AN EXPOSITION OF THE RIEMANN ROCH THEOREM FOR CURVES

DOMINIC L. WYNTER

Abstract. We introduce the concepts of divisors on nonsingular irreducible projective algebraic curves, the genus of such a curve, and differential forms on such a curve. We then state (without proof) the Riemann Roch theorem for curves, and give applications to the classification of nonsingular algebraic curves.

Contents

1. Introduction

The Riemann Roch theorem, though easy to state once the correct formalisms have been developed, is one of the most powerful tools one encounters when first studying the theory of algebraic curves. Making extensive use of the concept of a divisor, the Riemann Roch theorem provides an invaluable connection between the rational functions defined on a curve and the genus. Together with a discussion of rational morphisms associated to a divisor, the Riemann Roch theorem makes it possible to categorize smooth irreducible projective curves, giving additional information about possible embeddings of the curve in projective space.

We will work over smooth projective irreducible curves, which we will denote by the letter Y, and we will occasionally generalize to smooth irreducible varieties, which we will denote by the letter X. We consider a variety to be defined over an algebraically closed field k of arbitrary characteristic, and to be projective, smooth, and irreducible, unless stated otherwise. We begin by defining the concept of a divisor on a curve, a framework that allows us to investigate the zeros and poles of rational functions defined on the curve Y. We then use divisors to define rational maps into projective space, and show that given certain conditions on the divisor, the associated rational map can be made into an embedding. Next, we define the

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vector space of rational differential forms over a curve, which will give us the notion of a genus of a curve. Finally, we state the Riemann Roch theorem, and use it to give a classification of smooth irreducible projective curves by genus.

2. Divisors

We first define the concept of a divisor on an arbitrary irreducible smooth complete variety X. Since any projective variety is complete, and on curves, these two notions are equivalent, in the specific case where X is a curve, we require without loss of generality that X be projective. More precisely, if we wish to define the divisors on a curve Y, we require that Y be an irreducible variety of dimension 1 in some projective space $\mathbb{P}^n_k = \mathbb{P}^n$, over an algebraically closed field k, of any characteristic, such that all the local rings are discrete valuation rings.

Let X be an irreducible smooth complete variety. Then we define $\mathscr{C}$ as the set of all closed irreducible codimension-one subvarieties of Y (which we will refer to as prime divisors).

Definition 2.1. We let Div $Y = \mathbb{Z}[\mathscr{C}]$ be the free abelian group generated by the elements of $\mathscr{C}$. An element $D = \sum_{i=1}^{r} l_i C_i \in \text{Div } Y$, where $l_i \in \mathbb{Z}$ and $C_i \in \mathscr{C}$ for all $1 \leq i \leq r$, is called a *divisor* on Y. Whenever we write the divisor D as a sum $\sum_i l_i D_i$, we will require that D be given in *standard form*, that is, we require the C_i to be distinct, and the l_i nonzero. Alternately, we may consider a divisor D as being given by its coefficients $\text{ord}_C(D)$ at C, which we write as $\text{ord}_p(D)$ when p is a point. More precisely, we let $\text{ord}_{C_i}(\sum_i l_i D_i) = l_i$ and $\text{ord}_C(\sum_i l_i D_i) = 0$ for $C \neq C_i$ for all i, where $\sum_i l_i D_i$ is in standard form.

If $\text{ord}_C(D) \geq 0$ for all prime divisors $C \in \mathscr{C}$, then D is said to be *effective*, or an *effective divisor*. If $D - D'$ is an effective divisor, we say that $D \geq D'$ (this induces a partial ordering on Div Y).

For any divisor $D \in \text{Div } Y$, we define

$$D_0 = \sum_{\text{ord}_p(D) \geq 0} \text{ord}_p(D) \cdot p, \; D_\infty = \sum_{\text{ord}_p(D) \leq 0} \text{ord}_p(D) \cdot p$$

to be the zeroes and poles of the divisor D, respectively.

Note that when Y is a curve, that is to say, a variety of dimension 1, $\mathscr{C}$ consists precisely of the closed points of Y.

Recall that we define the field of *rational functions* $k(Y)$ on Y to be the set of equivalences classes $\langle U, f\rangle$, where $U \subseteq Y$ is a nonempty open subset and $f \in k[U]$ is a regular function, with field operations between $\langle U, f\rangle$ and $\langle V, g\rangle$ defined on the intersection $U \cap V$ (nonempty since Y is irreducible) in the obvious way, and setting $\langle U, f\rangle$ and $\langle V, g\rangle$ to be equivalent when $f\big|_{U\cap V} = g\big|_{U\cap V}$. We will now construct, for rational functions $f \in k(Y)$, the associated divisor. This can be thought of, more or less, as the zeros and the poles of f, with associated multiplicity effectively generalizing a concept already used in analysis. We first define, for any $p \in Y$, a valuation ν_p on $k(Y)$.

Now, for $f \in k(Y)^\times$, recall that we wish to define a valuation ν_p for all points p on the curve Y. Then we may consider $f\big|_U \in k(U)$ by restriction, where U is an open affine neighborhood of p, so we have that

$$f\big|_U = \frac{f_1}{f_2}, \; f_1, f_2 \in k[U]$$

Now, for $0 \neq g \in k[U]$, we look at the image of g in the local ring $\mathcal{O}_p$, which we know to be a DVR, and therefore, $\mathcal{O}_p$ has a maximal ideal of the form $\langle \pi \rangle$. We define $\nu_p(g)$ to be the maximal l such that $g \in \langle \pi^l \rangle$. Note that since $\mathcal{O}_p$ is Noetherian, and g nonzero, l must be finite.

