (\pi_1(x_{m_k}), \pi_2(x_{m_k}), ..., \pi_n(x_{m_k}))$. Since n-components of (z_m) converges, (z_m) converges as well. □

Corollary 2.2.1. *Every sequence in a closed and bounded set S in $\mathbb{R}^n$ has a convergent subsequence which converges to a point in S.*

Proof. Every sequence in a closed and bounded set is bounded, so it has convergent subsequence by Lemma 2.2. Since the set S is closed, such subsequence converges to a point in S. □

Theorem 2.4. *Kakutani's Fixed Point Theorem 1: If $x \to \Phi(x)$ is an upper semi-continuous point-to-set mapping of an r-dimensional closed simplex S into $\Omega(S)$, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Proof. Let S_n be the nth barycentric simplicial subdivision of S. The natural number n denotes the number of barycentric simplicial subdivisions that are done. The examples of S_0, S_1, and S_2 are Figure 2. Note that we only need the simplex S to be divided into smaller simplices, not necessarily through barycentric subdivision. Such subdivision was introduced as one of many possible subdivisions.

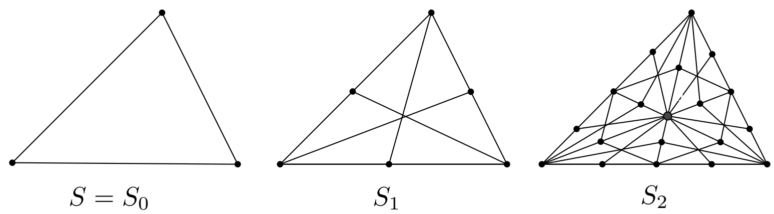

FIGURE 2.

Since S is an r-dimensional closed simplex, there are many smaller r-dimensional closed simplices in S_n. Choose one of those smaller simplices, which would have $r+1$ vertices, and denote it S_x. Denote each vertex as $x_0, x_1, ..., x_r$. Then, all the points in S_x can be represented by the equation $\theta_0 x_0 + \theta_1 x_1 + \cdots + \theta_r x_r, \theta_i \geq 0\,(i = 0, 1, 2, \ldots, r)$, $\sum_{i=0}^{r} \theta_i = 1$ by the definition of simplex. For each of those vertex x_i, $\Phi(x_i)$ would be a set of numbers since $\Phi(x_i)$ is a point-to-set mapping. Let y_i be an arbitrary point in $\Phi(x_i)$. Let the simplex formed by $y_0, y_1, ..., y_r$ be S_y. Then, all the points in S_y can also be represented by the equation $\theta_0 y_0 + \theta_1 y_1 + ... + \theta_r y_r, \theta_i \geq 0 (i = 0, 1, 2, ..., r), \sum_{i=0}^{r} \theta_i = 1$, where θ_i are exactly the same as those needed to express points in S_x. Therefore, there is a point-to-point mapping between S_x and S_y.

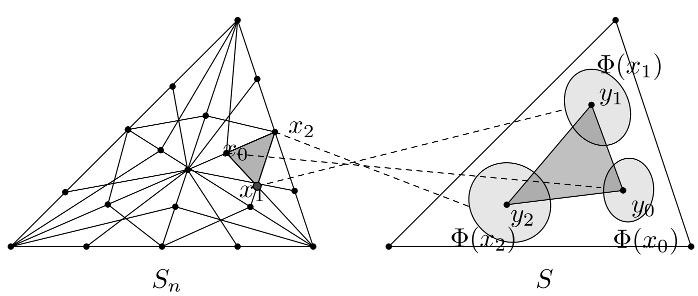

FIGURE 3.

If we repeat this process for all r-dimensional simplices in S_n, then all the points in S_n would have a corresponding point in S. Therefore, such mapping is a point-to-point mapping from S into itself. Denote such mapping as $x \to \varphi_n(x)$. Then, since $x \to \varphi_n(x)$ is a continuous point-to-point mapping of an r-dimensional closed simplex into itself, there exists a point $x_n \in S$ such that $x_n = \varphi_n(x)$ by Brouwer's fixed point theorem (Theorem 1.6). Repeat the whole process for $S_0, S_1, S_2, \ldots$, then there will be corresponding $x_0, x_1, x_2, \ldots$. Since barycentric subdivision can be done sufficiently many times, the sequence $x_0, x_1, x_2, \ldots$ is an infinite sequence. Since S is a closed and bounded set in $\mathbb{R}^n$, by the corollary of Bolzano-Weierstrass theorem (Corollary 2.2.1), the sequence x_0, $x_1, x_2, \ldots$ has a convergent subsequence which converges to a point $x^* \in S$. Such point is the point $x^* \in S$ such that $x_0 \in \Phi(x_0)$. Note here that there could be more than one x^*, if there are more than one convergent subsequences of $x_0, x_1, x_2, ...$

