GEOMETRIC INTERPRETATIONS OF CURVATURE

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Abstract. This is an expository paper on geometric meaning of various kinds of curvature on a Riemann manifold.

Contents

1. Notation and Summation Conventions

We assume knowledge of the basic theory of smooth manifolds, vector fields and tensors. We will assume all manifolds are smooth, i.e. C^∞, second countable and Hausdorff. All functions, curves and vector fields will also be smooth unless otherwise stated. Einstein summation convention will be adopted in this paper. In some cases, the index types on either side of an equation will not match and so a summation will be needed. The tangent vector field $\frac{\partial}{\partial x^i}$ induced by local coordinates (x^i) will be denoted as ∂_i.

2. Affine Connections

Riemann curvature is a measure of the noncommutativity of parallel transportation of tangent vectors. To define parallel transport, we need the notion of affine connections.

Definition 2.1. Let $\mathcal{M}$ be an n-dimensional manifold. An affine connection, or connection, is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, where $\mathfrak{X}(\mathcal{M})$ denotes the space of smooth vector fields, such that for vector fields $V_1, V_2, V, W_1, W_2 \in \mathfrak{X}(\mathcal{M})$ and function $f : \mathcal{M} \to \mathbb{R}$,

(1) $\nabla(fV_1 + V_2, W) = f\nabla(V_1, W) + \nabla(V_2, W)$,
(2) $\nabla(V, aW_1 + W_2) = a\nabla(V, W_1) + \nabla(V, W_2)$, for all $a \in \mathbb{R}$.
(3) $\nabla(V, fW) = V(f)W + f\nabla(V, W)$.

We write $\nabla(V, W) = \nabla_V W$.

Theorem 2.2. *For fixed W, $(\nabla_V W)|_p$ only depends on $V|_p$.*

Proof. It suffices to show if $V|_p = 0$ then $(\nabla_U W)|_p = 0$. The proof relies on the following lemma: if $V|_p = 0$ then there exists smooth scalar fields f_k and vector fields $\tilde{V_k}$ such that $f_k(p) = 0$ and $V = \sum_k f_k \tilde{V_k}$. This fact is easily proven using partitions of unity. Using this lemma, we then write $V = \sum_k f_k \tilde{V_k}$ where $f_k(p) = 0$. Then

$$(\nabla_{\sum_k f_k \tilde{V_k}} W)|_p = \sum_k f_k(p)(\nabla_{\tilde{V_k}} W)|_p = 0.$$

□

Remark 2.3. Since $\nabla_V W$ depends on V pointwise, we may write $\nabla_{\mathbf{v}} W := \nabla_V W|_p$ where $\mathbf{v} = V_p \in T_p\mathcal{M}$. We also note that $\nabla_V W|_p$ depends on the local values of W, that is, if $W_1 = W_2$ in some neighborhood of p, then $\nabla_{\mathbf{v}} W_1|_p = \nabla_{\mathbf{v}} W_2|_p$ for all $\mathbf{v} \in T_p\mathcal{M}$.

We now give an expression for the connection ∇ in local coordinates (x^i) on an open set U. Suppose we have two vector fields $W = w^j \partial_j$ and $V = v^i \partial_i$ on U. Then

$$\nabla_W V = w^j \frac{\partial v^i}{\partial x^j} \partial_i + w^j v^i \nabla_{\partial_j} \partial_i.$$

We define functions $\Gamma^k_{ij} : U \to \mathbb{R}$, for $1 \le i, j, k \le n$, by

$$\nabla_{\partial_j} \partial_i = \Gamma^k_{ij} \partial_k.$$

These functions are called the Christoffel symbols and depend on the coordinate system used to define them. Since covariant differentiation takes values in $\mathfrak{X}(\mathcal{M})$, we see immediately that the Christoffel symbols are smooth. We now show existence of a global affine connection on a manifold $\mathcal{M}$.

Theorem 2.4. *For any manifold $\mathcal{M}$, there exists an affine connection on $\mathcal{M}$.*

Proof. Let $\{U_\alpha\}_\alpha$ be an atlas of $\mathcal{M}$ and let $\{\rho_\alpha\}_\alpha$ be the associated partition of unity. On each open set U_α, we have a connection ∇_α given in the coordinates on U_α by $\Gamma^k_{ij} = 0$. We define ∇ by $\nabla_V W = \sum_\alpha (\nabla_\alpha)_{\rho_\alpha \cdot V}(\rho_\alpha \cdot W)$. Note that at each point p, $\nabla|_p$ is a finite sum. Some straightforward computations will show that ∇ satisfies the three properties in Theorem 2.1 and $\nabla_V W$ is smooth when V, W are smooth. □

Remark 2.5. The affine connection ∇ on a manifold need not be unique. If we vary the Christoffel symbols Γ^k_{ij} in each of the charts U_α of the proof, we may get different affine connections.

Remark 2.6. We know that $\nabla_{\mathbf{w}} V$ depends on W locally, however we can further show that if $\boldsymbol{\eta}$ is a smooth curve such that $\boldsymbol{\eta}(0) = p$ and $\boldsymbol{\eta}'(0) = \mathbf{w}$, then the value of $\nabla_{\mathbf{w}} V$ only depends on behavior of V on the curve $\boldsymbol{\eta}$. Thus, $\nabla_{\mathbf{w}} V$ is well-defined even if V is only defined along a curve through p and tangent to $\mathbf{w}$.

Definition 2.7. Let $\boldsymbol{\eta} : I \to \mathcal{M}$ be a smooth curve. A vector field V along $\boldsymbol{\eta}$ is a smooth map $V : I \to T\mathcal{M}$ such that for each $t \in I$, $V(t) \in T_{\boldsymbol{\eta}(t)}\mathcal{M}$.