Definition 2.2. For any rational function $f \in k(Y)$, letting U be some nonempty affine open subset of Y, and $f\big|_U = f_1/f_2$ given by the ratio of regular functions $f_1, f_2 \in k[U]$, we define the valuation of f at p to be $\nu_p(f) = \nu_p(f_1) - \nu_p(f_2)$.

If we fix the choice of affine open U, since $k[U]$ has fraction field $k(U) = k(Y)$, ν_p gives a well defined valuation on $k(Y)$. However, it is not necessarily clear that ν_p will be independent of the choice of U. Since the proofs of the following statements are easily worked out and not very interesting, we omit them.

Proposition 2.3. *We have that ν_p is independent of the choice of U.*

Proof. Omitted (Shafarevich pg. 128 [4]). □

Proposition 2.4. *For $f \in k(Y)$, $\nu_p(f) = 0$ for all but finitely many $p \in Y$.*

Proof. Omitted (Shafarevich pg. 129 [4]). □

Definition 2.5. We define $(f) = \sum_{p \in Y} \nu_p(f) \cdot p$ to be the divisor associated to the rational function $f \in k(Y)$. For any divisor $D \in \text{Div } Y$, If there exists some $f \in k(Y)$ such that $D = (f)$, then D is said to be a *principal* divisor, or a *rational* divisor. The set of all principal divisors D is defined as Prin Y.

Proposition 2.6. *In fact,* Prin Y *is a subgroup of* Div Y.

Proof. This follows from the fact that $(fg) = (f) + (g)$, and $(f^{-1}) = -(f)$. □

Definition 2.7. We let Cl Y be the quotient group $\text{Div}(Y)/\text{Prin}(Y)$. An element $C \in \text{Cl } Y$ is called a *class of divisors*, or a *divisor class.*

We have therefore defined the divisor associated to the function $f \in k(Y)$. Now we work with a few examples.

Example 2.8. Let $Y = \mathbb{P}^1$. We begin by investigating precisely which divisors in Div Y are given by rational functions $f \in k(Y)$. It will turn out that the divisors in Div Y given by rational functions are precisely those divisors $\sum_i l_i p_i$ such that $\sum_i l_i$.

First, we let $f \in k(Y)$ be any rational function on Y, then we cover Y with the open affine subsets U_0 and U_1, where

$$U_0 = Y \setminus \{[1:0]\},\ U_1 = Y \setminus \{[0:1]\}$$

We wish to find the divisor (f) associated with f . Our technique is to first look at the orders $\text{ord}_p(f)$ for $p \in U_0$ and show that they determine the order of f at $[1:0]$, and then we will show that for any divisor $D = \sum_i l_i p_i$ such that $\sum_i l_i$, we can assume that all but one of the p_i lie in U_0, and construct the rational function f on U_0 such that $D = (f)$ on Y.

We give Y the projective coordinates z and w. Fix $[z_0 : w_0] \in Y$, and suppose $[z_0 : w_0] \in \mathbb{A}_0$, which we will view as an element $\alpha = z_0/w_0 \in k$. We then restrict to $f\big|_{U_0} \in k(U_0) = k(z)$, and let

$$f\big|_{U_0} = \frac{f_1}{f_2},\ f_1, f_2 \in k[U_0] = k[z/w]$$

Since k is algebraically closed, we can write factor f_1 and f_2 as

$$f_1(z) = \prod_i \left(\frac{z}{w} - \alpha_i\right)^{e_i}, \quad f_2(z) = \prod_i \left(\frac{z}{w} - \alpha_i\right)^{e'_i}$$

with e_i and e'_i both nonnegative. Then it is easy to see that

$$\nu_\alpha(f) = \begin{cases} e_i - e'_i & \alpha = \alpha_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

It remains to find $\nu_{[1:0]}(f)$. Note that, switching to $k(U_1) = k(w/z)$, we get that

$$f = \prod_i \left(\frac{z}{w} - \alpha_i\right)^{e_i - e'_i} = \prod_i \left(\left(\frac{w}{z}\right)^{-1} - \alpha_i\right)^{e_i - e'_i} = \left(\frac{w}{z}\right)^{\sum_i (e'_i - e_i)} \prod_i \left(1 - \alpha_i \frac{w}{z}\right)^{e_i - e'_i}$$

so that

$$\nu_{[1:0]}(f) = - \sum_{[z_0:w_0] \neq [1:0]} \nu_{[z_0:w_0]}(f).$$

Therefore we have that

$$(f) = \sum_i (e_i - e'_i) \cdot \alpha_i + \left(\sum_i (e'_i - e_i)\right) \cdot [1:0].$$

Note then that $\sum_{y \in Y} \nu_y(f) = 0$.

Finally, it is simple to show that for any $D = \sum_i l_i p_i$ such that $\sum_i l_i = 0$, there exists some $f \in k(Y)$ such that $D = (f)$. We assume that $D = \sum_i l_i \cdot [\alpha_i : 1] + l_0 \cdot [1 : 0]$, and set

$$f = \prod_i \left(\frac{z}{w} - \alpha_i\right)^{l_i} \in k(U_0) = k(Y)$$

so that by similar calculations as above, $\text{ord}_{[1:0]}(f) = -\sum_{i \neq 0} l_i$.

We have checked then that the principal divisors of $Y = \mathbb{P}^1$ are precisely the divisors of degree 0. It will turn out that this result holds only in the case where $Y \cong \mathbb{P}^1$, and we will later use this fact in the classification of curves.

We formalize a technique used in the above example to define the *degree* of a divisor.

Definition 2.9. For any divisor $D = \sum_i l_i p_i \in \text{Div } Y$, where Y is a curve, we define $\deg D = \sum_i l_i$.

