RECURSIVE DERIVATION OF 2N GON TOPOLOGIES

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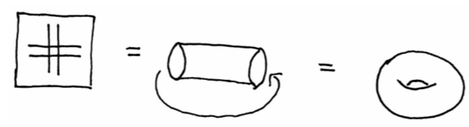
Abstract. A compact surface can be presented as a polygon whose edges are identified in pairs with orientation. Some presentations like this are commonly used, such as drawing a torus as a square with opposite edges identified. But there are many other ways to identify the $2n$ edges of a polygon with orientation. For a given n, the set of possible identifications define a set P_n of polygons with identified edges. Each element of P_n is a presentation of a compact surface, determined by two invariants: orientability and Euler Characteristic. We will prove various results about these invariants in the sets P_n, mostly using an indexed collection of maps from $P_n \to P_{n+1}$. Ultimately, we will find a computational formula for the number of elements of P_n that present a given compact surface S.

Contents

1. Introduction

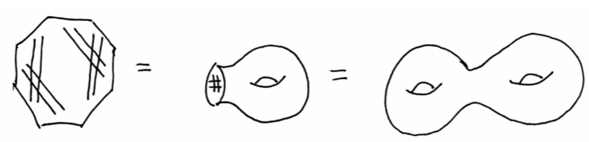
It is often useful to represent a more complicated topological object as something simpler with additional identifications. One example of this is presenting a compact surface as a polygon with "glued" edges. Here are a few examples of commonly used "glued polygons":

Example 1.1. The single torus T^2:



Example 1.2. The Klein Bottle K (equivalent to the double projective plane $2P^2$):

Example 1.3. The double torus $2T^2$:



The notation above isn't standard, but it will be useful later (it comes from Hilbert, [3] pg. 309).

These examples are constructed based on a set of rules about how to glue edges together: every edge must be glued to exactly one other edge (and not to itself) with one of two possible orientations (straight or twisted). To make this precise, we must label the $2n$ edges of the polygon X_n with $e_1, e_2, ..., e_{2n}$ marked clockwise from a basepoint a fixed at the upper left vertex of X_n. A gluing A_n is determined by the pairs of labelled edges it identifies, and the orientations of these identifications. We will call A_n an **identified polygon**, and define the set P_n to be the set of identified polygons A_n with 2n edges.

The purpose of this paper is to study the topology of identified polygons A_n, which are elements of the sets P_n. Given the rules mentioned above for gluing edges (which are the rules we will consider in this paper), every A_n is homeomorphic to a compact surface without boundary ([4], pg. 6). Every compact surface without boundary can be triangulated, and even more, can be presented as an identified polygon, and thus is presented by an element of P_n for some n ([4], pg. 6). A compact surface without boundary is determined by only two invariants: orientability and Euler Characteristic ([4], pg. 18). We can study these invariants in P_n (which are much easier to deal with computationally) to study the topology of identified polygons A_n.

In some sense this study might seem like a cataloging exercise, and it may not be clear why it is interesting. And it is not particularly interesting that there are 10 ways to present a torus as an identified hexagon. But there are 42 ways to present a klein bottle as an identified hexagon, and it is interesting that there are comparatively many more ways to present a klein bottle as an identified hexagon than a torus. Studying identified polygons A_n provides a different and interesting perspective on the surfaces they present.

Before outlining what we will study in more detail, know that the work described here is original to this paper, and aside from some relevant background in topology, does not (so far as I know) appear elsewhere.

In the next section, we will provide more rigorous definitions of identified polygons and the sets of them. In the third section, we will briefly mention a few relevant results in topology (without proof) that will be useful to us. In the fourth section, we will prove the following theorem about orientability in the sets P_n:

Theorem 1.4. *(Later* ***Theorem 4.4****.) Let $O_n \subset P_n$ be the set of orientable identified polygons. As $n \to \infty$,*

$$\frac{|O_n|}{|P_n|} \to 0.$$

In the fifth section, we will define an indexed collection of maps $f_{a_i,a_j,z} : P_n \to P_{n+1}$ which we will call **augment maps**. We will prove three theorems about properties of these maps which make them useful. This is arguably the most important section, as these maps will be the primary tool used to prove the rest of the results in this paper.

In the sixth section, we will study the behavior of the Euler Characteristic in P_n using augment maps, and develop a recursive formula that computes the number of elements of P_n with Euler Characteristic x. We will start by briefly explaining how to compute the Euler Characteristic of an identified polygon. Next, we will define graphs that correspond to each element of P_n. Together with augment maps, this allows us to show that

Theorem 1.5. *(Later* ***Theorem 6.14****.) Let A_n be an element of P_n. If $\chi(A_n) = x$, then $\chi(f_{a_i,a_j,z}(A_n))$ is x, $x-1$, or $x-2$.*

After that, we will define partitions which correspond to each element of P_n. Ultimately, we will recursively compute partitions using augment maps, constructing a tree of partitions. This tree of partitions has the following relationship with the distribution of Euler Characteristics in P_n, which is how we recursively compute the number of $A_n \in P_n$ with Euler Characteristic x:

Theorem 1.6. *(Later* ***Theorem 6.26****.) Let $C(n,x)$ be the number of $A_n \in P_n$ with Euler Characteristic x. Let $Q(n,k)$ be the number of partitions of length k at level n of the partition tree. Then*

$$C(n,x) = \frac{Q(n,k)}{(n-1)!},$$

where $x = 1 - n + k$.

After that, we note a property of the subset $O_n \subset P_n$ which allows us to show that

Corollary 1.7. *(Later* ***Corollary 6.29****.) For any n, given the tree of partitions and a compact surface without boundary S, we can compute the number of $A_n \in P_n$ that present S.*

In section seven, we apply these theorems and compute results with a computer. The code used to produce these results is included at the end of the paper. Finally, in section eight, we pose a few further questions.

2. Preliminary Definitions

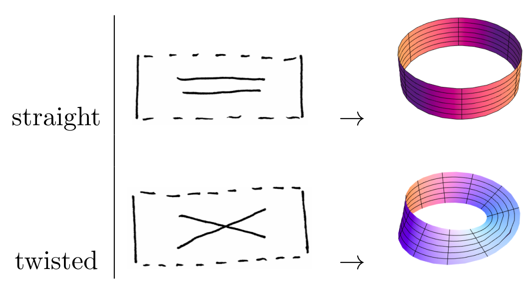
Definition 2.1. The space A is a **quotient space** of X if there exists an equivalence relation $\simeq$ on X such that the elements of A are the equivalence classes of $\simeq$ on X. This means that there is a surjective map $p : X \to A$ such that if $x \simeq y$, then $p(x) = p(y)$. The map p is strong continuous, meaning that $U \in A$ is open if and only if $p^{-1}(U) \in X$ is open. If $x \simeq y$, we will say that x is **identified** to y. ([2], pgs. 137, 139).

Definition 2.2. Let X_n be a polygonal region in $\mathbb{R}^2$ with $2n$ edges, $n \geq 1$ (when $n = 1$, we allow the edges to be curved so that they do not lie on top of each other). Label the edges of X_n clockwise with e_1, e_2, ... e_{2n}, fixing the left endpoint of e_1 at the upper left vertex of X_n. Denote this by putting a basepoint a at the upper left vertex of X_n. Define an **identified polygon** A_n to be a quotient space of X_n defined by a quotient map $p : X_n \to A_n$, where p follows two rules, mentioned in the introduction and explained further here:

(1) p identifies every edge e_i on X_n with exactly one other edge e_j on X_n, $i \neq j$.
(2) Every edge has an **initial point** and a **final point** induced by the clockwise labelling of X_n. When edges are identified, they can be identified with the **same orientation** (i.e. initial point to initial point and final point to final point), or with **opposite orientation** (i.e. initial point of one to final point of the other). If two edges with the same orientation are identified, this identification is **twisted**. If two edges with opposite orientation are identified, this identification is **straight**.

Ultimately, the map $p : X_n \to A_n$ which defines A_n is determined by which n pairs of labelled edges it identifies, and the orientations of those identifications.
([2], pg. 447-448.)

