THE FOURIER TRANSFORM FOR LOCALLY COMPACT ABELIAN GROUPS

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Abstract. An introduction to locally compact abelian algebraic groups and the usage of dual groups and Fourier transforms to study them.

Contents

1. Introduction

The Fourier Transform is commonly known as the mapping between every function $f : \mathbb{R} \to \mathbb{C}$ to a counterpart denoted $\hat{f} : \mathbb{R} \to \mathbb{C}$, where

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{-\pi ixy}dy$$

Via the Inversion Formula, the transform can be reversed so that well-behaved functions f can be represented as an infinite sum of trigonometric polynomials, the limit of which equals

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(y)e^{-\pi ixy}dy$$

This mapping is a very powerful tool in fields such as chemistry, physics, and computer engineering. For example, complicated sound waves take the form of periodic functions, and the infinite sums that represent them can be approximated

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very well by just a couple of leading Fourier coefficients. Plancharel's theorem is an application of the Fourier transform that is used to analyze particles in quantum physics.

This Fourier mapping and its characteristics do not stem from properties of the real numbers, but instead from certain mathematical spaces. The Fourier Transform can thus be generalized to sets other than the real line, such as the circle, the integers, and in fact any locally compact abelian group. Studying the Fourier transform of LCA groups allows us to explain many of the properties we take for granted about the everyday Fourier transform of real numbers.

2. Preliminaries on Topological Groups

Most of the spaces that we are interested in end up being topological groups. In this section we define the terms topology and group so that we can work with them. In addition, many of the topological spaces we work with are spaces of functions. In order to integrate and otherwise analyze function spaces, we introduce the Haar measure, which is a translation-invariant measure.

A set S becomes a **group** if an operator, say $+$, can be defined such that

- $x + (y + z) = (x + y) + z$ for all $x, y, z \in S$
- There exists an element 0, such that $x + 0 = 0 + x = x$ for all $x \in S$
- For each $x \in S$ there exists an inverse element $x^{-1} = -x$, such that $x + (-x) = (-x) + x = 0$

In addition, S is a commutative group if it is also true that

- $x + y = y + x$ for all $x, y \in S$

Given a set S, a **topology** T is a set of subsets on S that

- Contains S and the empty set $\emptyset$
- Is closed under finite intersections and infinite unions of subsets.

S is a **topological group** if it has a group operation and a topology such that the maps $\alpha : G \times G \to G$ and $\beta : G \to G$ are continuous, where $\alpha(x, y) = x + y$ and $\beta(x) = x^{-1}$.

If S is locally compact, that is, every point in S is contained in a compact neighborhood, and its group operation is commutative, then we call it a **locally compact abelian (LCA) group**.

In order to define the Fourier transform on LCA groups, we must be able to integrate over these groups. This is done with respect to the Haar Measure.

Given a topological space X, we define the Borel set as a set of subsets of X that

- Contains all subsets of the topology on X
- Is closed under complements, countable unions, and countable intersections of subsets
- Is the smallest set of subsets that meets these condition

A **measure** μ on X is a function on the Borel sets where

- $\mu(E) = \sum \mu(E_i)$ if $E \subset X$ and $E = \cup_{i \in I} E_i$, where E_i is a countable pairwise disjoint set

- $\mu(E)$ is finite for all $E \subset X$ where the closure of E is compact.

A measure μ is regular if for all Borel sets E we have $\mu(E) = \inf_{K \supset E} \mu(K) = \sup_{K \subset E} \mu(K)$. μ is invariant if $\mu(x + E) = \mu(E)$ for all $x \in X$.

Let $M(X)$ be the space of all complex-valued regular measures on X where $\|\mu\| = |\mu(S)|$ is finite.

A **Haar measure** is a measure which is nonnegative, regular, and invariant. In fact, Haar measures are unique up to a scalar, so we can call it *the* Haar measure. That is, if m_1, m_2 are both nonnegative, regular, translation invariant measures on S, then there exists $\lambda \geq 0$ such that $m_1 = \lambda m_2$. The corresponding integral is called the Haar integral, which is translation invariant. That is, integrals over a set E and $x + E$ are equivalent.