Remark 2.6 above says that for a vector $\mathbf{v}$, the map $W \mapsto \nabla_{\mathbf{v}} W$ is defined on vector fields W along curves $\boldsymbol{\eta}$ such that $\boldsymbol{\eta}(0) = p$ and $\boldsymbol{\eta}'(0) = \mathbf{w}$.

Definition 2.8. A vector field V along a curve $\boldsymbol{\eta}$ is parallel if $\nabla_{\dot{\boldsymbol{\eta}}(t)} V = 0$ for all t.

Theorem 2.9. *Let $\boldsymbol{\eta} : [0,1] \to \mathcal{M}$ be a smooth curve and let $\mathbf{v} \in T_{\boldsymbol{\eta}(0)}\mathcal{M}$. Then, there exists a unique parallel vector field V along $\boldsymbol{\eta}$ such that $V(0) = \mathbf{v}$.*

Proof. First, we consider the case when $\boldsymbol{\eta}$ is contained in a local coordinate chart. In local coordinates, $V(t) = V^k(t)\partial_k$ satisfies the system of ODEs

$$\frac{dV^k}{dt} + V^i \dot{\eta}^j \Gamma^k_{ij} = 0 \text{ for } k = 1, \dots, n$$

with initial conditions $V^k(0) = \mathbf{v}^k$. Observe that the map

$$(V^k, t) \mapsto -\Gamma^k_{ij}(\boldsymbol{\eta}(t)) V^j \dot{\boldsymbol{\eta}}^j(t)$$

is continuous; for fixed t, the map $(V^k) \mapsto -\Gamma^k_{ij} V^j \dot{\boldsymbol{\eta}}^j$ is linear and thus globally Lipchitz continuous in V^k. Picard-Lindelof theorem tells us that this equation has a unique solution $V(t)$.

For general $\boldsymbol{\eta}$, we let $0 = t_0 < \cdots < t_N = 1$ be a partition of $[0,1]$ such that, for each i, $\boldsymbol{\eta}|_{[t_{i-1},t_i]}$ is contained in a coordinate chart U_i. The previous paragraph implies that there is a parallel vector field V along $\boldsymbol{\eta}|_{[0,t_1]}$ such that $V(0) = \mathbf{v}$. If we have a parallel vector field V along $\boldsymbol{\eta}|_{[0,t_i]}$, we obtain a vector field $\tilde{V}$ along $\boldsymbol{\eta}|_{[t_i,t_{i+1}]}$ such that $\tilde{V}(t_i) = V(t_i)$, again using the previous paragraph. Then, V extends to a vector field along $\boldsymbol{\eta}|_{[0,t_{i+1}]}$ by defining $V(t) = \tilde{V}(t)$ for $t \in [t_i, t_{i+1}]$. By induction, we get a vector field V which is parallel along all of $\boldsymbol{\eta}$. □

3. Parallel Transport

One problem on general manifolds is, unlike Euclidean spaces, if $p, q \in \mathcal{M}$ are distinct points, then there is no natural identification $T_p\mathcal{M} = T_q\mathcal{M}$. In this section, we illustrate how we may construct such an isomorphism, called parallel transportation, using the notion of parallel vector fields. However, this isomorphism will depend on a choice of path between p and q.

Definition 3.1. Let $\boldsymbol{\eta} : [0,1] \to \mathcal{M}$ be a curve connecting two points $p, q \in \mathcal{M}$, i.e. $\boldsymbol{\eta}(0) = p$ and $\boldsymbol{\eta}(1) = q$. Let $\mathbf{v} \in T_p\mathcal{M}$ and let V denote the parallel vector field along $\boldsymbol{\eta}$ ensured by Theorem 2.9. The vector $V(1)$ is called the parallel transport of $\mathbf{v}$ along $\boldsymbol{\eta}$ and the map $\mathbf{v} \mapsto V(1)$ is denoted $T_{\boldsymbol{\eta}(0)\to\boldsymbol{\eta}(1)}$ or $T_{p\to q}$ or by T if the path $\boldsymbol{\eta}$ is clear from the context.

Remark 3.2. Since the equation defining parallel transportation is a linear ODE, for any path $\boldsymbol{\eta}$ from p to q, we have $T_{p\to q}(a\mathbf{v} + \mathbf{w}) = aT_{p\to q}\mathbf{v} + T_{p\to q}\mathbf{w}$. From the definition of parallel transport we see that $T_{p\to q}$ is the inverse of $T_{q\to p}$. Thus $T_{p\to q}$ is an isomorphism from $T_p\mathcal{M}$ to $T_q\mathcal{M}$.

Remark 3.3. Let $\boldsymbol{\eta}$ be a curve and $p = \boldsymbol{\eta}(0)$. If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $T_p\mathcal{M}$, then we can extend $\mathbf{v}_i$ to parallel vector fields $E_i(t)$ along $\boldsymbol{\eta}$. Since parallel transportation is invertible, $\{E_1(t), \dots, E_n(t)\}$ forms a basis of $T_{\boldsymbol{\eta}(t)}\mathcal{M}$ at each t.

We have defined parallel transport in terms of the affine connection. The following theorem shows that we may define the affine connection in terms of parallel transport.