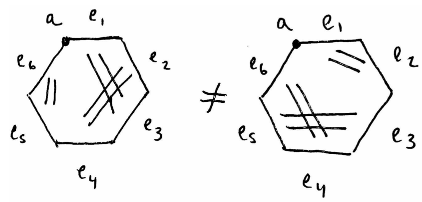
Remark 2.3. Calling an identification **straight** if it connects edges with opposite orientation and **twisted** if it connects edges with the same orientation might seem counterintuitive, but we use it because it lines up with what is happening geometrically:



Gluing edges with opposite orientation makes a regular band, whereas gluing edges with the same orientation makes a band with a half-twist called a **Möbius Strip**.

Definition 2.4. For each $n \in \mathbb{N}$, $n \geq 1$, let P_n be the set of distinct identified polygons A_n with $2n$ edges, where identified polygons A_n and B_n are distinct if they are images of different quotient maps p and q.

Remark 2.5. Note that these two identified polygons are different elements of P_3:



The hexagon on the left is the image of p_l which identifies e_1 to e_3, e_2 to e_4, and e_5 to e_6 (all straight), while the hexagon on the right is the image of p_r which identifies e_1 to e_2, e_3 to e_5, and e_4 to e_6, (all straight). This might seem strange, since they look the same and are homeomorphic to the same compact surface (which, as it so happens, is T^2). But as evidenced by p_l and p_r being different maps, they represent different presentations of T^2. Our goal is to study the set of presentations, and so we consider these different.

Proposition 2.6. *For a given n, the size of P_n is*

$$|P_n| = \frac{(2n)!}{n!} \tag{2.7}$$

Proof. Fix n. Then X_n has $2n$ labeled edges $(e_1, ..., e_{2n})$ which must be identified in pairs. It does not matter what order we choose pairs in, so there are

$$\frac{1}{n!}\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}...\binom{2}{2} = \frac{(2n)!}{2^n(n)!}$$

ways to identify edges in pairs. Next, each identification has two possible orientations (twisted or straight). There are n pairs of edges and so n identifications, and thus this equation should be multiplied by 2^n. Altogether, this gives

$$|P_n| = \frac{2^n(2n)!}{2^n(n)!} = \frac{(2n)!}{n!},$$

as above. □

Using this formula, the number of elements in P_n for the first few values of n are:

n	1	2	3	4	5	6
$\lvert P_n\rvert$	2	12	120	1,680	30,240	665,280

3. Topology

We want to study the topology of elements $A_n \in P_n$. As such, we need to know a little bit of topology. We will provide references instead of proofs here. Regardless, to understand what we are going to do, there are a few results and definitions from topology that we must note.

Theorem 3.1. *Every identified polygon is homeomorphic to a compact surface without boundary.* ([4], *pg. 6 and* [2] *Thm 74.1, pg. 450.)*

This theorem actually goes both ways, and the following is also true:

Theorem 3.2. *Every compact surface without boundary can be presented as an identified polygon $A_n \in P_n$ for some n.* ([4], *pg. 6.)*

This relies on the following proposition, which is also worth noting:

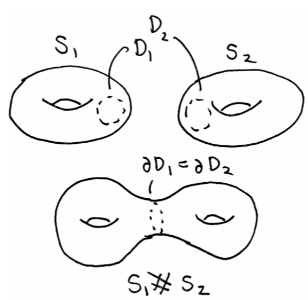
Proposition 3.3. *Every compact surface without boundary can be triangulated. (*[4]*, Appendix E.)*

Further, there are only a few types of compact surfaces without boundary:

Theorem 3.4. ***(The Classification Theorem)****. Let S be a compact surface without boundary. Then S is homeomorphic to one of the following:*
(1) the sphere S^2,
(2) the connected sum of g tori, denoted gT^2,
(3) or the connected sum of p projective planes, denoted pP^2.
([4]*, Ch 1 or Theorem 6.3, pg. 96, and* [2]*, ch. 12, sec. 77)*

Here the **connected sum** of two surfaces S_1 and S_2, denoted $S_1 \# S_2$, is the quotient of $(S_1 \setminus D_1) \cup (S_2 \setminus D_2)$ where D_1 and D_2 are disks on S_1 and S_2 respectively, and the quotient map identifies the boundaries of D_1 and D_2. ([4], pg. 95). In simpler terms, the connected sum of two surfaces is the result of cutting a piece off of each and gluing them together on their newly created boundaries.

Example 3.5. The connected sum of two tori $T^2 \# T^2$ is the double torus, $2T^2$:



Since every identified polygon is a compact surface without boundary, the Classification Theorem classifies every element of P_n. Further, if we range over every value of n, we find every compact surface without boundary, since all of them can be presented as an identified polygon. This is part of what makes the sets P_n interesting. Finally, the Classification Theorem leads to the following corollary:

Corollary 3.6. *Every compact surface without boundary is fully determined by two topological invariants: orientability and Euler Characteristic. (*[4]*, Thm 1.2, pg. 18.)*

This corollary gives us a computational way to classify every identified polygon, as we can easily compute the orientability and the Euler Characteristic of a given A_n. But more interestingly, we can ask questions about the property of orientability or the Euler Characteristic in P_n in general, which tells us about the topology of the elements of P_n without having to enumerate or compute for every element individually.

Remark 3.7. Before continuing, remember that when we refer to A_n, we refer to it as an identified polygon, defined by a quotient map p and with oriented edge identifications, not just the surface S that is homeomorphic to A_n.

4. Orientability in P_n

We can now prove a quick result about the behavior of orientability in the sets P_n.

Definition 4.1. A surface S is **nonorientable** if there exists a closed curve $C \in S$ for which it is not possible to choose a consistent orientation. A surface S is **orientable** if no such curve exists.

Lemma 4.2. *Let A_n be an identified polygon. Then A_n is orientable if and only if no pair of edges is identified with a twist.*

Proof. If a pair of edges is identified with a twist, then there are half-twist curves (like the Mobiüs Strip) on the surface S determined by A_n. Thus there exists a curve on the S which has no consistent orientation, and S is nonorientable. If every pair of edges on X_n is identified straight, then every curve on S has a consistent orientation, and S is orientable. A_n is homeomorphic to S, and therefore orientable if and only if no pair of edges is identified with a twist. □

Remark 4.3. gT^2 and S^2 are orientable for all g, and pP^2 is nonorientable for all p.

This lemma allows us to prove one of the theorems mentioned in the introduction.

Theorem 4.4. *Let $O_n \subset P_n$ be the set of orientable identified polygons. As $n \to \infty$,*

$$\frac{|O_n|}{|P_n|} \to 0.$$

Proof. By Lemma 4.2, A_n is orientable if and only if all of its edges are identified straight. Thus the number of elements of P_n which are orientable is

$$|O_n| = \frac{1}{2^n}|P_n|,$$

meaning that

$$\frac{|O_n|}{|P_n|} = \frac{1}{2^n}.$$

As $n \to \infty$,

$$\frac{1}{2^n} \to 0.$$

Therefore as $n \to \infty$, almost no element of P_n is orientable. □

The following corollary also follows immediately:

Corollary 4.5. *As $n \to \infty$, almost every element of P_n is nonorientable.*

5. Augment Maps $P_n \to P_{n+1}$

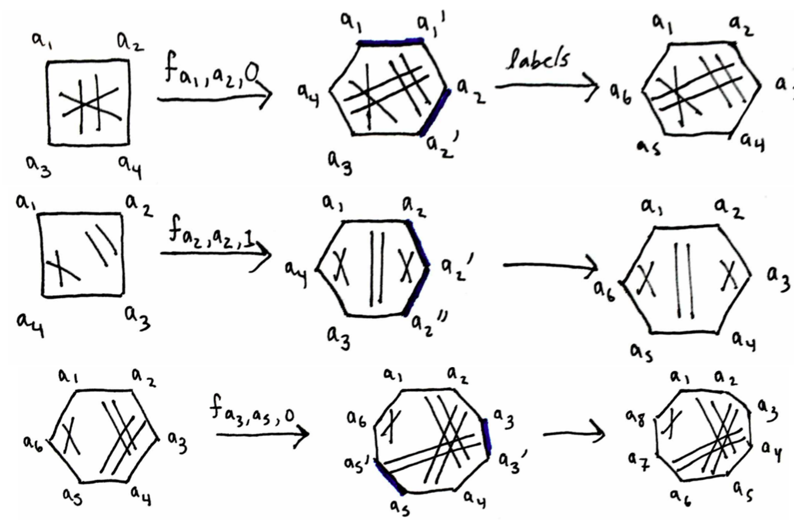
Here will we develop and study **augment maps**, which will be the primary tool in the rest of our investigations. While this section is not directly about the topology of elements of P_n, it is likely the most important. Without it, none of the results in later sections of this paper would be possible.