Given a LCA group G, we define an $L^p(G)$ space to be the space of all complex valued functions f on G such that the integral

$$\int_G |f|^p d\mu$$

exists with respect to the Haar measure. The **convolution** operator $*$ is defined over two functions $f, g \in L^p(G)$ as $f * g(x) = \int_G f(y)g(y^{-1}x)dy$. $L^p(G)$ becomes an algebra under convolution, which is an important characteristic later on.

3. Characters and the Dual Group

The Fourier transform of the real line often contains a function of x, $e^{\pi i x r}$ for some $r \in \mathbb{R}$. This function is actually part of a set of functions, called characters. Each LCA group will have a set of characters that has its properties intimately related to the group itself. We will define characters rigorously in this section to show that this set is a topological group. In later sections we will see that this set is in fact also locally compact and abelian.

Given a LCA group G, a **character** is a continuous group homomorphism from G to the circle group $\mathbb{T}$. That is, for every character $\chi : G \to \mathbb{T}$ and $x, y \in G$, we have

- $|\chi(x)| = 1$
- $\chi(x + y) = \chi(x)\chi(y)$

The **dual group** is the set of all characters on G, denoted $\widehat{G}$, along with the multiplication operation. We denote the inverse of $\chi \in \widehat{G}$, χ^{-1}, as $\overline{\chi}$. It is clear that $\widehat{G}$ is indeed a group as it inherits the group structure from the circle group.

Consider the LCA group $\mathbb{R}$, the real line. $\chi(x) = e^{2\pi i x}$ is a homomorphism from $\mathbb{R}$ to $\mathbb{T}$, so χ is a character of $\mathbb{R}$. In fact, all functions of the form $\chi_r(x) = e^{2\pi i r x}$ for $r \in \mathbb{R}$ are homomorphisms, and are the only continuous homomorphisms. Then $\widehat{\mathbb{R}}$ is $\{\chi_r\}$, which is isomorphic to $\mathbb{R}$!

Another common example is the circle group $\mathbb{T}$ itself. The dual group is $\{\chi_s(x) = x^s, s \in \mathbb{Z}\}$, with χ_s "wrapping" the circle around itself s number of times. This is clearly isomorphic to $\mathbb{Z}$. Interestingly, the the set of all continuous homomorphisms from $\mathbb{Z}$ to $\mathbb{T}$ is $\{\chi_\theta(x) = e^{i\theta x}, 0 \leq \theta \leq 2\pi i\}$, implying that $\widehat{\mathbb{Z}}$ is isomorphic to $\mathbb{T}$.

This illustrates two important facts about dual groups, which will be proved later on:

- **Compact-Discrete Duality**, stating dual groups of compact groups are discrete, and vice versa

- **Pontryagin Duality**, stating there is an isomorphism between G and $\widehat{\widehat{G}}$

We will first show that $\widehat{G}$ is a topological group for any LCA G by endowing it with the **compact-open topology**. Given a space X and the space $C(X)$ of all continuous maps $X \to \mathbb{C}$, consider a compact set $K \subset X$ and an open set $U \subset \mathbb{C}$. Letting $M(K, U)$ denote the set of functions that map K into U, the compact-open topology of $C(X)$ is generated by the set of all $M(K, U)$ as K and U vary over their respective spaces.

As a subset of $C(G)$, $\widehat{G}$ inherits the compact-open topology.

Theorem 3.1. *Under the compact-open topology, the dual group $\widehat{G}$ of a LCA group G is a topological group.*

Proof. To show that it is a topological group, it suffices to show that $\alpha : \widehat{G} \times \widehat{G} \to \widehat{G}$, where $\alpha(\chi_1, \chi_2) = -\chi_1 + \chi_2$, is continuous. Fix $\epsilon > 0$. Since χ_1, χ_2 are continuous, we can find convergent series $\chi_{1,j} \to \chi_1$ and $\chi_{2,j} \to \chi_2$, as well as a $n > 0$ such that for $k > n$ we have

$$|\chi_{1,k}(x) - \chi_1(x)| < \frac{\epsilon}{2} \text{ and } |\chi_{2,k}(x) - \chi_2(x)| < \frac{\epsilon}{2}$$

for $x \in G$. Then we also have

$$|(-\chi_{1,k}(x) + \chi_{2,k}(x)) - (-\chi_1(x) + \chi_2(x))| \leq |\chi_{1,k}(x) - \chi_1(x)| + |\chi_{2,k}(x) - \chi_2(x)| < \epsilon$$