Theorem 3.4. *Suppose we have a smooth vector field V and a smooth curve $\boldsymbol{\eta}$ such that $\dot{\boldsymbol{\eta}}(0) = \mathbf{v}$. Then $\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\big(T_{\boldsymbol{\eta}(\epsilon)\to\boldsymbol{\eta}(0)} V(\epsilon) - V(0)\big) = \nabla_{\dot{\boldsymbol{\eta}}(0)} V$.*

Proof. Let $E_1, \dots, E_n$ be parallel vector fields along $\boldsymbol{\eta}$ forming a basis at each point. Write $V(t) = V^k(t)E_k(t)$. Then, $T_{\boldsymbol{\eta}(\epsilon)\to\boldsymbol{\eta}(0)}V(\epsilon) = V^k(\epsilon)E_k(0)$, since $T_{\boldsymbol{\eta}(\epsilon)\to\boldsymbol{\eta}(0)}$ is a linear map. Then,

$$\frac{1}{\epsilon}(T_{\boldsymbol{\eta}(\epsilon)\to\boldsymbol{\eta}(0)}V(\epsilon) - V(0)) = \frac{1}{\epsilon}(V^k(\epsilon) - V^k(0))E_k(0).$$

Taking the limit $\epsilon \to 0$ gives us $\frac{d}{dt}|_{t=0}(V^k)(t)E_k(0)$. Since E_k is parallel, we also have that

$$\nabla_{\boldsymbol{\eta}'(0)}V(t)|_{t=0} = \boldsymbol{\eta}'(0)(V^k(t))E_k(0) + V^k(0)\nabla_{\boldsymbol{\eta}'(0)}E_k(t) = \boldsymbol{\eta}'(0)(V^k(t))E_k(0).$$

This is, by definition $\frac{d}{dt}|_{t=0}(V^k)(t)E_k(0)$, so the two quantities are the same. □

Remark 3.5. Note that this theorem states that the affine connection is equal to to the limit of a difference quotient. For this reason, for a fixed curve $\boldsymbol{\eta}$, the map $V(t) \mapsto \nabla_{\dot{\boldsymbol{\eta}}(t)}V(t)$, taking a vector field V along $\boldsymbol{\eta}$ to a vector field along $\boldsymbol{\eta}$, is sometimes called the covariant derivative along $\boldsymbol{\eta}$.

Definition 3.6. Let ∇ be an affine connection on $\mathcal{M}$. Given a vector $\mathbf{v}$ and an tensor field T, we define the tensor $\tilde{\nabla}_{\mathbf{v}}T$ of the same rank as T inductively by the following properties

1. if $f : \mathcal{M} \to \mathbb{R}$ is a smooth function, i.e. a $(0,0)$-tensor, then $\tilde{\nabla}_{\mathbf{v}}f = \mathbf{v}(f)$,
2. if W is any vector field, then $\tilde{\nabla}_{\mathbf{v}}W = \nabla_{\mathbf{v}}W$,
3. $\nabla_{\mathbf{v}}T$ is $\mathbb{R}$-linear in $\mathbf{v}$ and T,
4. (Leibniz) $\nabla_{\mathbf{v}}(T \otimes S) = (\nabla_{\mathbf{v}}T) \otimes S + T \otimes (\nabla_{\mathbf{v}}S)$,
5. If $\text{Contract}_a^b T$ denotes contracting T with respect to indices a and b. Then $\nabla_{\mathbf{v}}(\text{Contract}_a^b T) = \text{Contract}_a^b(\nabla_{\mathbf{v}}T)$.

Note that properties (1) and (2) indicate that $\tilde{\nabla}$ extends the connection ∇ to all tensor fields. This definition will be of interest in order to cut down on notation.

Theorem 3.7. *Given coordinates (x^i), we have $\nabla_{\partial_j}dx^i = -\Gamma^i_{kj}dx^k$.*

Proof. Let $\nabla_{\partial_j}dx^i = \Delta^i_{kj}dx^k$, where $\{dx^i|_p\}$ is the basis of $T^*_p\mathcal{M}$ dual to $\{\partial_j|_p\}$. Notice $dx^l(\partial_i) = \delta^l_i$ is a constant function, so

$$0 = \nabla_{\partial_j}[dx^l(\partial_i)] = (\nabla_{\partial_j}dx^l)(\partial_i) + dx^l(\nabla_{\partial_j}\partial_i) = \Gamma^l_{ij} + \Delta^l_{ij}.$$

Thus, $\Delta^l_{ij} = -\Gamma^l_{ij}$. □

4. Geodesics and the Exponential Map

Geodesics are the analogue of straight lines in Euclidean space and possess many of the same properties as straight lines.

Definition 4.1. A smooth curve $\boldsymbol{\eta} : (a,b) \to \mathcal{M}$ is called a geodesic if the tangent vector field $\dot{\boldsymbol{\eta}}$ along $\boldsymbol{\eta}$ is parallel, i.e. if $\nabla_{\dot{\boldsymbol{\eta}}(t)}\dot{\boldsymbol{\eta}}(t) = 0$ for all $t \in (a,b)$.

Theorem 4.2. *Given a point $p \in \mathcal{M}$ and a vector $\mathbf{v} \in T_p\mathcal{M}$, there exists an $\epsilon > 0$ and a geodesic $\boldsymbol{\eta} : (-\epsilon, \epsilon) \to \mathcal{M}$ such that $\boldsymbol{\eta}(0) = p$ and $\boldsymbol{\eta}'(0) = \mathbf{v}$.*

Proof. In terms of local coordinates, the geodesic equation $\nabla_{\dot{\boldsymbol{\eta}}(t)}\dot{\boldsymbol{\eta}}(t) = 0$ becomes

$$\frac{d^2\boldsymbol{\eta}^k}{dt^2} + \Gamma^k_{ij}\frac{d\boldsymbol{\eta}^i}{dt}\frac{d\boldsymbol{\eta}^j}{dt} = 0$$

along with the initial values $\boldsymbol{\eta}(0) = p$ and $\boldsymbol{\eta}'(0) = \mathbf{v}$. The theorem follows by existence and uniqueness of solutions to systems of ODEs. □

Note that the theorem does not ensure uniqueness of geodesics, since two geodesics may not be defined on the same interval around 0. We define a maximal geodesic to be a geodesic $\gamma : I \to \mathcal{M}$, where I is some interval, with the property that for any extension of γ to a new geodesic $\boldsymbol{\eta} : J \to \mathcal{M}$, we have $I = J$ and $\boldsymbol{\eta} = \gamma$.