Fix n, and label the vertices of the unidentified polygon X_n with $a_1, a_2, a_3, ...a_{2n}$, where a_1 is the basepoint a (i.e. the upper left vertex). Choose two of these vertices a_i and a_j, $i \leq j$ (they can be the same vertex) and an orientation, either straight or twisted. Denote this choice (a_i, a_j, z), where z is 0 if the chosen orientation is straight and 1 if it is twisted. The choice of orientation z is independent of the choice of vertices (a_i, a_j).

Definition 5.1. Let $\{\alpha_n\}$ be set of pairs of vertices (a_i, a_j) on X_n.

Definition 5.2. Define the **augment map** $f_{a_i,a_j,z} : P_n \to P_{n+1}$, where $(a_i, a_j) \in \{\alpha_n\}$ and z is 0 or 1, by the following process:
Split the vertices a_i and a_j, adding a new edge clockwise after each of them. This creates two new unidentified edges. Identify the new edges with the orientation determined by z. Note that this induces a shift in the labels of vertices (and edges) after a_i and a_j.

Examples 5.3. Here are three examples of augment maps, two from $P_2 \to P_3$ and one from $P_3 \to P_4$, including relabellings:



The bold edges are the ones added by $f_{a_i,a_j,z}$. Note that they always appear clockwise after the split vertices.

Proposition 5.4. *There are*

$$2n(2n + 1)$$

augment maps $f_{a_i,a_j,z} : P_n \to P_{n+1}$.

Proof. The number of ways to choose pairs of vertices (a_i, a_j) is

$$2n + \binom{2n}{2} = n(2n + 1),$$

since there are $2n$ ways to choose the same vertex twice, and $\binom{2n}{2}$ ways to choose two different vertices. There are two possible values of z (0 or 1), and so there are

$$2n(2n + 1)$$

augment maps from P_n to P_{n+1}. □

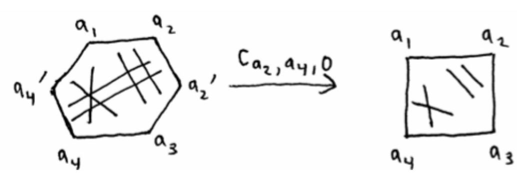
Theorem 5.5. $f_{a_i,a_j,z} : P_n \to P_{n+1}$ *is injective for every* $(a_i, a_j) \in \{\alpha_n\}$, z, *and* n.

Proof. Fix n, and pick arbitrary $(a_i, a_j) \in \{\alpha_n\}$, $z \in \{0, 1\}$. Suppose that $f_{a_i,a_j,z}(A_n) = f_{a_i,a_j,z}(B_n)$ for some $A_n, B_n \in P_n$. Then $f_{a_i,a_j,z}(A_n)$ and $f_{a_i,a_j,z}(B_n)$ are images of the same quotient map p, and have exactly the same identifications. The map $f_{a_i,a_j,z}$ adds one identification, identifying the edges which start with a_i and a_{j+1} (because of the label shift) with orientation determined by z on both $f_{a_i,a_j,z}(A_n)$ and $f_{a_i,a_j,z}(B_n)$. Thus the identification added by $f_{a_i,a_j,z}$ is the same on both, and the remaining identifications which make up A_n and B_n must be the same. Therefore A_n and B_n have the same identifications, and are images of the same quotient map. Hence $A_n = B_n$, and they are the same element of P_n. Thus $f_{a_i,a_j,z}$ is injective. $(a_i, a_j) \in \{\alpha_n\}$, $z \in \{0, 1\}$, and n were all arbitrary, and so this is true for all (a_i, a_j), z, and n. □

Theorem 5.6. *Every $A_{n+1} \in P_{n+1}$ has exactly n preimages in P_n (not necessarily distinct) under the indexed collection of augment maps $\{f_{a_i,a_j,z}\} : P_n \to P_{n+1}$, where the indexes range over all combinations of $(a_i, a_j) \in \{\alpha_n\}$ and $z \in \{0, 1\}$.*

Proof. Pick some $A_{n+1} \in P_{n+1}$. Then A_{n+1} is defined by $n + 1$ identifications of pairs of edges with orientation. To prove this theorem, we will show that exactly n of the $n + 1$ identifications that compose A_{n+1} could have been added by an augment map, and therefore that it has n preimages under the indexed collection of augment maps $P_n \to P_{n+1}$.

By Theorem 5.5, every augment map $f_{a_i,a_j,z}$ has an inverse on its image in P_{n+1}. Call this inverse the **collapse** map $c_{a_i,a_j,z}$ corresponding to $f_{a_i,a_j,z}$. As an example, fix $f_{a_i,a_j,z}$. Any element in the image of this map has an identification between the edge whose left endpoint is a_i and the edge whose left endpoint is a_{j+1} (augmenting induces a relabeling that shifts the labels of points after a_i, as $a_i^{'}$ becomes a_{i+1}). The collapse map deletes this identification, leaving n identifications and $2n$ edges. Here is an illustration of a collapse map $P_3 \to P_2$. Vertices on $A_{n+1} \in P_{n+1}$ are relabeled to avoid confusion.



Given that A_{n+1} has $n + 1$ identifications, it remains to show that exactly n of them can be collapsed.

The basepoint $a = a_1$ could never have been added by an augment map, since A_n must have a basepoint for every n to induce the labelling of its edges. Augment maps always add clockwise, and so always add the right endpoint of the edges they add. a_1 is the right endpoint of the edge $e_{2(n+1)}$ on A_{n+1}, and therefore this edge could never be added by an augment map (if it were, then the map would have added the basepoint, which is not possible). Hence whichever identification includes $e_{2(n+1)}$ cannot be collapsed; and one identification on A_{n+1} cannot be collapsed.

Every other pair of identified edges has right endpoints which are not a_1. Choose an arbitrary identification on A_{n+1} that does not have right endpoint a_1. Call the right endpoints of the identified edges a_i and a_{j+1} (a_{j+1} must be a vertex on X_{n+1} for $1 \leq j \leq 2n$, since we have excluded the last edge), and suppose the identification has orientation z. Then the collapse $c_{a_i,a_j,z} : P_{n+1} \to P_n$ is defined. We chose an arbitrary identification from the remaining n, and hence there are n identifications on A_{n+1} which can be collapsed. □

Theorem 5.7. *The indexed collection of augment maps $\{f_{a_i,a_j,z}\} : P_n \to P_{n+1}$ n-fold uniformly covers P_{n+1}.*

Proof. Every augment map $f_{a_i,a_j,z}$ is injective. Since every $A_{n+1} \in P_{n+1}$ has exactly n preimages in P_n by the indexed collection of augment maps, the indexed collection of maps must cover every element of P_{n+1} exactly n times. Therefore the indexed collection of maps $\{f_{a_i,a_j,z}\} : P_n \to P_{n+1}$ n-fold uniformly covers P_{n+1}. □

6. The Euler Characteristic in P_n

Studying the Euler Characteristic in P_n will be an application of the augment maps defined in the previous section. First, some definitions:

Definition 6.1. The **Euler Characteristic** of a surface S is

$$\chi(S) = F - E + V$$

where F is the number of faces in a triangulation of S, E is the number of edges, and V is the number of vertices.