For all $x \in G$, so that $-\chi_{1,j} + \chi_{2,j}$ converges to $-\chi_1 + \chi_2$, and α is continuous. $\square$

4. Structure Spaces and The Gelfand Transform

In this section we give an introduction to Banach algebras and the properties of algebras that contain a unit. This gives us insight as to the structure of the L^1 space of a LCA group, which is itself a Banach algebra. We finish with the definition of the Gelfand Transform, which will be later shown to be a generalized Fourier transform. A **Banach algebra** over a space X is a complex Banach space A, the product mapping $(a, b) \to ab$, and a norm $\|\cdot\|$ such that

- $a(bc) = (ab)c$ and $a(b + c) = ab + ac$ for $a, b, c \in A$
- $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for $a, b \in A, \lambda \in \mathbb{C}$
- A is complete with respect to $\|\cdot\|$
- $\|a \cdot b\| \leq \|a\|\|b\|$

In addition, A is

- Commutative if $ab = ba$ for $a, b \in A$
- Unital if there exists a unit element $1 \in A$ such that $1a = a1 = a$ for $a \in A$

The **structure space** of a commutative Banach algebra A is denoted Δ_A and is the set of all non-zero continuous algebra homomorphisms $m : A \to \mathbb{C}$. This space can be related to A through the Gelfand transform and is important for studying the $L^1(G)$ of LCA groups. In order to explore its properties, we need several properties about Banach algebras.

4.1. **The Spectrum.** If A is a unital Banach algebra and $a \in A$ has $\|a\| < 1$, then

$$(1-a)^{-1} = \frac{1}{1-a} = \sum_{n=0}^{\infty} a^n,$$

So the inverse of $1-a$ always exists in A.

Let $A^\times$ denote the set of invertible elements of A. We can prove that this is a topological group. The open unit ball $B_1(1) \subset A^\times$, so for $x \in A^\times$, $xB_1(1) \in A^\times$ is an open neighborhood of x, so $A^\times$ is open. Furthermore, for $y \in B_1(0)$, the mapping $y \to (1-y)^{-1}$ is continuous, so inversion is continuous on $B_1(1)$ and therefore continuous on $xB_1(1)$. Then inversion is continuous on $A^\times$, so it is a topological group that is open in A.

For $a \in A$, let $Res(a)$ denote the set of elements $\lambda \in \mathbb{C}$ where $\lambda\mathbf{1} - a$ is invertible. Since $\lambda \to \lambda\mathbf{1} - a$ is continuous and $A^\times$ is open, $Res(a)$ is also open. Then the spectrum of a, defined as

$$\sigma_A(a) = \mathbb{C} \setminus Res(a),$$

is closed. Define the spectral radius $r(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$. As it turns out, there is a formula for the radius, $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. Notably, because $\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|a\|^{(n)(\frac{1}{n})}$, we have $r(a) \leq \|a\|$.

For Banach algebras without a unit, we can obtain a unital Banach algebra through the product space $A^e = A \times \mathbb{C}$, the multiplication $(a,c)(b,d) = (ab + bc + ad, ad)$, and the norm $\|(a,b)\| = \|a\| + |b|$. A^e then has unit (0,1). Finally, for A without a unit, we embed A into A^e by $a \to (a,0)$ and define the spectrum of $a \in A$ as $\sigma_A(a) = \sigma_{A^e}(a)$.

4.2. **Banach-Alaoglu.** The **dual space** of a Banach space V is V', the set of all continuous linear maps from V to $\mathbb{C}$. Under the norm $\|\varphi\| = \sup_{v\in V\setminus 0} \frac{|\varphi(v)|}{\|v\|}$, V' is a Banach space.