We will see later that the degree of a divisor defines a very useful invariant on Div Y, and will play a central role in the Riemann Roch formula. Some of the basic properties of the degree are given below.

Proposition 2.10. *For any smooth variety Y and rational function $f \in k(Y)$, $(f) \geq 0$ if and only if $f \in k[Y]$.*

Proof. Omitted. □

Proposition 2.11. *For any smooth irreducible projective curve Y, and any rational function $f \in k(Y)$, $\deg(f) = 0$.*

Proof. Analogous to the case $Y = \mathbb{P}^1$ (Corollary III.2.1, Shafarevich [4]). □

This gives us an immediate generalization of the concept of degree to Cl Y.

Corollary 2.12. *For any $C \in \mathrm{Cl}\, Y$, $\deg C$ is well-defined, by letting $\deg C = \deg D$ for any representative D of the class C.*

Proof. This follows from Proposition 2.11. □

Definition 2.13. For any two divisors $D, D' \in \mathrm{Div}\; Y$, D and D' are said to be *linearly equivalent*, denoted by $D \sim D'$, if D and D' define the same class in $\mathrm{Cl}\; Y$, that is to say, when $D - D'$ is a principal divisor.

Definition 2.14. For any divisor $D \in \mathrm{Div}\; Y$, we denote by $\mathscr{L}(D)$ the set of all $f \in k(Y)$ such that $D + (f)$ is effective, as well as 0. More concisely, we set

$$\mathscr{L}(D) = \{f \in k(Y) \mid D + (f) \geq 0\} \cup \{0\}$$

Intuitively, $\mathscr{L}(D)$ has the following meaning: it is the linear space of all rational functions $f \in k(Y)$ such that the magnitudes of the orders of the poles of f are bounded above by D. Using the following propositions, we will obtain another very useful invariant on divisors, and classes of divisors.

Proposition 2.15. *The linear space $\mathscr{L}(D)$ is a finite-dimensional vector space over k.*

Proof. Omitted (Shafarevich, Theorem III.2.6)[4]. □

Proposition 2.16. *If D and D' are linearly equivalent divisors, then $\mathscr{L}(D) \cong \mathscr{L}(D')$ as k-vector spaces.*

Proof. Let $D - D' = (g)$, $g \in k(Y)$. For $f \in \mathscr{L}(D) \setminus \{0\}$, note that $D + (f) \geq 0$ implies $D' + (g) + (f) = D' + (gf) \geq 0$, so that $gf \in \mathscr{L}(D')$, defining the linear map $f \mapsto gf$. It is easy to check that $\mathscr{L}(D') \to \mathscr{L}(D)$, $f \mapsto g^{-1}f$ is a well defined inverse. □

Definition 2.17. We define the integer $l(D)$ as the dimension $\dim_k \mathscr{L}(D)$.

Some basic results follow.

Lemma 2.18. *For any two divisors D and D' such that $D \leq D'$, we have that $l(D) \leq l(D')$.*

Proof. Note that $(f) + D \geq 0$ implies that $(f) + D' \geq 0$, so that $\mathscr{L}(D) \leq \mathscr{L}(D')$ as linear spaces. The inequality $l(D) \leq l(D')$ follows. □

Proposition 2.19. *For any $C \in \mathrm{Cl}\; Y$, $l(C)$ is well defined, by setting $l(C) = l(D)$ for any representative D of C.*

Proof. We need only show that $l(D) = l(D')$ for any two linearly equivalent divisors D and D'. This is the statement of Proposition 2.16. □

Corollary 2.20. *Let Y be an irreducible smooth projective curve, then $l(0) = 1$.*

Proof. Firstly, note that for any $f \in k(Y)$, f is an element of $\mathscr{L}(0)$ if and only if (f) is effective, and by Proposition 2.10, this holds if and only if $f \in k[Y] = k$ is a regular function. Therefore $\mathscr{L}(0)$ consists precisely of the constant functions. □

Proposition 2.21. *If D is a divisor of negative degree, that is to say, $\deg D < 0$, then $l(D) = 0$.*

Proof. Note that if we had some nonzero $f \in \mathscr{L}(D)$, then $D+(f)$ would be effective, and therefore $\deg(D + (f)) = \deg D \geq 0$, a contradiction. □

Proposition 2.22. *If D is a divisor of Y, and p some point on Y, then we have the inequality $l(D) \geq l(D-p) \geq l(D) - 1$.*

Proof. The first inequality follows directly from Proposition 2.18. For the second inequality, let $f \in \mathscr{L}(D)$, and let n be the order of the divisor $-D$ at the point p, so that when $f \notin \mathscr{L}(D-p)$, then $\nu_p(f) \geq n$ but $\nu_p(f) < n+1$, so that $\nu_p(f) = n$. Conversely, $\nu_p(f) = n$ implies that $f \notin \mathscr{L}(D-p)$.