Remark 6.2. The Euler Characteristic has to be calculated based on a triangulation, but it does not depend on the triangulation.

Proposition 6.3. *The **genus** of A_n is g if A_n is homeomorphic to gT^2, p if A_n is homeomorphic to pP^2, and 0 if A_n is homeomorphic to S^2. The Euler Characteristic of A_n has the following relationship with its genus:*

$$\chi(A_n) = 2 - 2g$$

if A_n is orientable and has genus g, and

$$\chi(A_n) = 2 - p$$

if A_n is nonorientable and has genus p.

Remark 6.4. Note that since the minimum genus of A_n is 0, the maximum Euler Characteristic is 2.

In this section, we will develop a computable recursive formula for the number of elements of P_{n+1} with a given Euler Characteristic x.

6.1. Calculating the Euler Characteristic of an Identified Polygon.

Calculating the Euler Characteristic of A_n requires choosing a triangulation of A_n (being careful to choose one that is still a triangulation of the surface A_n presents). Some calculation from a triangulation shows that

$$\chi(A_n) = 1 - n + k, \tag{6.5}$$

where k is the number of vertices of A_n. The k vertices of A_n are equivalence classes of the vertices of X_n induced by the map $p : X_n \to A_n$. Each of these k equivalence classes is a loop: every vertex a_i of X_n is part of two edges, each of which is identified to another edge. As such, a_i is identified to two vertices (possibly the same vertex twice), one by each of its edges. There are only finitely many edges, and so this process must terminate, leaving k loops which contain all $2n$ vertices of X_n.

Remark 6.6. The number of vertices k of A_n is always between the bounds

$$1 \leq k \leq n+1$$

Every vertex could be in the same equivalence class, giving us the lower bound. Then the upper bound comes from the fact that the Euler Characteristic of a surface is never greater than 2.

Given how to calculate the Euler Characteristic, we can prove that which surfaces are presented by elements of P_n depends on n:

Proposition 6.7. *The maximum genus of a surface presented by an element of P_n is $\frac{n}{2}$ if the surface is orientable, and n if the surface is nonorientable.*

Proof. Let n be given, and choose $A_n \in P_n$. $\chi(A_n) = 1 - n + k$, where k is the number of vertices of A_n. Recall that

$$\chi(A_n) = 2 - 2g$$

if A_n is orientable and has genus g, and

$$\chi(A_n) = 2 - p$$

if A_n is nonorientable and has nonorientable genus p. Hence the smallest possible value of k gives the largest possible genus. The smallest possible value of k is $k = 1$. Thus for an orientable surface, the largest possible genus is

$$\begin{aligned} 2 - 2g &= 1 - n + 1 \\ g &= n/2 \end{aligned}$$

And for a nonorientable surface,

$$\begin{aligned} 2 - p &= 1 - n + 1 \\ p &= n \end{aligned}$$

□

6.2. **Graphs.**

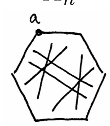
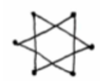
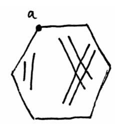
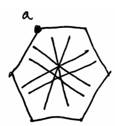
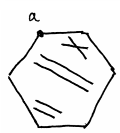
Definition 6.8. Given an identified polygon $A_n = p(X_n)$, let $G(A_n)$ be a graph on the $2n$ vertices of X_n where two points in $G(A_n)$ are connected by a single line if and only if they were connected by a single identification in A_n.

Remark 6.9. Let $p : X_n \to A_n$ be the quotient map which defines A_n. $G(A_n)$ is a graph on $2n$ points with $2n$ identifications, where each of these points is a vertex of X_n. Every point in $G(A_n)$ is connected to another point if and only if it is identified to that point by p, and so each component of $G(A_n)$ is a vertex of A_n. Note that this means that every component of $G(A_n)$ is a loop, and more importantly that

$$\#\text{ of vertices of } A_n = k = \#\text{ of components of } G(A_n).$$

So instead of trying to count the vertices of A_n as equivalence classes of vertices on X_n, we can count components of $G(A_n)$.

Example 6.10. Here are a few examples of identified hexagons and their graphs:

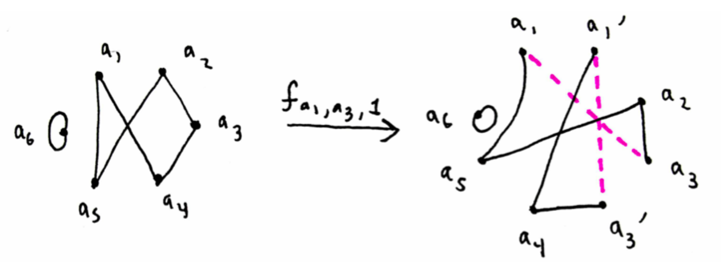
A_n	$G(A_n)$	# of components	$\chi(A_n)$	surface
		2	0	$2P^2$
		2	0	T^2
		1	-1	$3P^2$
		3	1	P^2

6.3. **Augments of Graphs.** Graphs give us a visual way to determine the Euler Characteristic of an identified polygon. The question we will answer in this section is:

Question 6.11. *What is the relationship between $G(A_n)$ and $G(f_{a_i,a_j,z}(A_n))$? If $G(A_n)$ has k components, how many components can $G(f_{a_i,a_j,z}(A_n))$ have?*

When we augment the identified polygon that lies under the graph, we add two points by splitting two existing points a_i and a_j into a_i and $a_i^{'}$ and a_j and $a_j^{'}$. Then we identify one of the a_i's to one of the a_j's, where which one is identified to which depends on the orientation determined by z. On the graph, this looks like we picked two points a_i and a_j, and cut a hole in a component of the graph at each of them. Then we connect the holes in one of two possible ways as determined by z.

Example 6.12. Consider this example of an augment map $P_3 \to P_4$ in the graph presentation:



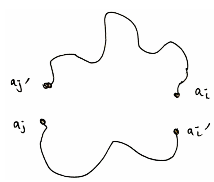
The pink dotted lines are the ones added by the augment map. In this case, the augment map took a graph with 2 components to one with 3 components.

In the graph presentation, there are two possible relationships that the points a_i and a_j can have: either they are in the same component of $G(A_n)$ or they are in different components. This amount of information about a_i and a_j is enough to know what happens when they are augmented:

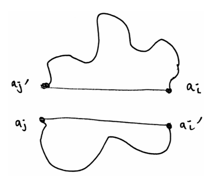
Theorem 6.13. *Let $A_n \in P_n$ and $G(A_n)$ be its graph. Suppose $G(A_n)$ has k components. If a_i and a_j are in the same component, then one of $G(f_{a_i,a_j,0}(A_n))$ and $G(f_{a_i,a_j,1}(A_n))$ has k components, and the other has $k+1$. If a_i and a_j are in different components, then $G(f_{a_i,a_j,0}(A_n))$ and $G(f_{a_i,a_j,1}(A_n))$ both have $k-1$ components.*

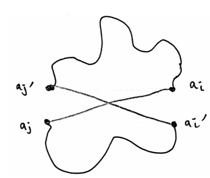
Proof. Suppose $G(A_n)$ has k components. There are two cases; consider them one at a time.

(1) Suppose a_i and a_j are in the same component. Then the new edges create two holes in the same loop, making something that looks like this:



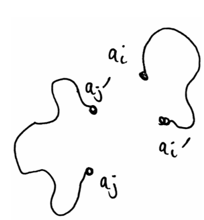
One endpoint of the a_i hole must be identified to one endpoint of the a_j hole, since edges cannot be identified to themselves. Both pieces of the divided component have one endpoint that is an a_i point and the one that is an a_j point. Thus one type of identification splits the original component in two, and the other joins it back together again:



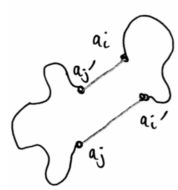


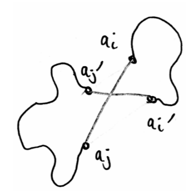
In the example illustrated above, a twist rejoins and a straight identification breaks apart. But this is not always true (if the two disjoint pieces crossed once, for example, it would be the other way around). Regardless, in every situation, one identification joins endpoints of the same piece, and the other joins endpoints of opposite pieces. As a result, one of $f_{a_i,a_j,0}$ and $f_{a_i,a_j,1}$ takes $G(A_n)$ to a graph with k components and one takes it to a graph with $k+1$ components.