For each $v \in V$, we can define a function $\delta_v : V' \to \mathbb{C}$ where $\delta_v(\varphi) = \varphi(v)$. The topology induced on V' by the set of mappings δ_v is called the **weak-* topology**. Note that a sequence φ_j converges to φ with respect to the weak-* topology if and only if it converges pointwise, that is, $\varphi_j(v)$ converges to $\varphi(v)$ for all $v \in V$.

We now state the **Banach-Alaoglu** theorem.

Theorem 4.1. *For a complex vector space V, the closed unit ball in V', $\bar{B}' = \{f \in V' : \|f\| \leq 1\}$, is a compact Hausdorff space under the weak-* topology.*

Proof. Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$ be the closed disk centered at the origin with radius r. For $\varphi \in \bar{B_1}'(0)$ and $v \in V$, we have $|\varphi(v)| \leq \|\varphi\|\|v\| \leq \|v\|$, which means

$\varphi(v) \in \mathbb{D}_{\|v\|}$. Mapping each $\varphi \in V'$ to $\prod_{v \in V} \varphi(v)$, we get the injection

$$\bar{B}' \to \prod_{v \in V} \mathbb{D}_{\|v\|}$$

$\mathbb{D}_{\|v\|}$ is compact Hausdorff and so a countable product of it is also compact Hausdorff by Tychonov's Theorem. Since a sequence converges only if it converges pointwise, the weak-* topology coincides with the subspace topology induced by the injection of $\bar{B}'$ into the product, and thus $\bar{B}'$ is Hausdorff. A closed set in a compact Hausdorff space is compact, so it suffices to show $\bar{B}'$ is closed. Since elements of $\bar{B}'$ are linear transformations, any x in the product space lies in $\bar{B}'$ if and only if its coordinates satisfy $x_{v+w} = x_v + x_w$ and $x_{\lambda v} = \lambda x_v$. These conditions define a closed subset of the product, and thus $\bar{B}'$ is compact. □

Recall that the structure space Δ_A of commutative Banach algebra A is the set of all non-zero continuous algebra homomorphisms $m : A \to \mathbb{C}$. We show that $\|m\| \leq 1$ for all $m \in \Delta_A$, and in particular, $\|m\| = 1$ for all m if A is unital.

Suppose A is unital. For every $a \in A$ and $m \in \Delta_A$ we have

$$m(a - m(a) \cdot 1) = m(a) - m(m(a) \cdot 1) = m(a) - m(a)m(1) = 0,$$

so that $a - m(a)$ doesn't have an inverse in A, and $m(a) \in \sigma_A(a)$. From the spectral radius $r(a) \leq \|a\|$, we then have $|m(a)| \leq \|a\| \Rightarrow \|m\| \leq 1$. Since m is a homomorphism we have $m(1) = 1$, and thus $\|m\| = 1$ for all $m \in \Delta_A$.

Suppose A is not unital. Then all $m^e : A^e \to \mathbb{C}$ in Δ_A will have $\|m^e\| = 1$. Restricting these functions back to A by $m = m^e|_A$ will give $\|m\| \leq 1$ for $m \in \Delta_A$.

We can now prove important properties about the structure of Δ_A.

Theorem 4.2. *Δ_A is a locally compact Hausdorff space.*

Proof. We have $\Delta_A \subset \bar{B}' \subset A'$. Let m_j be a sequence in Δ_A that converges pointwise to a function $f \in A'$. Then for $a, b \in A$ we have

$$f(ab) = lim_{n\to\infty} m_n(ab) = lim_{n\to\infty} m_n(a) lim_{n\to\infty} m_n(b) = f(a)f(b).$$

This means that f is an algebra homomorphism. Since Δ_A contains all nonzero algebra homomorphisms, either $f \in \Delta_A$ or $f = 0$. Thus $\overline{\Delta_A} = \Delta_A \cup \{0\}$ We can imbed this closure with $\overline{\Delta_A} \subset \bar{B}'_1(0)$, which is a compact Hausdorff space by Banach-Alaoglu. $\overline{\Delta_A}$ is closed and therefore compact, while Δ_A is locally compact. □

Note that if A is unital, since all algebra homomorphisms $m \in \Delta_A$ will have $m(1) = 1$, any $f \in A'$ with a sequence m_j approaching it will also have $f(1) = 1$, so f will be nonzero. Then $\overline{\Delta_A} = \Delta_A$, and Δ_A is in fact compact.