To prove such characteristics of x^*, let $\triangle_n$ be an r-dimensional simplex of S_n which contains the point x^*. The point x^* may be a part of more than one r-dimensional simplex if it lies on the vertices or edges of simplices. In that case, choose any one of such simplices. Let $r+1$ vertices of $\triangle_n$ be $v_0, v_1, ..., v_r$. Then, since $\triangle_n$ is a simplex and $x_n \in \triangle_n$, $x_n = \sum_{i=0}^r \theta_i v_i$ for suitable values of $\theta_0, \theta_1, ..., \theta_r$ that satisfy $\theta_i \geq 0$, $\sum_{i=0}^r \theta_i = 1$. Now, let $q_i = \varphi(v_i), (i = 0, 1, 2, ..., r)$. Then, $q_i \in \Phi(v_i)$ since $\varphi(x)$ was defined to choose one of the elements in $\Phi(x)$. x^* is a fixed point, so $x^* = \varphi(x^*)$. Also, $\varphi(x^*)$ is now in the new simplex made by $q_1, q_2, ..., q_r$, so $\varphi(x^*) = \sum_{i=0}^r \theta_i q_i$ for θ_i that were used to represent $x_n = \sum_{i=0}^r \theta_i v_i$. We can repeat such process for S_n so that

$$x^* = \varphi_n(x^*) = \sum_{i=0}^{r} \theta_i^n v_i^n$$

Now take a further subsequence (n'_v) of (n_v) such that $(\theta_i^{n'_v})$ and $(v_i^{n'_v})$ converge for $i = 0, 1, ..., r$ and put $\lim_{v\to\infty} \theta_i^{n'_v} = \theta_i^0$ and $\lim_{v\to\infty} v_i^{n'_v} = v_i^0$. Then, by the upper semi-continuity of $\Phi(x)$, $y_i^0 \in \Phi(x_0)$ for $i = 0, 1, 2, ..., r$ and this implies, by the convexity of $\Phi(x_0)$, that $x_0 = \sum_{i=0}^r \theta_i^0 v_i^0 \in \Phi(x_0)$. This completes the proof. □

We can generalize Kakutani's fixed point theorem (Theorem 2.4) to an arbitrary bounded closed convex set in Euclidean space.

Definition 2.5. A **retraction** is a continuous mapping of a space onto a subspace leaving each point of the subspace fixed.

Corollary 2.2.2. *Theorem 2.4 is also valid even if* S *is an arbitrary bounded closed convex set in a Euclidean space.*

Proof. Let S' be a closed simplex such that $S \subset S'$. Let a continuous retracting of S' onto S be denoted $x \to \psi(x)$, which is a point-to-point mapping. One example of such retraction would be mapping each point in S' to the closest point in S. Let $x \to \Phi(x)$ be an upper semi-continuous point-to-set mapping from S into $\Omega(S)$, where $\Omega(S)$ is the family of all closed convex subsets of S, just like in Theorem 1. Then, $x \to \Phi(\psi(x))$ is an upper semi-continuous point-to-set mapping of S' into $\Omega(S)$, which satisfies $\Omega(S) \subset \Omega(S')$ since $S \subset S'$. Therefore, $x \to \Phi(\psi(x))$ is also a mapping of S' into $\Omega(S')$, thereby satisfying the conditions of Theorem 1. Since then, by Theorem 1, there exists a point $x_0 \in S'$ such that $x_0 \in \Phi(\psi(x_0))$. Note that $\Phi(\psi(x_0)) \subset S$, since we defined mapping $x \to \psi(x)$ as of S' onto S and $x \to \Phi(x)$ as of S onto S. Therefore, $\Phi(\psi(x_0)) \subset S$ so $x_0 \in \Phi(\psi(x)) \subset S$. Since $x_0 \in S$ and each point of the subspace is fixed after retraction, $x_0 = \psi(x_0$, so

$x_0 \in \Phi(x_0) = \Phi(\psi(x_0)) \subset S$. Since there exists x_0 such that $x_0 \in S$ and $x_0 \in \Phi(x_0)$, this completes the proof.

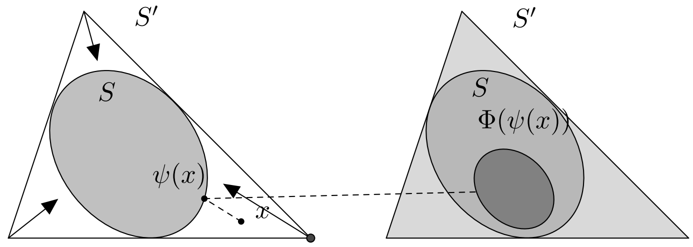

FIGURE 4.