(2) Suppose a_i and a_j are in different components. Then we add one new edge to each component, effectively cutting a hole in each and create something like this:



The endpoints of these pieces are both from the same edge, and so cannot be identified together. Thus no matter which orientation we identify the edges with, we combine the two components:





Hence both $f_{a_i,a_j,0}$ and $f_{a_i,a_j,1}$ take $G(A_n)$ to graphs with $k-1$ components.

□

This theorem shows that the number of components in the graph of $f_{a_i,a_j,z}(A_n) = A_{n+1} \in P_{n+1}$ depends on the number of components in the graph of A_n. Since $\chi(A_n)$ is determined by the number of components in $G(A_n)$, this gives us the following result about $\chi(f_{a_i,a_j,z}(A_n))$:

Theorem 6.14. *Let A_n be an element of P_n. If $\chi(A_n) = x$, then $\chi(f_{a_i,a_j,z}(A_n))$ is x, $x-1$, or $x-2$.*

Proof. We know that

$$x = \chi(A_n) = 1 - n + k.$$

Augments of A_n have $n+1$ edges instead of n. Further, by the previous theorem, we have that since $G(A_n)$ has k components, $G(f_{a_i,a_j,z}(A_n))$ has $k-1$, k, or $k+1$ components. Thus

$$\chi(f_{a_i,a_j,z}(A_n)) = 1 - (n+1) + v$$

where $v = k-1, k$, or $k+1$. Therefore

$$\chi(f_{a_i,a_j,z}(A_n)) = x$$

if $v = k+1$, and

$$\chi(f_{a_i,a_j,z}(A_n)) = x - 1$$

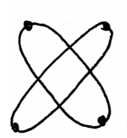
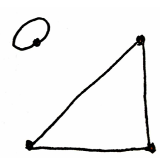
if $v = k$, and

$$\chi(f_{a_i,a_j,z}(A_n)) = x - 2$$

if $v = k-1$. □

6.4. **Partitions.** Augment maps and graphs together show that the Euler Characteristics of elements of P_{n+1} depend on the Euler Characteristics of elements of P_n. But they also depend on how many augment maps are based on vertices in the same component and how many are based on vertices in the different components, which clearly depends on individual graphs and how they are composed. This is still information that we have and can generate recursively, and in this section we will explain how.

Example 6.15. Consider the following two graphs, which correspond to elements of P_2:

Both of these graphs have $k = 2$ components, which means that they both have Euler Characteristic $x = 1$. But there are 6 ways to chose two points in the same component from the graph on the left, and 7 ways to choose two points in the same component from the graph on the right. As a result, augments of these graphs in P_3 will not have the same distribution of Euler Characteristics.

Definition 6.16. Fix $A_n \in P_n$. The graph $G(A_n)$ has k components, which together contain $2n$ points. We can write the graph as a **partition** $(p_1, p_2, ..., p_k)$ of $2n$ of length k, where each p_i in the partition is the number of points in a component of $G(A_n)$. We will call each p_i a **piece** of the partition.

Examples 6.17. The graphs in Example 6.10 have the following partitions: (3,3), (1,5), (6), and (1,2,3), respectively. The two graphs in Example 6.15 have partitions (2,2) and (1,3).

Proposition 6.18. *Given a partition $(p_1, p_2, ..., p_k)$ of $2n$, the number of ways to choose two points in the same piece of this partition (and so the same component of the graph it represents) is*

$$S(A_n) = n + \sum_{i=1}^{k} \frac{p_i^2}{2} \tag{6.19}$$

Consequently, the number of ways to choose two points in different pieces of this partition is

$$D(A_n) = n(2n+1) - \big(n + \sum_{i=1}^{k} \frac{p_i^2}{2}\big) = 2n^2 - \sum_{i=1}^{k} \frac{p_i^2}{2}. \tag{6.20}$$

It is already clear that knowing the partitions for P_n is sufficient to know the distribution of Euler Characteristics in P_n, as the number of partitions of length k corresponds to the number of elements of P_n with Euler Characteristic $x = 1-n+k$. But we want to show that we can recursively compute partitions. As such, our goal with be to prove that

Claim 6.21. The partition of A_n together with the augment map $f_{a_i,a_j,z}$ determines the partition of $f_{a_i,a_j,z}(A_n)$, and therefore augment maps recursively generate partitions.

Given that, we will lay out the process for computing the partitions for P_{n+1} from the partitions for P_n, and apply this to count elements of P_{n+1} with a specified Euler Characteristic.

There are three possible situations an augmented partition can be in, corresponding to a partition (or graph) of length k being mapped to one of length $k-1$, k, or $k+1$. Two correspond to (a_i, a_j) being in the same piece of the partition, and one to them being in different pieces.

(1) Suppose a_i and a_j are in the same partition piece p_i. Then
   (a) The partition of $f_{a_i,a_j,z}(p_1, ..., p_i, ..., p_k)$ is $(p_1, ..., p_i + 2, ..., p_k)$, a partition of $2n+2$ of length k,
   (b) The partition of $f_{a_i,a_j,z}(p_1, ..., p_i, ..., p_k)$ is $(p_1, ..., q_1, q_2, p_{i+1}, ..., p_k)$, a partition of $2n+2$ of length $k+1$, where $q_1 + q_2 = p_i + 2$.

   Both of these outcomes occur, one for $z = 0$ and one for $z = 1$.

(2) Suppose (a_i, a_j) are in different partition pieces p_i and p_j. Then the partition of $f_{a_i,a_j,z}(p_1, ..., p_k)$ is $(p_1, ..., p_i + p_j + 2, ..., p_k)$, a partition of $2n + 2$ of length $k - 1$, for both $z = 0$ and $z = 1$.

These calculations are a consequence of Theorem 6.13 (about graphs) which we proved in the previous section. Given this, to make partitions for P_{n+1} computable from partitions for P_n, we have to determine the frequency with which these three situations occur. Consider them one at a time.

Proposition 6.22. *The partition of* $f_{a_i,a_j,z}(p_1, ..., p_i, ..., p_k)$ *is* $(p_1, ..., p_i+2, ..., p_k)$ *with frequency*

$$p_i + \binom{p_i}{2} = \frac{p_i(p_i + 1)}{2}$$

for each p_i.

Proposition 6.23. *For every pair of positive integers* q_1, q_2 *with* $q_1 + q_2 = p_i + 2$, *the partition of* $f_{a_i,a_j,z}(p_1, ..., p_i, ..., p_k)$ *is* $(p_1, ..., q_1, q_2, p_{i+1}, ..., p_k)$ *with frequency*

$$p_i$$

if $q_1 \neq q_2$, *and*

$$\frac{p_i}{2}$$

if $q_1 = q_2$.

Proposition 6.24. *The partition of* $f_{a_i,a_j,z}(p_1, ..., p_k)$ *is* $(p_1, ..., p_i + p_j + 2, ..., p_k)$ *with frequency*

$$2p_ip_j$$

for every pair p_i, p_j.

Example 6.25. Consider the graphs in Example 6.15 and their respective partitions, (2,2) and (1,3). Each has 20 augments.

Augments of (2,2):	(2,4) 6 times, (1,2,3) 4 times, (2,2,2) 2 times, (6) 8 times.
Augments of (1,3):	(3,3) 1 time, (1,5) 6 times, (1,1,4) 3 times, (1,2,3) 4 times, (6) 6 times.

Given these formulas which construct the partitions for P_{n+1} from the partitions for P_n, we can construct a tree of partitions. The first layer of this tree is the partitions for P_1,

(1,1) and (2).

Each following layer is composed of all the augments of the partitions on the layer before it, preserving multiplicities. The second layer is then

(1,3), (1,2,1), (4), (4), (1,3), (1,2,1), (1,3), (4), (2,2), (4), (1,3), (4).