4.3. **The Gelfand Transform.** For $a \in A$, define the function $\hat{a} : \Delta_A \to \mathbb{C}$ where $\hat{a}(m) = m(a)$. Convergence over the structure space is pointwise, so $\hat{a}$ is continuous. For noncompact Δ_A, $\hat{a}$ vanishes over the closure $\overline{\Delta_A} = \Delta_A \cup \{0\}$. We can then define the **Gelfand transform** $\psi : A \to C_0(\Delta_A)$, the mapping $\psi(a) = \hat{a}$. Since m is an algebra homomorphism, for $a, b \in A$ we have

$$\psi(ab)(m) = (\hat{ab})(m) = m(ab) = m(a)m(b) = \varphi(a)(m)\psi(b)(m).$$

Thus the Gelfand transform is an algebra homomorphism. Moreover, we have

$$|\hat{a}(m)| = |m(a)| \leq \|m\|\|a\| = \|a\|,$$

So the Gelfand transform is also continuous. Note that we use similar notation to denote the dual group $\widehat{G}$ and the Gelfand transform $\hat{g}$ because as we will show later, every Gelfand transform corresponds to a mapping between a character on $\widehat{G}$ and its Fourier transform.

5. C* Algebras

For an algebra A, an **involution** is a map $A \to A$ that maps $a \in A$ to $a*$ and has the properties

- $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for $a, b \in A$
- $(\lambda a)^* = \bar{\lambda}a^*$ for $a \in A$, $\lambda \in \mathbb{C}$

A **Banach-* algebra** is a Banach algebra along with an involution that also satisfies

- $\|a^*\| = \|a\|$

A **C* algebra** is a Banach-* algebra where the involution also satisfies

- $\|a^*a\| = \|a\|^2$

We like working with C* algebras because they are very well-behaved. The classic example is the set of bounded operators on a Hilbert space X, denoted $B(X)$, along with the involution A^* of A equivalent to the adjoint operator, the unique operator such we have the following equivalency between inner products: $\langle Am, n\rangle = \langle m, A^*b\rangle$. More relevant to our study is the space of continuous vanishing functions, a continuous function that approaches 0 at infinity, on a locally compact group G. Denoted $C_0(G)$, this space can have an involution defined making it a C* algebra. We shall prove this in later sections.

Theorem 5.1. *Every C* algebra has a unique C* norm.*

In order to prove this, we need the concept of self-adjoint. If A is a Banach-* algebra, an element $a \in A$ is **self-adjoint** if $a^* = a$.

Proof. If a is self adjoint in a C* algebra A then $\|a^2\| = \|a^*a\| = \|a\|^2 \Rightarrow \|a^{2^n}\| = \|a\|^{2^n}$. Recall the spectral radius is the largest magnitude over all elements in $\sigma_A(a)$ and is fixed at

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|,$$

meaning the norm is unique for self adjoint a. In fact, for arbitrary a we have $\|a\|^2 = \|a^*a\| = r(a^*a)$, making the norm unique for all $a \in A$. □

If A is a C* algebra, we can equip A^e with a norm that makes it a C* algebra too. Let $L : A^e \to B(A)$ be $L_{(a,\lambda)}(b) = ab + \lambda b$, and define the norm on A^e for (a, λ) to be the operator norm on $L_{(a,\lambda)}$. It can be checked that this norm satisfies all the properties of C* algebras, and in fact makes the embedding of A into A^e isometric.This is an important fact that allows us to prove properties about algebras without a unit by first showing them in their unital embedding.

A commutative Banach-* algebra A is **symmetric** if $m(a^*) = \overline{m(a)}$ for all $a \in A$, $m \in \Delta_A$.