Now, since $f \in k(Y)$ is a rational function, we can restrict to some open affine neighborhood U of p, and we get $f = f_1/f_2$ in U, for $f_1, f_2 \in k[U]$, and since $\mathcal{O}_{Y,p}$ is a UFD, if we let π generate the maximal ideal of $\mathcal{O}_{Y,p}$, so that intuitively, π has a zero at p, then when $f \notin \mathscr{L}(D-p)$, we have that $f = u\pi^n$ for some unit $u \in \mathcal{O}_{Y,p}$. We define therefore the linear function $\alpha : \mathscr{L}(D) \to k$ as $\alpha(f) = u(p)$ for $f \notin \mathscr{L}(D-p)$, where $u(p)$ is the value of u at p, and we set $\alpha(f) = 0$ otherwise. Then α has kernel precisely $\mathscr{L}(D-p)$, while $\dim_k k = 1$, so by counting dimension, we have the second inequality. □

3. Maps associated to a divisor

Let Y_1, Y_2 be curves, and $\varphi : Y_1 \to Y_2$ a nonconstant regular map. Then it can be shown that φ is a dominant morphism (that is to say, it has dense image). We have the following lemma:

Lemma 3.1. *For $\varphi : Y_1 \to Y_2$ a nonconstant rational map of curves, $k(Y_1)$ is a finite algebraic extension of $\varphi^*(k(Y_2))$.*

Proof. Note that $\varphi^* : k(Y_2) \to k(Y_1)$ is an injective field homomorphism, so we may consider $\varphi^*(k(Y_2))$ as a subfield of $k(Y_1)$. Next, since $\varphi^*(k(Y_2))$ is isomorphic to $k(Y_2)$ over k, note that $\varphi^*(k(Y_2))$ has transcendence degree 1 over k – then if follows that $k(Y_1)$ also has transcendence degree 1 over k. By Proposition II.6.8 of Hartshorne [2], φ must be a finite map, and therefore, $k(Y_1)/\varphi^*(k(Y_2))$ must be a finite algebraic extension of fields. □

Definition 3.2. For $\varphi : Y_1 \to Y_2$ a nonconstant rational map of curves, we define the degree of φ, $\deg \varphi$, as the degree of field extensions $[k(Y_1) : \varphi^*(k(Y_2))]$.

We have several useful corollaries:

Corollary 3.3. *For φ defined as before, if $\deg \varphi = 1$, then φ is an isomorphism.*

Corollary 3.4. *For any rational map $f \in k(Y)$, viewed as a map $f : Y \to \mathbb{P}^1$, then defining $(f)_0$ and $(f)_\infty$ as in Definition 2.1, we have that $\deg f = \deg(f)_0 = -\deg(f)_\infty$, where the first is the degree of a rational map, and the others are the degrees of the divisors.*

We now define, for any divisor $D \in \operatorname{Div} Y$, the associated rational map φ_D into projective space.

Definition 3.5. Let $f_0, \ldots, f_n$ be a basis of $\mathscr{L}(D)$, so that $n = l(D) - 1$. We define $\varphi_D : Y \dashrightarrow \mathbb{P}^n$ to be the rational map given by $x \mapsto [f_0(x) : \cdots : f_n(x)]$.

We still have potential issues with the map φ_D being well defined, independent of the choice of basis. However, these are resolved by noting that $\mathbf{PGL}_n(k)$ consists of isomorphisms of $\mathbb{P}^n$, and that for any two choices $\{f_0, \ldots, f_n\}$ and $\{f_0', \ldots, f_n'\}$ for a basis, giving maps φ and φ' respectively, we have that for some $\sigma \in \mathbf{PGL}_n(k)$, $\varphi = \varphi' \circ \sigma$. Therefore φ_D is defined up to isomorphism.

We generalize the definition of φ_D for divisor classes. Recall that for a linear equivalence of divisors $D = D' + (g)$, we have an isomorphism of $\mathscr{L}(D)$ to $\mathscr{L}(D')$ given by $f \mapsto fg$. This gives us the following definition:

Definition 3.6. For a class of divisors $C \in \mathrm{Cl}\, Y$, let D be a representative of C. Then we define φ_C to be the map φ_D. This is independent of the choice of D, since for some other $D + (g)$, given a basis $\{f_0, \ldots, f_n\}$ for $\mathscr{L}(D)$, we have a basis $\{gf_0, \ldots, gf_n\}$ for $\mathscr{L}(D + (g))$, and $[f_0(y) : \cdots : f_n(y)] = [gf_0(y) : \cdots : gf_n(y)]$ for all $y \in Y$ such that the morphism is defined, and $g(y) \neq 0$.

It will turn out that the map φ_D is incredibly useful, by sometimes providing an embedding of the curve in projective space of a specific dimension, which can give us valuable information about the curve. For a divisor $D \in \mathrm{Div}\, Y$, we introduce the concept of a *base point* of a divisor.

Definition 3.7. We say that a point $p \in Y$ is a *base point* of the divisor $D \in \mathrm{Div}Y$ when we have that $l(D - p) = l(D)$. We say that D is *base-point free* if it has no base points.

Proposition 3.8. *For any base-point free divisor D, the map $\varphi_D : Y \dashrightarrow \mathbb{P}^n$ is in fact a regular morphism, and for any point $y_0 \in Y$, we can choose a basis $\{f_0, \ldots, f_n\}$ of $\mathscr{L}(D)$ such that $\nu_{y_0}(f_0) = \mathrm{ord}_{y_0} D$ and $\nu_{y_0}(f_i) > \mathrm{ord}_{y_0} D$ for $i > 0$.*

Before we can prove this statement, we briefly include a discussion on rational and regular maps from Y to $\mathbb{P}^n$. Specifically, we consider rational maps of the form

$$\varphi(y) = [f_0(y) : \cdots : f_n(y)]$$

where $f_i \in k(Y)$ are rational maps for all i, not all zero. Right now, it may not be clear precisely where φ is defined, and when φ determines a regular map. For φ to be defined for all $y \in Y$, we first need that the f_i are not all zero at y, and that none of the f_i have a pole at y. Under these conditions, φ will in fact be a regular morphism. However, if we require that not all of the f_i are zero at y, this is in fact enough.