Note that since we preserve multiplicities, each (4) in the second layer of the tree is its own unique node, and the third layer of the tree will include every augment of all five (4)s. This tree has a relationship with the Euler Characteristic that makes the distribution of Euler Characteristics in P_{n+1} computable:

Theorem 6.26. *Let* $C(n, x)$ *be number of* $A_n \in P_n$ *with Euler Characteristic* x. *Let* $Q(n, k)$ *be the number of partitions of length* k *at level* n *of the partition tree. Then*

$$C(n, x) = \frac{Q(n, k)}{(n - 1)!},$$

where $x = 1 - n + k$.

Proof. Since $x = 1 - n + k$, a partition of length k at level n corresponds to an element of P_n with Euler Characteristic x. Since the next level of the partition tree is constructed by applying augment maps to the partitions at the previous level, by Theorem 5.7 we must divide by $(n-1)!$. □

6.5. **Restriction to Orientables.**

Lemma 6.27. *Suppose that A_n is orientable and $G(A_n)$ has k components. If a_i and a_j are in the same component, then $G(f_{a_i,a_j,0}(A_n))$ has $k+1$ components. If a_i and a_j are in different components, then $G(f_{a_i,a_j,0}(A_n))$ has $k-1$ components.*

Proof. Let $G(A_n)$ be a graph with k components.

(1) Suppose that a_i and a_j are in the same component. A_n is orientable, and so the two pieces of this component have endpoint a_i and $a_j^{'}$ and a_j and $a_i^{'}$. $f_{a_i,a_j,0}(A_n)$ adds a straight identification, and so identifies a_i to $a_j^{'}$ and a_j to $a_i^{'}$. Hence $G(f_{a_i,a_j,0}(A_n))$ has $k+1$ components.
(2) Suppose that a_i and a_j are in different components. Then by Theorem 6.13, $G(f_{a_i,a_j,0}(A_n))$ has $k-1$ components.

□

Theorem 6.28. *The tree of partitions corresponding to elements of $O_n \subset P_n$ is a subtree of the tree of partitions.*

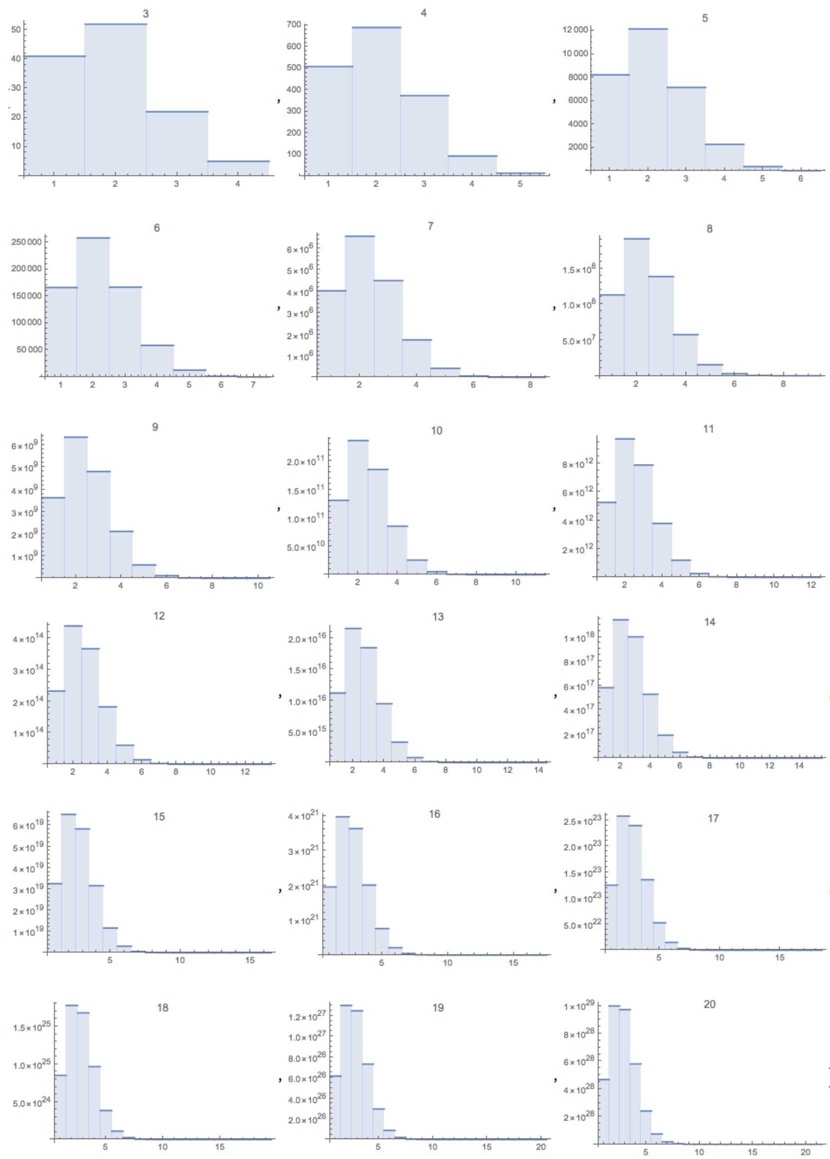
Proof. By Lemma 4.2, $A_n \in P_n$ is orientable if and only if none of its identifications are twists. Therefore elements of O_n must follow a path down the tree of only straight augments $f_{a_i,a_j,0}$. By Lemma 6.27, these paths are always determined, taking a $k-1$ path if a_i and a_j are in different components and the $k+1$ path if they are in the same component. □

Corollary 6.29. *For any n, given the tree of partitions and a compact surface without boundary S, we can compute the number of $A_n \in P_n$ that present S.*

7. Results

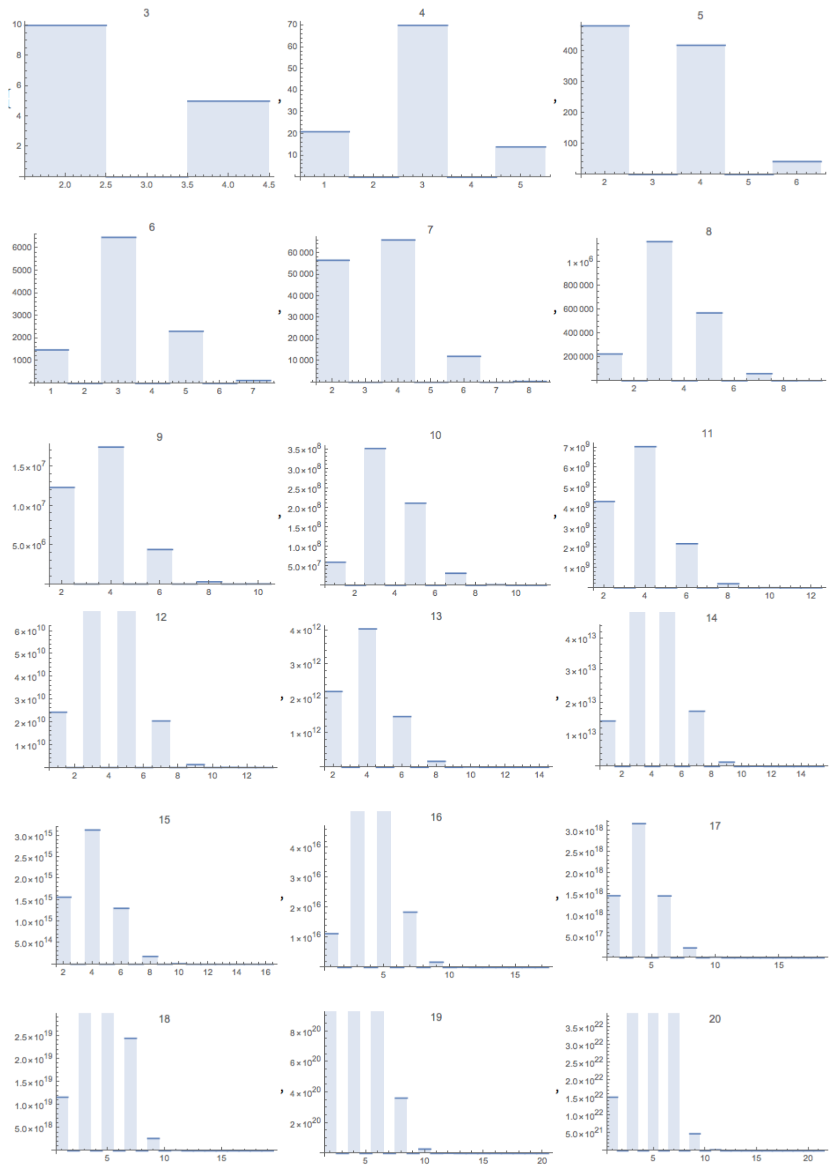
For $n > 3$, it isn't realistic to do these computations by hand. But they are well-suited to be done by a computer. Using the initial conditions defined by P_1 (two elements, with partitions $(1,1)$ and (2). $(1,1)$ is orientable, and (2) is not), we can compute the number of $A_n \in P_n$ with a given Euler Characteristic, and the number that present each compact surface S. Here are some interesting results found by a computer.