Theorem 5.2. *Every commutative C* algebra is symmetric.*

Theorem 5.3. *Note that because the real and imaginary parts of $a \in A$ are self-adjoint, that if $m(a)$ is real for every self-adjoint a and $m \in \Delta_A$, then A is symmetric. Assume A is unital; if it isn't, we can use A^e. Take $m \in \Delta_A$ and a self-adjoint $a \in A$. Let $m(a) = x + iy$ with $x, y \in \mathbb{R}$. If $y = 0$ then we are done.*
Define $a_t = a + it$ for $t \in \mathbb{R}$, so that $m(a_t) = x + i(y + t)$ and

$$x^2 + y^2 + 2yt = x^2 + (y+t)^2 = |m(a_t)|^2 \leq \|a_t\|^2 = \|a_t^* a_t\| = \|a^2 + t^2\| \leq \|a\|^2 + t^2$$

Since $x^2 + y^2 = \|a\|$, we have that $2yt \leq t^2$ for all $t \in \mathbb{R}$, which implies that $y = 0$ and A is symmetric.

This property allows us to relate the norms of A and $\widehat{A}$. For a commutative C* algebra A, the Gelfand map becomes an isometric map:

$$\|\hat{a}\|^2 = \|\bar{\hat{a}}\hat{a}\| = \|\widehat{a^*a}\| = r(a^*a) = \|a^*a\| = \|a\|^2$$

This allows us to prove the **Gelfand-Naimark** theorem, which relates A to continuous vanishing functions on its structure space.

Theorem 5.4. *Given a symmetric commutative Banach-* algebra A, the space of Gelfand maps $\widehat{A}$ is a dense subalgebra of $C_0(\Delta_A)$. If A is also a C* algebra then the Gelfand transform is an isometric *-isomorphism of A to $C_0(\Delta_A)$.*

Proof. By the Stone-Weierstrass Theorem, to prove that $\widehat{A}$ is dense in $C_0(\Delta_A)$ it suffices to prove the following:

- $\widehat{A}$ separates the points in Δ_A
- For all $m \in \Delta_A$ there exists $\hat{a} \in \widehat{A}$ where $f(x) \neq 0$
- $\widehat{A}$ is closed under complex conjugation

Since $\hat{a}(m) = m(a)$, if $m_1, m_2 \in \Delta_A$ such that $\hat{a}(m_1) = \hat{a}(m_2)$ for every $\hat{a} \in \widehat{A}$, then we have $m_1 = m_2$. Then $\widehat{A}$ is separating in Δ_A. The second condition is trivially satisfied because by definition, all $m \in \Delta_A$ are nonzero. Since A is symmetric we also have that $\bar{\hat{a}} = \hat{a^*}$ for all $\hat{a} \in \widehat{A}$, so $\widehat{A}$ is also closed under conjugation. Then by Stone-Weierstrass, $\widehat{A}$ is dense in $C_0(\Delta_A)$ when A is a symmetric commutative Banach-* algebra.

If furthermore A is a C* algebra, then we have A isometric to $\widehat{A}$, so $\widehat{A}$ is not only dense, but closed in $C_0(\Delta_A)$, so we have $\widehat{A} = C_0(\Delta_A)$. Then the Gelfand transform mapping A to $\widehat{A}$ is an isometric *-isomorphism between A and $C_0(\Delta_A)$. □

This then implies that $\widehat{A}$ is isometric to A, so for all $a \in A$ and $m \in \Delta_A$ we have $\|a\| = \|\hat{a}\|_{\Delta A}$.

6. The Fourier Transform

We can now define the Fourier Transform and relate it to the structure space. For a LCA group G, every function f defines a function, its **Fourier transform**, as $\hat{f} : \widehat{G} \to \mathbb{C}$ where

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}dx$$

We first want to identify the Fourier transform with the Gelfand transform from $L^1(G)$ to $\Delta_{L^1(G)}$. Consider the map $\psi : \widehat{G} \to \Delta_{L^1(G)}$ where $\psi(\chi)(f) = \hat{f}(\chi)$. We will show that ψ is a homeomorphism, so that each function on the structure space can determines a unique character on $\widehat{G}$, and so the two transforms are identical.