More precisely, we have the following lemma:

Lemma 3.9. *Let $\varphi : Y \dashrightarrow \mathbb{P}^n$ be a rational morphism given by the rational functions $f_0, \ldots, f_n \in k(Y)$, that is to say, $\varphi(y) = [f_0(y) : \cdots : f_n(y)]$ for all $y \in Y$. Now suppose that not all f_i are identically zero. Then we may extend φ to a regular morphism $\varphi : Y \to \mathbb{P}^n$.*

Proof. It is sufficient to check that φ is a regular map at each point y_0 in Y. Recall that we have defined the valuation ν_{y_0} at y_0. We let $N = -\min_i \nu_{y_0}(f_i)$, and then we let π_{y_0} be the regular function defined in an affine neighborhood U of y_0 such that the f_i have no other poles or zeros in U. If we let the local ring $\mathcal{O}_{Y,y_0}$ have maximal ideal given by $\langle \pi_{y_0} \rangle$, then we note that π_{y_0} is nonzero in $U \setminus \{y_0\}$ (if not, U can be chosen as such).

Therefore, we have that

$$\varphi(y) = [\pi_{y_0}^N f_0(y) : \cdots : \pi_{y_0}^N f_n(y)]$$

agrees with the previous definition in $U \setminus \{y_0\}$, and that since for some i, $\nu_{y_0}(f_i) = -N$, then $\nu_{y_0}(\pi_{y_0}^N f_i) = 0$, and therefore $\pi_{y_0}^N f_i$ is nonzero at y_0, and none of the $\pi_{y_0} f_i$ have a pole at y_0. $\square$

Now we are ready to prove Proposition 3.8.

Proof. Having fixed the point p, we first choose an arbitrary basis $f_1, \dots, f_n$ for $\mathscr{L}(D-p)$, and let $f_0 \in \mathscr{L}(D) \setminus \mathscr{L}(D-p)$ be arbitrary, then it is easy to see that the condition is satisfied. $\square$

Definition 3.10. We say that the divisor D is *very ample* if D is base-point free, and the map φ_D is an embedding.

Our next goal will be to find a sufficiently strong condition on the divisor D that ensures that φ_D is an embedding. In fact, our condition will be necessary and sufficient to ensure that φ_D is an embedding.

Theorem 3.11. *The morphism φ_D is an embedding if and only if for any two points $p, q \in Y$ (not necessarily distinct), $l(D-p-q) = l(D) - 2$.*

Note that this implies that D is base-point free, by Proposition 2.22. Since the proof is long, we break it into several parts.

We first check for injectivity.

Lemma 3.12. *The morphism φ_D is injective if and only if for any two distinct points $p, q \in Y$, we have that $l(D-p-q) = l(D) - 2$. Note the slightly weaker condition.*

Proof. We first prove an equivalence. Suppose $\varphi_D(p) = \varphi_D(q)$, and choose a basis for $\mathbb{P}^n$ satisfying the condition in Proposition 3.8 at the point p, so that $\varphi_D(p) = \varphi_D(q) = [1 : 0 : \cdots : 0]$. Now, if $\varphi = [f_0 : \cdots : f_n]$, then this condition implies that $\nu_q(f_0) \leq \nu_q(f_i)$ for all $i > 0$. This implies that $f_i \in \mathscr{L}(D-q)$ for all $i \geq 1$ as linearly independent rational functions, and since D is base-point free, $f_1, \dots, f_n$ forms a basis for $\mathscr{L}(D-q)$, and therefore $\mathscr{L}(D-q) = \mathscr{L}(D-p)$. Similarly, if $\mathscr{L}(D-q) = \mathscr{L}(D-p)$, it is easy to see that $\varphi_D(p) = \varphi_D(q)$, using the previous basis.

Now, if $l(D-p-q) = l(D-2)$ for all points $p, q \in Y$, then if we choose p and q to be distinct, we get that if $\mathscr{L}(D-p) = \mathscr{L}(D-q)$, then $\mathscr{L}(D-p) = \mathscr{L}(D-q) = \mathscr{L}(D-p-q)$, a contradiction with the previous equivalence, and therefore φ is injective. Conversely, if φ is injective, then if $\mathscr{L}(D-p) = \mathscr{L}(D-q)$, we get a contradiction. $\square$

Finally, we must prove that φ is an embedding. Here, we strengthen our condition to include the case where $p = q$, and we use the following proposition:

Proposition 3.13. *For any divisor $D \in \operatorname{Div} Y$, assuming the associated map φ_D is injective, we have that φ_D is an embedding if and only if $l(D-2p) = l(D) - 2$ for all points $p \in Y$.*

Proof. We first prove the if statement. In order to define the morphism $\varphi_D = [f_0 : \cdots : f_n]$, we choose a basis $f_2, \dots, f_n$ of $\mathscr{L}(D-2p)$, and we let

$$f_1 \in \mathscr{L}(D-p) \setminus L(D-2p), \;\; f_0 \in \mathscr{L}(D) \setminus \mathscr{L}(D-p)$$

so that $\nu_p(f_1) = -\mathrm{ord}_p D + 1$, and therefore $\mathrm{ord}_p(f_1/f_0) = 1$, and $\mathrm{ord}_p(f_i/f_0) > 2$ for $i > 2$, so that applying the implicit function theorem, we see that the image of φ_D has the local coordinate f_1/f_0 at p, and using injectivity, we get that φ_D is an embedding.

The converse statement follows from Lemma V.4.19 from Miranda [3]. $\square$

To summarize, we have just proved the following statement (which we will make enormous use of, once we know how to use the Riemann Roch theorem to find a suitable divisor D):

Proposition 3.14. *Let $D \in \mathrm{Div}\ Y$ be a divisor such that for any two points $p, q \in Y$ not necessarily distinct, $l(D - p - q) = l(D) - 2$ (note that this implies a fortiori that D is basepoint free). Then the associated morphism $\varphi_D : Y \to \mathbb{P}^{l(D)-1}$ is an embedding.*

In fact, the converse of the above statement is also true.