First, here are histograms of the number of elements of P_n with each possible number of graph components k for $n = 3$ to $n = 20$. Note that to $k = 1$ corresponds to A_n which is homeomorphic to a surface with the maximum possible genus presentable in P_n, and $k = n+1$ corresponds to A_n which is homeomorphic to the sphere S^2.

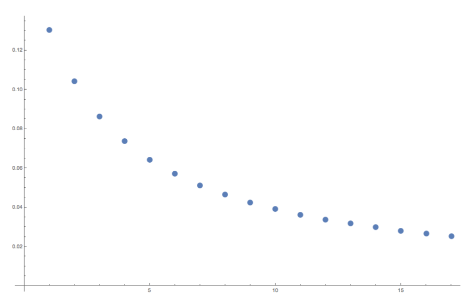


Interestingly, observe that the number of elements of P_n that present the maximal genus surface is lower than the number that present second maximal surfaces. Also observe that by $n = 20$, $k = 2$ and $k = 3$ are almost equally frequent.

Here are similar histograms for the subset of orientables, O_n (note that the scale on these histograms is different from the ones for all of P_n). Combining this data with the data for all of P_n is what would allow us to compute the number of elements of P_n which present S for a given compact surface S.



Finally, observe a plot of the difference between the expected number of components in P_n and P_{n+1} as a function of n.



This plot suggests that difference between the expected number of components for P_n and P_{n+1} might converge to 0, and so the expected number of components might converge as $n \to \infty$. If it does converge, that would imply that the average Euler Characteristic decreases by 1 from P_n to P_{n+1} for large n.

8. Other Questions

The results in the previous section ask more questions than they answer. A few of the questions they ask are asked here. There are also other interesting ways to look at the sets P_n, other ways to filter these sets with additional restrictions, or generalize them by taking some of those restrictions away. A few of these questions are also asked here.

1. Is there a distribution that fits the distribution of Euler Characteristics in P_n? If it converges, what does the distribution of Euler Characteristics in P_n limit to as $n \to \infty$?

2. Does the second maximal genus continue to be the most frequent as $n \to \infty$? Does it become second and third maximal, or does the most frequent band continue to widen? Why?

3. Does the expected number of components k of $A_n \in P_n$ converge as $n \to \infty$? Does it have any otherwise predictable behavior?

4. What are the answers to questions 1, 2, and 3 for $O_n \subset P_n$? How are the answers for O_n similar (or not) to the answers for P_n?

5. A set with a preorder is a finite topological space. Is there an interesting preorder structure that we can put on the sets P_n to turn them into spaces? One possible preorder is to order the elements of P_n based on how many twists they have. Is this the most interesting structure we can put on the sets P_n?

6. What happens if we allow the identification of different numbers of edges (such as identifying three edges together, or letting the number of edges identified together vary), or allow some edges to not be identified at all?

7. What happens if we invoke other restrictions on the orientation of identifications, such as requiring that every identified polygon have exactly one twist?

8. The augment maps $f_{a_i,a_j,z}$ from P_n to P_{n+1} define limiting "paths" through the space of compact surfaces, based on a sequence of choices of (a_i, a_j) and z. Using a sequence of choices (which could be something simple like choosing $(a_1, a_1, 0)$ every time, or something more complicated), we can iterate the maps $f_{a_i,a_j,z}$ determined by the chosen sequence. Each sequence defines a "limit" of $A_n \in P_n$ as $n \to \infty$. What are the "limit" surfaces $\{f_{a_i,a_j,z}\}^{\infty}(A_n)$? Does every infinite path defined by a sequence of augment maps have a unique resulting object, or are some of them the same? What are they?

9. As $n \to \infty$, each A_n approaches a circle with infinitely many identifications of different "points" (edges with side lengths $< \epsilon$). If we direct the graph of vertices induced by the identified polygon, we can think of the components of $G(A_n)$ as orbits on the circle. What sort of dynamical systems on the circle do the identifications of A_n define? Do most points have a dense orbit?

Acknowledgments. It is my pleasure to thank my mentors, Sean Howe and Yun Cheng, for listening to many ideas, reading multiple drafts, and keeping me on track. I would also like to thank my brother, Christopher Wolfram, for turning these ideas into code, allowing us to compute much more interesting results. Finally, I am happy to thank Peter May for organizing the REU program at the University of Chicago and making this possible.