Theorem 6.1. *ψ is a homeomorphism between $\widehat{G}$ and $\Delta_{L^1(G)}$.*

Proof. ψ is injective:

$$\psi(\chi_1) = \psi(\chi_2) \Rightarrow \int_G f(x)\overline{(\chi_1(x) - \chi_2(x))}dx = 0 \text{ for all } f \in C_c(G) \Rightarrow \chi_1 = \chi_2$$

ψ is surjective: For any $m \in \Delta_{L^1(G)}$ there exists $g \in C_c(G)$ such that $m(g) \neq 0$ since m is nonzero. It can be shown that $|\overline{m(x^{-1}g)}| = |\overline{m(g)}|$ for all $x \in G$. We can then construct a continuous function $\chi : G \to \mathbb{T}$ such that $\psi(\chi) = m$.

$$\text{Let } \chi(x) = \frac{\overline{m(x^{-1}g)}}{\overline{m(g)}}.$$

Then $|\chi| = 1$ so $\chi \in \widehat{G}$. Furthermore,

$$\begin{aligned}
\psi(\chi)(f) &= \int_G f(x)\overline{\chi(x)}dx \\
&= \frac{1}{m(g)} m \int_G f(x)x^{-1}gdx \\
&= \frac{1}{m(g)} m(f * g) \\
&= \frac{m(f)m(g)}{m(g)} = m(f)
\end{aligned}$$

So for each m there exists χ such that $\psi(\chi) = m$, and thus ψ is surjective. □

Since $\widehat{G}$ is now homeomorphic to $\Delta_{L^1(G)}$, we have the following corollary.

Corollary 6.2. *$\widehat{G}$ is a locally compact abelian group.*

With this homeomorphism, we now also have that the Fourier transform is continuous over $\widehat{G}$ and vanishing at infinity.

We can now present a rigorous proof of the duality between compact and discrete groups.

Theorem 6.3. *The dual group of a compact group is discrete, and the dual group of a discrete group is compact.*

Proof. If G is discrete then define the function $f : G \to \mathbb{C}$

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

$f(x)$ is a continuous function because G is discrete. Furthermore, for any function $g \in L^1(G)$ we have $g * f = g$, so that f is a unit of $L^1(G)$, and $L^1(G)$ is unital. Then $\Delta_{L^1(G)}$ is equal to its closure and therefore compact. $\widehat{G}$ is isomorphic to $\Delta_{L^1(G)}$ and thus also compact.

If G is compact then let P be the set of all $\varphi \in \widehat{G}$ such that $\varphi(G) \subset \{Re(\cdot) > 0\}$. P must be an open unit neighborhood of $\widehat{G}$ since G is compact and all φ are continuous. However, for $\varphi \in \widehat{G}$, $\varphi(G)$ is a subgroup of $\mathbb{T}$, and the only $x \in \mathbb{T}$ such that both x and x^{-1} are in $\{Re(\cdot)\}$ is $x = 1$, so the only group in $\{Re(\cdot)\}$ is $\{1\}$. Then P is discrete in $\widehat{G}$ and so $\widehat{G}$ is discrete. $\square$

We wish to use the Fourier transform to cast a C* algebra onto $L^1(G)$. For a LCA group G, $L^1(G)$ is a Banach-* algebra under the involution $f^*(x) = \overline{f(x^{-1})}$. It is not a C* algebra, however. To achieve this, we would like to embed it into a space of bounded operators, which by definition is a C* algebra. Fix $f \in L^1(G)$, $\varphi, \phi \in L^2(G)$. For every $y \in G$ we have $|\langle y^{-1}\varphi, \phi\rangle|$ is bounded by

$$|\langle y^{-1}\varphi, \phi\rangle| \leq \|y^{-1}\varphi\|\|\phi\| = \|\varphi\|\|\phi\|,$$

so that the integral $\int_G f(y)\langle y^{-1}\varphi, \phi\rangle dy$ is bounded and therefore continuous by

$$\int_G f(y)\langle y^{-1}\varphi, \phi\rangle dy \leq \|f\|\|\varphi\|\|\phi\|$$