Definition 4.3. Given $p \in \mathcal{M}$ and $\mathbf{v} \in T_p\mathcal{M}$, there is a unique maximal geodesic $\boldsymbol{\eta}$ passing through p with velocity $\mathbf{v}$. The exponential map at p, $\mathrm{Exp}_p : T_p\mathcal{M} \to \mathcal{M}$, is defined by $\mathbf{v} \mapsto \boldsymbol{\eta}(1)$. Note that this map is only defined on a subset of $T_p\mathcal{M}$.

Theorem 4.4. *There is an open neighborhood U around $0 \in T_p\mathcal{M}$ such that $\mathrm{Exp}_p|_U$ is a diffeomorphism onto its image.*

Remark 4.5. The exponential map is a local diffeomorphism, and so we can parametrize our manifold $\mathcal{M}$ near p by neighborhood of origin in $T_p\mathcal{M}$ as follows. Let U be a neighborhood of p in $\mathcal{M}$ and $\tilde{U}$ a neighborhood of 0 in $T_p\mathcal{M}$ such that $\mathrm{Exp}_p|_{\tilde{U}}$ is a diffeomorphism onto U. If $\{\mathbf{v_1}, \dots, \mathbf{v_2}\}$ is an orthonormal basis for $T_p\mathcal{M}$, then we can define a map taking $q \in U$ to the components of $(\mathrm{Exp}_p|_{\tilde{U}})^{-1}(q)$ in the basis $\{\mathbf{v}_i\}$. This coordinate chart is called the normal coordinate chart at p associated to the basis $\{\mathbf{v}_k\}$. It is a useful computational tool that will be employed frequently.

5. Riemannian Curvature Tensor

Parallel transportation is path-dependent; given two paths $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ with the same endpoints, parallel transportation along $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are, in general, not the same. Riemann curvature measures how much parallel transport depends on the path.

Definition 5.1. Given an affine connection ∇ and two vector fields X, Y, we can define another vector field $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. The map T is called the torsion tensor.

Theorem 5.2. *The torsion tensor is a tensor field.*

Proof. We just have to show that T is C^∞-linear in its arguments.

$$\begin{aligned} T(fX,Y) &= \nabla_{fX} Y - \nabla_Y fX - [fX,Y] \\ &= (f\nabla_X Y) - (f\nabla_Y X + Y(f)X) - (f[X,Y] - Y(f)X) \\ &= f(\nabla_X Y - \nabla_Y X) - f[X,Y] \\ &= fT(X,Y) \end{aligned}$$

□

Definition 5.3. An affine connection ∇ is called torsion-free if $T = 0$.

Remark 5.4. In local coordinates, the torsion-tensor T has an expression given as follows. Let $X = X^i\partial_i$ and $Y = Y^j\partial_j$,

$$\nabla_X Y - \nabla_Y X - [X,Y] = \left((\Gamma^k_{ji} - \Gamma^k_{ij}) X^i Y^j \right) \partial_k.$$

Notice that the expression only involves the components of X and Y and not their derivatives. Thus $T = 0$ is equivalent to $\Gamma^k_{ji} = \Gamma^k_{ij}$, i.e. the Christoffel symbol is symmetric in its lower two indices.

On a Riemannian manifold $(\mathcal{M}, g)$, there is a unique affine connection satisfying two additional properties, torsion-freeness and compatibility with the metric.

Theorem 5.5 (Fundamental Theorem of Riemannian Geometry)**.** *Let $(\mathcal{M}, g)$ be a Riemannian manifold. Then there exists a unique torsion-free affine connection ∇ on $\mathcal{M}$ such that $\nabla_{\mathbf{v}} g = 0$ for all tangent vectors $\mathbf{v}$.*

The second property $\nabla_{\mathbf{v}} g = 0$ is equivalent, by definition, to

$$\mathbf{v}(g(Y,Z)) = g(\nabla_{\mathbf{v}} Y, Z) + g(Y, \nabla_{\mathbf{v}} Z)$$

for any vector fields Y, Z and any tangent vector $\mathbf{v}$. This is also equivalent to requiring for all parallel vector fields $V(t), W(t)$ along a curve $\boldsymbol{\eta}$, their inner product is preserved, i.e. for all t, $g(V(t), W(t)) = g(V(0), W(0))$.

Proof. To prove uniqueness, let ∇ be a torsion-free affine connection such that $\nabla_{\mathbf{v}} g = 0$ for all tangent vectors $\mathbf{v}$. We first compute

$$\begin{aligned} \partial_i g_{jk} &= \partial_i (g(\partial_j, \partial_k)) \\ &= g(\nabla_{\partial_i} \partial_j, \partial_k) + g(\partial_j, \nabla_{\partial_i} \partial_k) \\ &= g_{lk} \Gamma^l_{ji} + g_{jl} \Gamma^l_{ki}. \end{aligned}$$

Notice g_{ij} is symmetric in i and j and Γ^k_{ij} is symmetric in i and j by torsion-freeness. By permuting the indices i, j, k, we obtain the following three equations.

$$g_{kl} \Gamma^l_{ji} + g_{jl} \Gamma^l_{ki} = \partial_i g_{jk}$$

$$g_{jl} \Gamma^l_{ik} + g_{il} \Gamma^l_{jk} = \partial_k g_{ij}$$

$$g_{il} \Gamma^l_{kj} + g_{kl} \Gamma^l_{ij} = \partial_j g_{ik}.$$

If we add the first two equations and subtract the last we find

$$\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = 2 g_{kl} \Gamma^l_{ij}.$$

Therefore

$$\Gamma^l_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right).$$

This shows that ∇ is unique.