Proposition 3.15. *Let $D \in \mathrm{Div}\ Y$ be a divisor of degree $\deg D = 0$, then either $l(D) = 1$ or $l(D) = 0$. Furthermore, $l(D) = 1$ if and only if D is a principal divisor, that is to say, D is linearly equivalent to 0.*

Proof. The fact that $l(D) \leq 1$ follows from Propositions 2.22 and 2.21, since for any point $p \in Y$, $\deg(D-p) < 0$ implies that $l(D-p) = 0$. Now, suppose that D is a principal divisor $D = (f)$, then since $(f) + (f^{-1}) = 0$, we have that $f^{-1} \in \mathscr{L}(D)$, so that $l(D) > 0$, and therefore $l(D) = 1$.

Conversely, if $l(D) = 1$, then if we let $f \in \mathscr{L}(D)$, then $D + (f) \geq 0$, and since $\deg(f) = 0$, we must have that $D = -(f) = (f^{-1})$. □

Proposition 3.16. *For any very ample divisor D with associated morphism $\varphi_D : Y \to \mathbb{P}^n$, we have the equality $\deg\ \varphi_D = \deg D = \deg \varphi_D(Y)$, where the last degree is the degree of the algebraic variety.*

Proof. It follows from Lemmas 5.1.17 and 5.4.13 from Miranda [3]. □

Lemma 3.17. *For any point $p \in Y$, $l(p) = 1$ or 2. We have that $l(p) = 2$ if and only if Y is isomorphic to $\mathbb{P}^1$.*

Proof. We have shown in Example 2.8 that if $Y = \mathbb{P}^1$, then any degree zero divisor $D \in \mathrm{Div}\ Y$ is effective, so that $\mathscr{L}(p)$ contains nonconstant functions, and therefore $l(p) = 2$ for all $p \in Y$.

Conversely, suppose that $l(p) = 2$ for all $p \in Y$. We use the associated morphism φ_p. By the previous proposition, we have that φ_p is of degree 1. If $l(p) = 2$, then since $l(p-q-r) = 0$ for any points $q, r \in Y$, we have that p is a very ample divisor, and therefore, $\varphi_p : Y \to \mathbb{P}^1$ is an embedding. Letting $f \in \mathscr{L}(p)$ be a nonconstant function, we have that $\varphi_p = [1 : f]$ which is nonconstant, and therefore we must have that φ_p is an isomorphism. □

4. Differential forms

To begin, we let X be a smooth irreducible affine variety. We will define the notion of a differential form on X, by analogy with the concept of the same name in differential geometry – eventually, we will be able to generalize this to the case where X is any smooth irreducible variety, which we will cover with affine open varieties. We start by defining *regular differential forms*.

Reducing even further, we work over $\mathbb{A}^n$, and we define the notion of a differential operator on $k[\mathbb{A}^n]$. For $f \in k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$, we define

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i \in \bigoplus_{i=1}^{n} k[\mathbb{A}^n] \mathrm{d}x_i$$

in the free module over $k[\mathbb{A}^n]$ generated by the $\mathrm{d}x_i$.

Let $X \subseteq \mathrm{A}^n$ have the associated ideal $\langle f_1, \ldots, f_m \rangle$. Then we define the set of differential q-forms over X as follows:

Definition 4.1. We define the $k[X]$-module of regular differential forms $\Omega^1[X]$ as the quotient of $k[X]$-modules

$$\bigoplus_{i=1}^{n} k[X]\mathrm{d}x_i \Big/ \langle \mathrm{d}f_1, \ldots, \mathrm{d}f_m \rangle$$

where $\langle \mathrm{d}f_1, \ldots, \mathrm{d}f_m \rangle$ is the submodule of

$$\bigoplus_{i=1}^{n} k[X]\mathrm{d}x_i$$

generated by the terms

$$\mathrm{d}f_j \in \bigoplus_{i=1}^{n} k[X]\mathrm{d}x_i.$$

This is the set of differential 1-forms on X. We define $\Omega^q[X]$ as the exterior algebra $\bigwedge^q \Omega^1[X]$ over $k[X]$. Finally, we define the differential operator

$$\mathrm{d} : k[X] \to \Omega^1[X],\ f \mapsto \mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Note that this implies that $\Omega^1[\mathbb{A}^n] = \langle \mathrm{d}x_1, \ldots \mathrm{d}x_n \rangle$.

Definition 4.2. For a smooth irreducible variety X, we define the $k(X)$-vector space of rational differential 1-forms $\Omega^1(X)$ as the set of pairs $\langle U, \omega \rangle$ for affine open sets U, where $\omega \in \Omega^1[U]$, with the equivalence relation that $\langle U, \omega \rangle = \langle V, \omega' \rangle$ when $\omega\big|_{U \cap V} = \omega'\big|_{U \cap V}$. More generally, we define the $k(X)$-vector space of rational differential q-forms as the exterior algebra $\Omega^q(X) = \bigwedge^q \Omega^1(X)$ over $k(X)$.

Proposition 4.3. *Let X be an n-dimensional smooth projective variety. Then* $\dim_{k(X)} \Omega^q(X) = \binom{n}{q}$

Proof. Omitted (Shafarevich Theorem III.4.3 [4]). □

Corollary 4.4. *Suppose that Y is a smooth irreducible projective curve. Then* $\dim_{k(Y)} \Omega^1(Y) = 1$.