□

Theorem 2.6. *Kakutani's Fixed Point Theorem 2: Let K and L be two bounded closed convex sets in the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let us consider their Cartesian product $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of all $y \in L$ such that $(x_0, y) \in U$, is non-empty, closed and convex, and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is non-empty, closed and convex. Under these assumptions, U and V have a common point.*

Proof. Let $S = K \times L$. Define a point-to-set mapping $z \to \Phi(z)$ of S into $\Omega(S)$ as $\Phi(z) = V_y \times U_x$ if $z = (x, y)$. Such function $\Phi(z)$ is upper semi-continuous for the following reasons. For a sequence of $t_i \in S(i = 0, 1, 2, ...)$ that converges to t_0, let $q_i \in \Phi(t_i)$ such that (q_i) converges to $q_0 \in S$. To show that $\Phi(z)$ is upper semi-continuous, we need to show that $q_0 \in \Phi(t_0)$. It suffices to show that $\phi_1(q_0) \in V_{y_0}$ and $\phi_2(q_0) \in U_{x_0}$, because then $q_0 \in V_{y_0} \times U_{x_0} = \Phi(t_0)$. Since π_1 is continuous, $\pi(q_i)$ is a convergent sequence in K converging to $\pi_1(q_0)$. By the definition of q_i, $\pi_1(q_1) \in V_y$, so $(\pi_1(q_i), y_i) \in V$. Now, $\pi_1(q_i)$ is convergent and y_i is convergent (since t_i is convergent), so $(\pi_1(q_i), y_i)$ is convergent to $(\pi_1(q_0, y_o))$. Since V is closed, $\pi(q_0), y_0) \in V$. Therefore, $\pi_1(q_0) \in V_{y_0}$. Similarly, $\pi_2(q_0) \in U_{x_0}$. Therefore, $\Phi(z)$ is upper semi-continuous function. Now, since S is bounded closed convex set, by the corollary, there exists a point $z_0 \in S = K \times L$ such that $z_0 \in \Phi(z_0)$. Since $(x_0, y_0) = z_0 \in \Phi(z_0) = V_{y_0} \times U_{x_0}$, $x_0 \in V_{y_0}$ and $y_0 \in U_{x_0}$, so $(x_0, y_0) \in V_{y_0} \times U_{x_0}$. Since $V_{y_0} \times U_{x_0} \subset U \cap V$, such $z_0 = (x_0, y_0)$ is the common point of U and V The process is illustrated in Figure 5.

□

Imagine a game in which a player can always take at least one strategy given other players' strategies. Also, let each closed sets U and V be a set of strategies individuals can take. Then Kakutani's fixed point theorem guarantees that there exists a tuple of expected outcomes. Economic interpretations of Kakutani's fixed point theorem will be illustrated in the next section.

3. The Minimax Theorem in Zero-Sum Games

Definition 3.1. A **zero-sum game** is a game in which the sum of all players' payoffs equal zero for every outcomes.