For existence, we substitute the coordinate expression for Γ^k_{ij} into the proof of Theorem 2.4. It is then easy to check that the resulting affine connection ∇ is torsion-free and $\nabla_{\mathbf{v}} g = 0$ for all tangent vectors $\mathbf{v}$. □

We thus have the following definitions.

Definition 5.6. The Levi-Civita connection ∇ on a Riemannian manifold $(\mathcal{M}, g)$ is the affine connection satisfying the additional properties:

(1) ∇ is torsion-free,
(2) $\nabla_{\mathbf{w}} g = 0$ for all tangent vectors $\mathbf{w}$.

Definition 5.7. Given vector fields U, V, W, we define the Riemann curvature R by

$$R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W,$$

where ∇ is the Levi-Civita connection.

Theorem 5.8. *The Riemann curvature R is a tensor field.*

Proof. It suffices to show that R is $C^\infty(\mathcal{M})$-linear in each argument. If $f \in C^\infty(\mathcal{M})$, then

$$\begin{aligned}
\nabla_U \nabla_V fW &= U(V(f)) + V(f)\nabla_U W + U(f)\nabla_V W + f\nabla_U\nabla_V W \\
\nabla_V \nabla_U fW &= V(U(f)) + U(f)\nabla_V W + V(f)\nabla_U W + f\nabla_V\nabla_U W \\
\nabla_{[U,V]} fW &= [U,V](f)W + f\nabla_{[U,V]}W.
\end{aligned}$$

Substituting these values into the definition of Riemann curvature gives

$$R(U,V)fW = f\nabla_U\nabla_V W - f\nabla_V\nabla_U W - f\nabla_{[U,V]}W = fR(U,V)W$$

Thus R is $C^\infty(\mathcal{M})$-linear in the third variable W. Similar computations show that R is also $C^\infty(\mathcal{M})$-linear in first two variables. □

Remark 5.9. Since R is a tensor field, $R(U,V)W|_p$ depends only on the values of U, V, and W at the point p. In terms of local coordinates, if $U = U^i\partial_i$, $V = V^j\partial_j$, and $W = W^k\partial_k$, then $R(U,V)W = R^l_{ijk}U^iV^jW^k$. We have

$$\begin{aligned}
R^l_{ijk} &= dx^l\Big(R(\partial_i,\partial_j)\partial_k\Big) \\
&= dx^l\Big(\nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{[\partial_i,\partial_j]}\partial_k\Big) \\
&= \frac{\partial\Gamma^l_{kj}}{\partial x^i} - \frac{\partial\Gamma^l_{ki}}{\partial x^j} + \Gamma^\alpha_{kj}\Gamma^l_{\alpha i} - \Gamma^\alpha_{ki}\Gamma^l_{\alpha j}
\end{aligned}$$

Theorem 5.10. *The Riemann curvature tensor satisfies, for all tangent vectors U, V, W, X, Y based at the same point,*

(1) $R(V,U)=-R(U,V)$
(2) (Bianchi identity) $R(U,V)W+R(V,W)U+R(W,U)V=0$
(3) $g(R(U,V)X,Y)=-g(X,R(U,V)Y)$

Proof. The first equality follows directly from the definition of R. The last follows by substituting $g(X,Y)$ for W in the expression for R:

$$0 = UVg(X,Y) - VUg(X,Y) - [U,V]g(X,Y)$$

and noting that this is zero and applying the Leibniz rule (property 2 in Definition 5.6). This is equal to

$$\begin{aligned}
0 &= Ug(\nabla_V X, Y) + Ug(X,\nabla_V Y) - Vg(\nabla_U X, Y) \\
&\quad - Vg(X,\nabla_U Y) - g(\nabla_{[U,V]}X,Y) - g(X,\nabla_{[U,V]}Y) \\
&= g(R(U,V)X,Y) + g(X,R(U,V)Y).
\end{aligned}$$

The last second identity is checked by direct computation and is omitted. □

The definition of Riemann curvature has been extremely abstract. We now see one of it's very nice geometric meaning. Suppose we have two linearly independent vectors $\mathbf{u}, \mathbf{v} \in T_p\mathcal{M}$. Then we can find local coordinates (x^i) in which $\partial_1 = \mathbf{u}$ and $\partial_2 = \mathbf{v}$. Consider the four vertices $\{(0,0),(s,0),(s,r),(0,r)\}$ of a rectangle in the x^1x^2-plane, illustrated by the following diagram. We denote its edges by A, B, C, D, starting from $(0,0)$ in the counterclockwise direction.

Let $\mathbf{w}$ be a tangent vector at $(0,0)$. We let T denote parallel transportation around the loop $ABCD$. Note that this operation is dependent on s and t.

Theorem 5.11. $\lim_{s,t\to 0}\frac{\mathbf{w}-T\mathbf{w}}{st} = R(\mathbf{u},\mathbf{v})\mathbf{w}$

Proof. By abuse of notation, A, B, C, D will not only denote the paths but also the operation of parallel transportation along these paths, i.e. A will denote parallel transportation along the path A, B will denote parallel transportation along B, etc. Thus, $T = DCBA$.