Now we define the divisor associated to a differential form.

Definition 4.5. Let X be an n-dimensional smooth variety, $\omega \in \Omega^n(X)$. Then we cover X with finitely many affine open sets U_i, and we get the representation $\omega\big|_{U_i} = g^{(i)} \mathrm{d}u_1^{(i)} \wedge \ldots \wedge \mathrm{d}u_n^{(i)}$ for all i, where $u_1^{(i)}, \ldots, u_n^{(i)}$ are the local paramaters of X in U_i. Then we get that

$$g^{(j)} = g^{(i)} \mathrm{J} \left(\frac{u_1^{(i)}, \ldots, u_n^{(i)}}{u_1^{(j)}, \ldots, u_n^{(j)}} \right)$$

and since the Jacobian is a regular and nonzero function in $U_i \cap U_j$, each g_i determines a divisor in U_i, and these divisors are compatible, giving us the divisor (ω) of the form ω.

Lemma 4.6. *We have the following properties of (ω), for $\omega \in \Omega^n(X)$, where X is a smooth complete variety:*

(1) $(f\omega) = (f) + (\omega)$ *for* $f \in k(X)$.
(2) $(\omega) \geq 0$ *if and only if* $\omega \in \Omega^n[X]$.

Proof. Omitted. □

Proposition 4.7. *If we let* Y *be a smooth irreducible projective curve, then there is a unique element* $K \in \mathrm{Cl}\, Y$ *such that for all nonzero* $\omega \in \Omega^1(Y)$, (ω) *corresponds to* K.

Proof. Since $\dim_{k(X)} \Omega^1(X) = 1$, for any two nonzero $\omega, \omega' \in \Omega^1(Y)$, we have, for some $f \in k(X)$, that $\omega' = f\omega$ and therefore $(\omega') = (f) + (\omega)$ which maps to the same divisor class in $\mathrm{Cl}\, X$. □

Definition 4.8. The class $K \in \mathrm{Cl}\, Y$ defined by any nonzero $\omega \in \Omega^1(Y)$ is called the *canonical class* of the curve Y.

We will give a simple example.

Example 4.9. Let $Y = \mathbb{P}^1$, so as in Example 2.8, we cover $\mathbb{P}^1$ with affine patches U_0 and U_1. Then we must find a basis element for $\Omega^1(Y)$. If $\omega \in \Omega^1(Y)$ is nonzero, then we must have that $\omega\big|_{U_0}$ is expressible in the form $f(x)\mathrm{d}x$, and $\omega\big|_{\mathbb{A}_1}$ is expressible in the form $g(y)\mathrm{d}y$, where $f(x) \in k(x)$ and $g(y) \in k(y)$, and that these two expressions agree on $\mathbb{A}_0 \cap \mathbb{A}_1$. Otherwise stated,

$$f(x)\mathrm{d}x = g(y)\mathrm{d}y = -x^{-2}g(1/x)\mathrm{d}x$$

which implies that $x^2 f(x) + g(1/x) = 0$. Then we can conveniently set $f(x) = x^{-2}$ and $g(y) = -1$, giving us a basis element ω for $\Omega^1(Y)$. This gives us the divisor $(\omega) = 2 \cdot [0 : 1] \in \mathrm{Div}\, Y$, which induces a canonical class

$$K_Y = -2 \cdot [0 : 1] + \mathrm{Prin}\, Y \in \mathrm{Cl}\, Y$$

We are now able to define algebraically the concept of the *genus* of a curve. In fact, though we will not prove this, for the case $k = \mathbb{C}$, the algebraic concept of genus coincides precisely with the topological concept of genus.

Definition 4.10. For a smooth projective irreducible curve Y over an algebraically closed field k, we define the *genus* of Y as the integer $g = l(K)$.

5. Riemann-Roch Theorem

As before, we let Y be a smooth projective irreducible curve over an algebraically closed field k.

Theorem 5.1. *Let* K *be the canonical class of the curve* Y, $D \in \mathrm{Div}\, Y$ *any divisor, and* g *the genus of the curve* Y. *Then we have the following:*

$$l(D) - l(K - D) = \deg D + 1 - g$$

Proof. Omitted (Theorem IV.1.3, Hartshorne [2]). □

We have some immediate corollaries. Firstly, we calculate the degree of the canonical divisor K:

Corollary 5.2. *If* Y *has canonical divisor* K, *then the degree of* K *is determined by the genus* g *of* Y. *More precisely, we have that* $\deg K = 2g - 2$.

Proof. First set $D = K$, then note that by Proposition 2.20, we have that $l(K - K) = l(0) = 1$, and therefore

$$l(K) - 1 = g - 1 = \deg K + 1 - g$$

so that $\deg K = 2g - 2$. □

Note that this agrees with our earlier calculations for $Y = \mathbb{P}^1$. Next, note that for $\deg D$ sufficiently large, $l(K - D) = 0$. More precisely,

Proposition 5.3. *If D is a divisor such that $\deg D \geq 2g - 1$, then $l(D) = \deg D + 1 - g$.*

Proof. Note that $l(K - D) = 0$, since $\deg(K - D) < 0$. This follows from Proposition 2.21. □

6. Applications

Proposition 6.1. *Let $D \in \mathrm{Div}\ Y$ be a divisor. Then if $\deg D \geq 2g + 1$, we have that the associated morphism φ_D is an embedding, that is to say, D is a very ample divisor.*