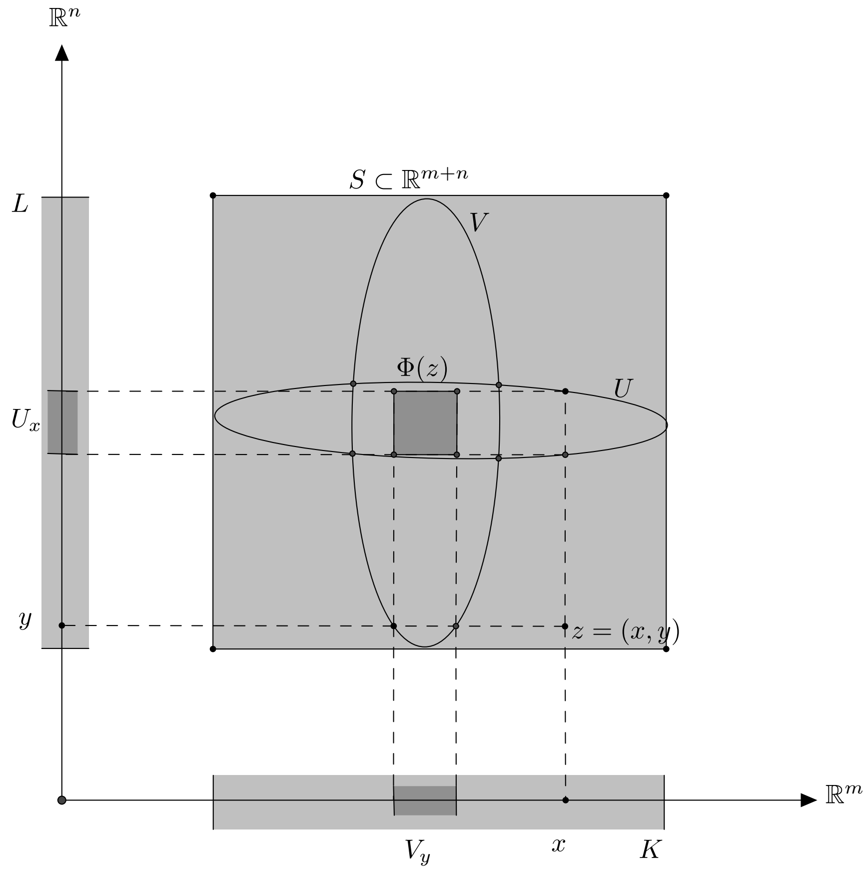

FIGURE 5.

Notice that zero-sum game is a special case of constant-sum game. we can easily convert constant-sum games into zero-sum games through manipulation of utilities. If there is a game with a fixed aggregate utility k, then subtract $k/2$ for each players' utility outcomes. Since the main goal of economics is to determine the effective way to distribute scarce resources for maximum utility, there are many applications of zero-sum games in economic models.

The minimax theorem is applicable in various types of game models, but it is especially significant in analysis of zero-sum games. The basic principle of the minimax theorem is that players try to maximize utilities given the other players will try to minimize his or her outcome or, conversely, try to minimize the loss of utility assuming that the other player will try to maximize it. Moreover, there exists a set of strategies one could take that achieves both objectives. Notice here that the minimax theorem assumes players to be very conservative. Each players assume that the other players will try to worsen his or her utility. Such nature of the minimax theorem explains why it is of special interest in zero-sum games. Since one's benefit exactly equals others' loss in zero-sum games, rational players would tend to act in more conservative way. Since then, the minimax theorem can even be used to trace back the expected strategies of the players by starting from the expected outcome.

Theorem 3.2. *Minimax Theorem: Let $f(x, y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x\in K}\min_{y\in L} f(x, y) = \min_{y\in L}\max_{x\in K} f(x, y). \tag{3.3}$$

[1]

Proof. Let U and V be the sets that consist of all $z_0 = (x_0, y_0) \in K \times L$ such that $f(x_0, y_0) = min_{y\in L} f(x_0, y)$ for set U and $f(x_0, y_0) = max_{x\in K} f(x, y_0)$ for set V. Therefore, by Kakutani's fixed point theorem 2 (Theorem 2.6), there exists a point $z_0 = (x_0, y_0) \in K \times L$ such that $z_0 \in U \cap V$. In other words, there exists (x_0, y_0) such that $f(x_0, y_0) = min_{y\in L} f(x_0, y) = \max_{x\in K} f(x, y_0)$. Therefore, since

$$\min_{y\in L}\max_{x\in K} f(x, y) \leq \max_{x\in K} f(x, y_0) = f(x_0, y_0) = \min_{y\in L} f(x_0, y) \leq \max_{x\in K}\min_{y\in L} f(x, y),$$

$$\min_{y\in L}\max_{x\in K} f(x, y) \leq \max_{x\in K}\min_{y\in L} f(x, y).$$

Now, for all $x_0 \in K$ and $y_0 \in L$, $f(x_0, y_0) \geq \min_{y\in L} f(x_0, y)$. Therefore, for all $y_0 \in L$, $\max_{x\in K} f(x, y_0) \geq \max_{x\in K} \min_{y\in L} f(x_0, y)$. Since then,

$$\min_{y\in L}\max_{x\in K} f(x, y) \geq \max_{x\in K}\min_{y\in L} f(x, y).$$

Therefore, we conclude

$$\min_{y\in L}\max_{x\in K} f(x, y) = \max_{x\in K}\min_{y\in L} f(x, y).$$

□

Note that the closed, convex sets K and L in minimax represent sets of strategies for each players. In general, convexity of the set of strategies can be assumed since players can take certain strategies continuously with no special restrictions other than budget constraints. Also note that function $\Phi(x)$ in the use of the minimax theorem gives a new tuple where a player's expected utility is maximized given the other players' strategy x. Since there can be more than just one optimum strategy given the others' strategies, such function is better represented as a point-to-set mapping and Kakutani's fixed point theorem ensures that a fixed point still holds.

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