We may assume that the paths A, B, C, and D are contained in a single coordinate chart. Extend $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to vector fields U, V, W near $(0,0)$. We may assume that U and V are coordinate vector fields: $U = \frac{\partial}{\partial s}$ and $V = \frac{\partial}{\partial r}$ such that $[U, V] = 0$. Notice

$$W - TW = DC(C^{-1}D^{-1}W - BAW)$$

and

$$\begin{aligned} &C^{-1}D^{-1}W - BAW \\ &\quad = C^{-1}(D^{-1}W - W) + (C^{-1}W - W) - B(AW - W) - (BW - W) \end{aligned}$$

We observe

$$\lim_{r\to 0} \frac{C^{-1}(D^{-1}W - W)}{r} = -C^{-1}\nabla_V W$$
$$\lim_{r\to 0} \frac{BW - W}{r} = -\nabla_V W$$

Then

$$\lim_{s,r\to 0} \frac{C^{-1}(D^{-1}W - W) - (BW - W)}{sr} = \lim_{s\to 0} \frac{-C^{-1}\nabla_V W + \nabla_V W}{s} = \nabla_U \nabla_V W$$

We also have

$$\lim_{s\to 0} \frac{C^{-1}W - W}{s} = -\nabla_U W$$
$$\lim_{s\to 0} \frac{B(AW - W)}{s} = -B\nabla_U W$$

Then

$$\lim_{s,r\to 0} \frac{(C^{-1}W - W) - B(AW - W)}{sr} = \lim_{r\to 0} \frac{-\nabla_U W + B\nabla_U W}{r} = -\nabla_V \nabla_U W$$

Therefore if we combine the above expressions we have

$$\begin{aligned} \lim_{s,r\to 0} \frac{W - TW}{sr} &= \lim_{s,r\to 0} DC(\nabla_U \nabla_V W - \nabla_V \nabla_U W) \\ &= \nabla_U \nabla_V W - \nabla_V \nabla_U W \\ &= R(U, V)W \end{aligned}$$

□

Remark 5.12. If ∇ is the Levi-Civita connection of a Riemann manifold, then $R(U, V)$ is a skew-symmetric operator, i.e. for all X and Y,

$$g\big(R(U, V)X, Y\big) = -g\big(X, R(U, V)Y\big).$$

This was proved in Theorem 5.10, (2). However, this can also be seen because parallel transportation T preserves the inner product g, therefore is an element of the orthogonal group $O(T_p\mathcal{M}) \cong O(n)$. The operator $R(U, V)$ is the derivative of this parallel transportation and the tangent space $T_{\mathrm{Id}}O(n)$ of the space orthogonal transformations $O(n)$ is the space of skew-symmetric transformations.

We now give two additional tensor fields which are also called curvature.

Definition 5.13. We define the following:

(1) The Ricci curvature Ric is given by specifying a local orthonormal frame $\{E_i\}_{i=1}^n$ and computing
$$\mathrm{Ric}(X,Y)=\sum_{i=1}^n g(R(X,E_i)Y,E_i).$$
This does not depend on the choice of E_i,
(2) The scalar curvature S is the trace of the Ricci curvature, i.e. for an orthonormal frame E_i, we have $S=\sum_{i=1}^n \mathrm{Ric}(E_i,E_i)$.

Locally, the Ricci curvature is given by the contraction $\mathrm{Ric}_{ij}=R^k_{ikj}$ and scalar curvature is the contraction $S=\mathrm{Ric}^i_i=R^j_{iji}$.

6. Taylor Expansion of the Metric in Normal Coordinates and the Geometric Interpretation of Ricci and Scalar Curvature

In this section, we will see how Ricci curvature measures the deviation of the Riemannian metric from the standard Euclidean metric and how scalar curvature measures the deviation in the volume of a geodesic ball from the volume of a Euclidean ball of the same radius.

Choose an orthonomal basis $\{e_k\}_k$ of $T_p\mathcal{M}$ and a neighborhood U of p, on which the exponential map Exp_p is a diffeomorphism. Let (U,x^k) denote the normal coordinates at p associated to the basis $\{e_k\}$. In this coordinate chart, we have

(1) $g_{ij}|_p=\delta_{ij}$
(2) $\Gamma^k_{ij}|_p=0$
(3) $\frac{\partial g_{ij}}{\partial x^k}\Big|_p=0$

We prove (1) by
$$g_{ij}|_p=\langle\partial_i,\partial_j\rangle=\langle e_i,e_j\rangle=\delta_{ij}$$
We prove (2) by using the fact that for each $v\in T_p\mathcal{M}$, $\boldsymbol{\eta}(t)=\mathrm{Exp}_p(tv)$ is a geodesic and thus satisfies the geodesic equation
$$\frac{d^2\boldsymbol{\eta}^k}{dt^2}+\Gamma^k_{ij}\frac{d\boldsymbol{\eta}^i}{dt}\frac{d\boldsymbol{\eta}^j}{dt}=0$$
Therefore we have
$$0+\Gamma^k_{ij}v^iv^j=0$$
where $v=v^ke_k$. We restrict the equation to the point p and choose an appropriate value of v to get
$$\Gamma^k_{ij}|_p+\Gamma^k_{ji}|_p=0,\ \text{for all } i,j,k.$$
Since the Levi-Civita connection is torsion-free, we conclude
$$\Gamma^k_{ij}|_p=0$$
Equation (3) follows by computing
$$\frac{\partial g_{ij}}{\partial x^k}\bigg|_p=\partial_k\langle e_i,e_j\rangle=\langle\nabla_{\partial_k}e_i,e_j\rangle+\langle e_i,\nabla_{\partial_k}e_j\rangle=\left(\Gamma^l_{ki}g_{lj}+\Gamma^l_{kj}g_{il}\right)|_p$$
Since the Christoffel symbol vanishes at p, we conclude the derivative of g vanishes in normal coordinates.