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[1] Hatcher, Allen. Algebraic Topology. Cambridge: Cambridge UP, 2002.
[2] Munkres, James R. Topology. 2nd ed. Upper Saddle River, NJ: Prentice Hall, 2000.
[3] Hilbert, David, and S. Cohn-Vossen. Geometry and the Imagination. Providence, RI: AMS Chelsea, 1999.
[4] Gallier, Jean H., and Dianna Xu. A Guide to the Classification Theorem for Compact Surfaces. Heidelberg: Springer, 2013.
[5] Gallian, Joseph A. Contemporary Abstract Algebra. Lexington, MA: D.C. Heath, 1986.

```
In[312]:= partitions[partition : {__Integer}] := Merge[Join[
     Flatten[MapIndexed[{p, i} ↦
          Join[
           {Sort@ReplacePart[partition, i → p + 2] → (1 / 2 * p * (p + 1))},
           Sort@Flatten[ReplacePart[partition, i → #], 1] →
              If[#[[1]] == #[[2]], p / 2, p] & /@ IntegerPartitions[p + 2, {2}]
          ]
         , partition], 1],
      (*Pairs*)
     Map[
       j ↦
        Sort@Append[Delete[partition, j[[All, 2]]], j[[1, 1]] + j[[2, 1]] + 2] →
         j[[1, 1]] * j[[2, 1]] * 2,
       DeleteDuplicates[Sort /@ DeleteCases[
          Tuples[MapIndexed[List, partition], 2], {n_, n_}]]]
    ],
    Total]

Table[Block[{counts = Total /@
      GroupBy[Normal@Nest[p ↦ Merge[KeyValueMap[partitions[#1] * #2 &, p], Total],
         partitions[{{1, 1}, {2}}], n], Length@*First → Last]},
   Table[Lookup[counts, v, 0] / (n + 1) !, {v, Min@Keys[counts], Max@Keys[counts]}]],
  {n, 1, 18}]

Table[Block[{counts = Total /@
      GroupBy[Normal@Nest[p ↦ Merge[KeyValueMap[partitions[#1] * #2 &, p], Total],
         partitions[{{1, 1}, {2}}], n], Length@*First → Last]}, DiscretePlot[
    Lookup[counts, v, 0] / (n + 1) !, {v, Min@Keys[counts], Max@Keys[counts]},
    ExtentSize → Full, PlotLabel → n + 2, ImageSize → 300]], {n, 1, 18}]

In[314]:= orientablepartitions[orientablepartition : {__Integer}] := Merge[Join[
     Flatten[
      MapIndexed[{p, i} ↦ Sort@Flatten[ReplacePart[orientablepartition, i → #], 1] →
            If[#[[1]] == #[[2]], p / 2, p] & /@
         IntegerPartitions[p + 2, {2}], orientablepartition], 1],
      (*Pairs*)
     Map[
      j ↦ Sort@Append[Delete[orientablepartition, j[[All, 2]]],
          j[[1, 1]] + j[[2, 1]] + 2] → j[[1, 1]] * j[[2, 1]],
      DeleteDuplicates[Sort /@ DeleteCases[Tuples[
          MapIndexed[List, orientablepartition], 2], {n_, n_}]]]
    ],
    Total]
```

```
Table[Block[{counts = Total /@ GroupBy[Normal@
       Nest[p ↦ Merge[KeyValueMap[orientablepartitions[#1] * #2 &, p], Total],
        orientablepartitions[{1, 1}], n], Length@* First → Last]},
  Table[Lookup[counts, v, 0] / (n + 1) !, {v, Min@Keys[counts], Max@Keys[counts]}]],
 {n, 1, 18}]
```

```
Table[Block[{counts = Total /@ GroupBy[Normal@
       Nest[p ↦ Merge[KeyValueMap[orientablepartitions[#1] * #2 &, p], Total],
        orientablepartitions[{1, 1}], n], Length@* First → Last]},
  DiscretePlot[Lookup[counts, v, 0] / (n + 1) !, {v, Min@Keys[counts],
    Max@Keys[counts]}, ExtentSize → Full,
   PlotLabel → n + 2, ImageSize → 300]], {n, 1, 18}]
```

```
Table[Block[{counts = Total /@
      GroupBy[Normal@Nest[p ↦ Merge[KeyValueMap[partitions[#1] * #2 &, p], Total],
         partitions[{{1, 1}, {2}}], n], Length@* First → Last]},
   Mean[WeightedData @@ Transpose@Table[{v, Lookup[counts, v, 0] / (n + 1) !},
       {v, Min@Keys[counts], Max@Keys[counts]}]]], {n, 1, 18}] // N
```

```
ListPlot[Table[Block[{counts = Total /@
       GroupBy[Normal@Nest[p ↦ Merge[KeyValueMap[partitions[#1] * #2 &, p], Total],
          partitions[{{1, 1}, {2}}], n], Length@* First → Last]},
    Mean[WeightedData @@ Transpose@Table[{v, Lookup[counts, v, 0] / (n + 1) !},
        {v, Min@Keys[counts], Max@Keys[counts]}]]], {n, 1, 18}] // N]
```

```
ListPlot[Differences[Table[Block[{counts =
       Total /@ GroupBy[Normal@Nest[p ↦ Merge[KeyValueMap[partitions[#1] * #2 &, p],
             Total], partitions[{{1, 1}, {2}}], n], Length@* First → Last]},
     Mean[WeightedData @@ Transpose@Table[{v, Lookup[counts, v, 0] / (n + 1) !},
         {v, Min@Keys[counts], Max@Keys[counts]}]]], {n, 1, 18}] // N]]
```

1 Appendix: Table of Results for Number of Orientable Gluings of a $2n$-gon

I learned in August 2018 that the number of orientable gluings of a $2n$-gon was also computed (by a different method) in [1] in order to calculate euler characteristics of moduli spaces of curves. The original version of this paper only presented the numerical results we computed as histograms. Below is a table of the precise values computed with the method outlined in this paper, for comparsion with the table of precise values in [1] computed using Gaussian integrals.

genus g (column) , 2n−gon (row)	10	9	8	7	6	5	4	3	2	1	0
3	0	0	0	0	0	0	0	0	0	10	5
4	0	0	0	0	0	0	0	0	21	70	14
5	0	0	0	0	0	0	0	0	483	420	42
6	0	0	0	0	0	0	0	1485	6468	2310	132
7	0	0	0	0	0	0	0	56628	66066	12012	429
8	0	0	0	0	0	0	225 225	1 169 740	570 570	60060	1430
9	0	0	0	0	0	0	12 317 877	17 454 580	4 390 386	291 720	4862
10	0	0	0	0	0	59 520 825	351 683 046	211 083 730	31 039 008	1 385 670	16796
11	0	0	0	0	0	4 304 016 990	7 034 538 511	2 198 596 400	205 633 428	6 466 460	58786
12	0	0	0	0	24 325 703 3⋱ 25	158 959 754⋱ 226	111 159 740⋱ 692	20 465 052 6⋱ 08	1 293 938 646	29745 716	208 012
13	0	0	0	0	2 208 143 02⋱ 8 375	4 034 735 95⋱ 9 800	1 480 593 01⋱ 3 900	174 437 377⋱ 400	7 808 250 450	135 207 800	742 900
14	0	0	0	14 230 536 4⋱ 45 125	100 940 771⋱ 124 360	79 553 497 7⋱ 60 100	17 302 190 6⋱ 25 720	1 384 928 66⋱ 6 550	45 510 945 4⋱ 80	608 435 100	2 674 440
15	0	0	0	1 564 439 68⋱ 6 929 000	3 130 208 76⋱ 9 783 780	1 302 772 71⋱ 8 028 600	182 231 849⋱ 209 410	10 369 994 0⋱ 05 800	257 611 421⋱ 340	2 714 556 600	9 694 845
16	0	0	11 288 163 7⋱ 62 500 62⋱ 5	85 775 385 8⋱ 35 387 80⋱ 0	74 520 697 7⋱ 07 149 58⋱ 0	18 475 997 0⋱ 06 212 20⋱ 0	1 763 184 57⋱ 1 730 010	73 920 866 3⋱ 62 200	1 422 156 20⋱ 2 740	12 021 607 8⋱ 00	35 357 670
17	0	0	1 463 987 08⋱ 9 109 939⋱ 625	3 160 912 92⋱ 2 719 805⋱ 880	1 457 897 21⋱ 6 520 222⋱ 060	233 454 817⋱ 237 201 5⋱ 60	15 894 791 3⋱ 12 284 17⋱ 0	505 297 829⋱ 133 240	7 683 009 54⋱ 4 980	52 895 074 3⋱ 20	129 644 790
18	0	11 665 426 0⋱ 77 721 04⋱ 0 625	94 035 726 1⋱ 63 975 53⋱ 8 250	88 656 726 5⋱ 30 014 12⋱ 7 100	24 464 684 5⋱ 45 968 00⋱ 4 800	2 682 208 75⋱ 1 185 413⋱ 450	134 951 136⋱ 993 773 1⋱ 00	3 331 309 74⋱ 1 059 300	40 729 207 2⋱ 26 400	231 415 950⋱ 150	477 638 700
19	0	1 749 439 02⋱ 8 845 202⋱ 483 250	4 031 976 19⋱ 8 236 643⋱ 244 665	2 027 209 09⋱ 4 015 371⋱ 081 200	362 610 922⋱ 3 100 400⋱ 35 940	28 449 551 6⋱ 53 853 22⋱ 9 900	1 088 243 82⋱ 6 731 751⋱ 690	21 280 393 6⋱ 66 593 60⋱ 0	212 347 275⋱ 857 640	1 007 340 01⋱ 8 300	1 767 263 190
20	15 230 046 9⋱ 89 184 65⋱ 5 753 125	129 268 273⋱ 737 506 8⋱ 15 518 75⋱ 0	130 723 600⋱ 701 707 4⋱ 04 561 17⋱ 0	39 470 026 9⋱ 30 000 17⋱ 7 711 200	4 848 655 67⋱ 9 592 076⋱ 350 570	281 858 111⋱ 998 039 4⋱ 76 900	8 391 311 31⋱ 6 938 069⋱ 520	132 216 351⋱ 453 357 6⋱ 00	1 090 848 50⋱ 5 817 070	4 365 140 07⋱ 9 300	6 564 120 420

References

[1] J. Harer, D. Zagier. The Euler characteristic of the moduli space of curves. Invent. Math. 85 (1986), 457–485.