Proof. Note that by Proposition 3.10, φ_D is an embedding when for any two points p and q of Y, we have that $l(D - p - q) = l(D) - 2$, and if D is base-point free (in fact, the first condition implies the second, by Proposition 2.22). Now, by Proposition 5.3, since $\deg D > \deg(D - p - q) \geq 2g - 1$, we have the following form of Riemann Roch:

$$l(D) = \deg D + 1 - g$$

and

$$l(D - p - q) = \deg(D - p - q) + 1 - g = \deg D - 1 - g$$

so that $l(D) - 2 = l(D - p - q)$. □

Proposition 6.2. *If Y is a curve with genus $g = 0$, then Y is isomorphic to $\mathbb{P}^1$.*

Proof. Let $p \in Y$ be any point. Then note that since $\deg(K - p) = -3$, by Proposition 2.21, we have that $l(K - p) = 0$, so that by Riemann Roch, $l(p) = \deg p + 1 - g = 2$, and by Lemma 3.17, Y must be isomorphic to $\mathbb{P}^1$. □

Proposition 6.3. *If Y has genus $g = 1$, then Y is a plane cubic curve (up to isomorphism).*

Proof. This time, we set $D = p + q + r$ for some points $p, q, r \in Y$. Again by Proposition 6.1, the associated morphism $\varphi_D : Y \to \mathbb{P}^{l(D)-1}$ is an embedding . Note that by Riemann Roch, since $\deg(K - D) < 0$, we get that $l(D) = 3$, so φ_D is an embedding into $\mathbb{P}^2$, so Y is a plane curve. It remains to show that Y is a cubic curve, that is to say, $\deg Y = 3$. This follows directly from Proposition 3.16. □

Definition 6.4. We define a *hyperelliptic curve* to be a curve Y (smooth irreducible projective) such that there exists some regular map $\varphi : Y \to \mathbb{P}^1$ that is of degree 2.

Remark 6.5. In fact, for $k = \mathbb{C}$, any such curve is isomorphic to a projective plane curve given by some equation $y^2 = f(x)$ where $f(x)$ is a polynomial with no repeated roots. This is the definition given in Miranda [3].

Proposition 6.6. *If Y has genus $g = 2$, then Y is a hyperelliptic curve.*

Proof. Note that $l(K) = g = 2$, so that we may assume that K is effective. Since $\deg K = 2$, we must have that $K = p + q$ for some points $p, q \in Y$. Let $f \in \mathscr{L}(K)$ be nonconstant. Then f has either one or two poles, which must be at the points p or q. Suppose f has only one pole, at p, say. Then we have that $f \in \mathscr{L}(p)$, and therefore $l(p) = 2$, which implies that Y is isomorphic to $\mathbb{P}^1$ by Lemma 3.17, a contradiction of genus. So f must have either a double pole or two simple poles, which gives us a morphism $f : Y \to \mathbb{P}^1$ of degree 2. □

Lemma 6.7. *For any curve Y of genus $g \geq 1$ with canonical divisor K, we have that K is base-point free.*

Proof. Let $p \in Y$ be an arbitrary point, and suppose that $l(K - p) = l(K) = g$. Then by Riemann Roch, we have that

$$l(p) - l(K - p) = l(p) - l(K) = l(p) - g = 2 - g$$

so that $l(p) = 2$, but by Lemma 3.17, this implies that Y is isomorphic to $\mathbb{P}^1$, a contradiction of genus. So K must be basepoint-free. □

Proposition 6.8. *Let $g \geq 3$, then φ_K is an embedding if and only if Y is not hyperelliptic.*

Proof. We already know that K is base-point free, so we need only check the condition in Proposition 3.7. We also have that $\varphi_K : Y \to \mathbb{P}^2$ is a regular morphism of degree $2g - 2$. We already know that φ_K is an embedding if and only if for any two points $p, q \in Y$, $l(K - p - q) = l(K) - 2$, and by Riemann Roch,

$$l(K - p - q) = \deg(K - p - q) + 1 - g + l(p + q) = g - 3 + l(p + q)$$

so that φ_K fails to be an embedding if and only if for some points p, q of Y, $l(p + q) = 2$.

Now, supposing that $l(p + q) = 2$ for some points $p, q \in Y$, then letting $f \in \mathscr{L}(p + q)$ be any nonconstant function, then f must have more than one pole, counting multiplicity, otherwise we get a contradiction with Lemma 3.17 and the fact that Y has genus $g \neq 0$. Therefore f must have two polls counting multiplicity, so that $\deg(f)_0 = 2 - \deg(f) = 2 = \deg f$ by Corollary 3.4, that is to say, $f : Y \to \mathbb{P}^1$ has degree 2, and therefore, Y is hyperelliptic.

Conversely, suppose that Y is hyperelliptic, then if $f : Y \to \mathbb{P}^1$ is any degree 2 regular map (and therefore f is nonconstant), considering f as a rational map $f \in k(Y)$, by Corollary 3.4, we have that $-\deg(f)_\infty = \deg f = 2$, so letting $-(f)_\infty = p + q$, then $f \in \mathscr{L}(p + q)$ is nonconstant, so that $l(p + q) = 2$, and therefore φ_K cannot be an embedding. □

Corollary 6.9. *If Y is not hyperelliptic, then φ_K embeds Y in $\mathbb{P}^{g-1}$ as a curve of degree $2g - 2$.*

Proof. By Proposition 6.8, the morphism $\varphi_K : Y \to \mathbb{P}^{l(K)-1} = \mathbb{P}^{g-1}$ is an embedding. Therefore, using Proposition 3.16, we have that $\deg \varphi_K(Y) = \deg K = 2g - 2$. □

Corollary 6.10. *If $g = 3$ and Y is not hyperelliptic, then Y is a plane curve of degree 4.*

Proof. Immediate by the previous corollary. □

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