Theorem 6.1 (Gauss Lemma)**.** *Let (x^i) be normal coordinates around a point p. Then g satisfies*

$$x^k(q) = \sum_j g_{kj}|_q x^j(q),$$

for all points q in the coordinate neighborhood and for all k.

Before we give the proof, we need a short lemma.

Lemma 6.2. *Let q be a point in a normal coordinate chart at p, with coordinates $x^i(q) = x^i$. Then $\sum_i (x^i)^2 = g_{ij}|_q x^i x^j$.*

Proof. The curve $\boldsymbol{\eta}^i(t) = x^i t$ in the normal coordinate is a geodesic on a Riemannian manifold. Therefore,

$$g(\dot{\boldsymbol{\eta}}(0), \dot{\boldsymbol{\eta}}(0)) = g(\dot{\boldsymbol{\eta}}(1), \dot{\boldsymbol{\eta}}(1))$$

which shows that $\delta_{ij} x^i x^j = g_{ij}|_q x^i x^j$. □

Proof of Gauss' Lemma. First observe that if the coordinate system (x^i) is normal then the straight path $\boldsymbol{\eta}^k(t) = (x^k t) = (x^1 t, \dots, x^n t)$ is a geodesic through p. If $\boldsymbol{\eta}(t) = (x^k t)$ is a geodesic, substituting $\boldsymbol{\eta}(t)$ into the geodesic equation we get

$$\Gamma^k_{ij}(\boldsymbol{\eta}(t)) x^i(t) x^j(t) = 0.$$

We substitute in the formula for Γ^k_{ij} and since g^{ij} is invertible, we get

$$\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) x^i x^j = 0.$$

For convenience, we drop the argument showing that the expression on the left is evaluated along $\boldsymbol{\eta}(t)$. We will continue to do this, keeping in mind that all expressions are to be evaluated along this curve. We note that $\frac{\partial g_{ik}}{\partial x^j} x^i x^j = \frac{\partial g_{jk}}{\partial x^i} x^i x^j$, and so

$$\left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\right) x^i x^j = 0. \tag{6.3}$$

We introduce the function

$$\bar{x}^\beta = \sum_\alpha g_{\alpha\beta} x^\alpha \tag{6.4}$$

and we wish to show that $\bar{x}^\beta = x^\beta$ for each β. Differentiating with respect to x^δ gives

$$\frac{\partial \bar{x}^\beta}{\partial x^\delta} = \sum_\alpha \frac{\partial g_{\beta\alpha}}{\partial x^\delta} x^\alpha + g_{\beta\delta}$$

We substitute 6.3 into this equality to obtain, after some minor algebraic manipulation,

$$\begin{aligned} 0 &= \sum_j \left(\frac{\partial \bar{x}^k}{\partial x^j} - g_{kj}\right) x^j - \frac{1}{2}\sum_i \left(\frac{\partial \bar{x}^i}{\partial x^k} - g_{ik}\right) x^i \\ &= \sum_j \frac{\partial \bar{x}^k}{\partial x^j} x^j - \frac{1}{2}\frac{\partial(\sum_i x^i \bar{x}^i)}{\partial x^k} \end{aligned} \tag{6.5}$$

Now by Lemma 6.2 and Equation 6.4 we have

$$\sum_i x^i \bar{x}^i = \sum_{i,j} g_{ij} x^i x^j = \sum_i (x^i)^2$$

and so substituting this into Equation 6.5 gives

$$\sum_j \frac{\partial \bar{x}^k}{\partial x^j} x^j - x^k = \sum_j \frac{\partial(\bar{x}^k - x^k)}{\partial x^j} x^j = 0.$$

Since this holds for all geodesics $\boldsymbol{\eta}(t)$, we get that

$$\frac{\partial(\bar{x}^k - x^k)}{\partial x^j} = 0.$$

At p, we have $\bar{x}^k|_p = x^k|_p$, therefore we have proven that at any point in the normal coordinate we have $x^k = \bar{x}^k = \sum_j g_{kj} x^j$. □

The derivatives of the Gauss Lemma can tell us useful information about the derivatives of the metric. Starting from

$$x^k = \sum_j g_{kj} x^j,$$

differentiate with respect to ∂_i:

$$\delta_{ki} = \frac{\partial g_{kj}}{\partial x^i} x^j + g_{ki}.$$

Evaluating at the origin p we get

$$g_{ki}\Big|_p = \delta_{ki}.$$

Differentiating again with respect to ∂_l,

$$0 = \frac{\partial^2 g_{kj}}{\partial x^i \partial x^l} x^j + \frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^l}.$$

Evaluating at the origin p we get

$$\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} \right)\Bigg|_p = 0.$$

Repeating the process we get

$$\left(\frac{\partial^2 g_{km}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{ki}}{\partial x^l \partial x^m} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} \right)\Bigg|_p = 0.$$

A consequence of this constraint on the second derivatives of g is that

$$\frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^l \partial x^k}\Bigg|_p x^k x^l = -\frac{1}{3} R_{ikjl}\Bigg|_p x^k x^l. \tag{6.6}$$

See Section 3 and 4 of [1] for details of derivation of 6.6.

Therefore, we have the Taylor expansion of the Riemannian metric g in local coordinates as

$$\begin{aligned} g_{ij} &= g_{ij}\Bigg|_p + \frac{\partial g_{ij}}{\partial x^k}\Bigg|_p x^k + \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\Bigg|_p x^k x^l + O(|x|^3) \\ &= \delta_{ij} - \frac{1}{3} R_{ikjl}\Bigg|_p x^k x^l + O(|x|^3) \end{aligned}$$

Definition 6.7. The Riemann volume element is the unique volume element ω on a oriented Riemann manifold such that for any positively oriented orthonomal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in a tangent space we have $\omega(\mathbf{v}_1, \ldots, \mathbf{v}_n) = 1$.

Remark 6.8. In local coordinates (x^i),

$$\omega = \sqrt{\mathrm{Det}(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

We now study the Taylor expansion of the Riemann volume element. Recall the formula

$$\mathrm{Det}(\mathrm{Id} + A) = 1 + \mathrm{Tr}(A) + O(|A|^2),$$

where $|A| = \max\{|a_{ij}|\}$ is the maximum absolute value of all entries of A. We have

$$\mathrm{Det}(g_{ij}) = 1 - \mathrm{Tr}\left(\frac{1}{3} R_{ikjl} x^k x^l\right) + O(|x|^3).$$

Therefore,

$$\begin{aligned}
\mathrm{Det}(g_{ij}) &= 1 - \mathrm{Tr}\left(\frac{1}{3} R_{ikjl}\Big|_p x^k x^l\right) + O(|x|^3) \\
&= 1 - \frac{1}{3}\mathrm{Ric}_{kl}\Big|_p x^k x^l + O(|x|^3) \\
&= 1 - \frac{1}{3}\mathrm{Ric}(x,x)\big|_p + O(|x|^3)
\end{aligned}$$

Where Ric is the Ricci curvature. Because $g_{ij}|_p = \delta_{ij}$, raising and lowering indices does not have an effect on the component of the curvature tensor. Finally, using the Taylor expansion of the square root function,

Theorem 6.9. *In normal coordinates* (x^i) *around* p*, we have the local expression*

$$\begin{aligned}
\omega &= \sqrt{\mathrm{Det}(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n \\
&= \left(1 - \frac{1}{6}\mathrm{Ric}\Big|_p (x,x) + O(|x|^3)\right) dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

Therefore we observe a nice geometric meaning of the Ricci curvature. It tells us how the Riemann volume element ω deviates from the standard Euclidean volume element $\omega_E = dx^1 \wedge \cdots \wedge dx^n$ in the normal coordinates.

Definition 6.10. A geodesic ball $B(p,r)$ is collection of points $\mathbf{x} = (x^i)$ in the normal coordinate chart at p such that $\sum_i (x^i)^2 \le r^2$. (Assuming r is small enough such that the exponential map is injective on the ball)

We investigate the volume of the geodesic ball $B(p,r)$. We let $B_E(0,r)$ denote the standard Euclidean ball of radius r and $V(B)$ the volume of a ball B and by $V(\partial B)$ the volume of its boundary ∂B.

$$\begin{aligned}
V(B(p,r)) &= \int_{B(p,r)} \omega \\
&= \int_{x \in B_E(0,r)} \left(1 - \frac{1}{6}\mathrm{Ric}\Big|_p (x,x) + O(|x|^3)\right) dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

Since Ricci curvature is a symmetric bilinear form, it has orthonormal principal axis $\{v_1, \ldots, v_n\}$ and corresponding eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Let (y^i) be the normal coordinates corresponding to the basis v_i. Therefore,

$$\begin{aligned}
&\int_{x\in B_E(0,r)} \mathrm{Ric}(x,x)\, dx^1\wedge\cdots\wedge dx^n \\
&\qquad = \int_{y\in B_E(0,r)} \Big(\lambda_1(y^1)^2+\cdots+\lambda_n(y^n)^2\Big)\, dy^1\wedge\cdots\wedge dy^n \\
&\qquad = \sum_i \lambda_i \int_{y\in B_E(0,r)} (y^i)^2\, dy^1\wedge\cdots\wedge dy^n \\
&\qquad = \left(\sum_i \lambda_i\right)\cdot \frac{1}{n}\left(\int_{y\in B_E(0,r)} \rho^2\, dy^1\wedge\cdots\wedge dy^n\right),
\end{aligned}$$

where $\rho^2=\sum_i (y^i)^2$ and the sum $\sum_i \lambda_i = S|_p$, the scalar curvature at the point p. The integral can be evaluated using (hyper)spherical coordinates in n-dimensions:

$$\int_{y\in B_E(0,r)} \rho^2 dy^1\wedge\cdots\wedge dy^n = \int_{B_E(0,r)} \rho^2\cdot\rho^{n-1}\, d\rho\wedge d\Omega.$$

Here, $d\Omega$ is the volume form of $\partial B_E(0,1)$. This is evaluated:

$$\begin{aligned}
\int_{B_E(0,r)} \rho^2\cdot\rho^{n-1}\, d\rho\wedge d\Omega &= \left(\int_0^r \rho^{n+1} d\rho\right)\left(\int_{\partial B_E(0,1)} d\Omega\right). \\
&= \frac{r^{n+2}}{n+2}\cdot V(\partial B_E(0,1)) \\
&= \frac{r^{n+2}}{n+2}\cdot \frac{n}{r^n} V(B_E(0,r)).
\end{aligned}$$

Therefore in conclusion,

Theorem 6.11. *Given a point p, we have*

$$\frac{V(B(p,r))}{V(B_E(0,r))} = 1 - \frac{S|_p}{6(n+2)} r^2 + O(r^4).$$

Therefore we see that scalar curvature at a point describes how the volume of geodesic ball centered at that point deviates from the volume of standard euclidean ball with the same radius.

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References

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