Then we can find a unique element determined by f and φ, call it $L(f)\varphi \in L^2(G)$, such that $\langle L(f)\varphi, \phi\rangle = \int_G f(y)\langle y^{-1}\varphi, \phi\rangle dy$. Then we have

$$\begin{aligned} \langle L(f)\varphi, \phi\rangle &\leq \|f\|\|\varphi\|\|\phi\| \\ \Rightarrow \|L(f)\varphi\|^2 &\leq \|f\|\|\varphi\|\|L(f)\varphi\| \\ \Rightarrow \|L(f)\varphi\| &\leq \|f\|\|\varphi\| \end{aligned}$$

So that the map from φ to $L(f)\varphi$ is bounded and therefore continuous. If $\varphi \in C_c(G)$ we can use Fubini's Theorem to show that $L(f)\varphi = f * \varphi$. We can imbed $L^1(G)$ into the bounded mappings on $L^2(G)$, denoted $B(L^2(G))$, with the mapping $f \to L(f)\varphi$. Since $C_c(G)$ is dense in $L^2(G)$, it can be shown that $f \to f * \varphi$ is an injective homomorphism of Banach-* algebras, and therefore so is the mapping from $L^1(G)$ into $B(L^2(G))$.

We now obtain a commutative C*-algebra on $L^1(G)$ by letting it inherit the C*-algebra on $B(L^2(G))$. It is commutative because G is abelian and $L^1(G)$ is commutative. We refer to this algebra as $C^*(G)$.

Consider the mapping $L^* : \Delta_{C^*(G)} \to \Delta_{L^1(G)}$ where $L^*(m) = m \circ L$. It can be shown that this map is a homeomorphism. Remembering that $\widehat{G}$ is also homeomorphic to $\Delta_{L^1(G)}$, we thus have a homeomorphism between $\Delta_{C^*(G)}$ and $\widehat{G}$. By Gelfand-Naimark, we have that the Fourier transform on the Banach-* algebra $L^1(G)$ is dense in $C_0(\widehat{G})$. By extending to our constructed C* algebra, we have that the Fourier transform is an isometric *-isomorphism between $C^*(G)$ and $C_0(\widehat{G})$. This extension also shows that the Fourier transform, mapping f to $\hat{f}$, is injective from $L^1(G) \to C_0(\widehat{G})$.

We now wish to work with the space $C_0^*(G)$, obtained as an extension of $L^1(G) \cap C_0(G)$. Define the norm of a function on this space as

$$\|f\|_0^* = \max(\|f\|_G, \|\hat{f}\|_{\hat{G}}).$$

We can embed $L^1(G) \cap C_0(G)$ into both $C_0(G)$ and $C^*(G)$ via the identity. Call these maps i_0 and i_*, respectively. For $f \in L^1(G) \cap C_0(G)$ we have $\|i_0(f)\|_G \leq \|f\|_0^*$ and $\|i_*(f)\|_G \leq \|f\|_0^*$, and so we can continuously extend $\|i_0(f)\|_G \leq \|f\|_0^*$ to $C_0^*(G)$. Remembering that $C^*(G)$ and $C_0(\widehat{G})$ are isomorphic, we can now treat

$f \in C_0^*(G)$ as an element of both $C_0(G)$ and $C_0(\widehat{G})$.

Fix a function $f \in L^1(G) \cap C_0(G)$ and define $K_n \subset A$ to be a compact set with $|f| < \frac{1}{n}$ outside K_n. Choose a $\chi_n \in C_c(G)$ such that $0 \leq \chi_n \leq 1$ and $\chi_n(x) = 1$ for $x \in K_n$. Then the sequence $f_n = \chi_n f$ converges to f, so that $C_c(G)$ is dense in $L^1(G) \cap C_0(G)$. Then we also have $C_c(G)$ dense in $C_0^*(G)$ since $L^1(G) \cap C_0(G)$ is. This is an important fact in proving the Inversion Theorem in the next section.

7. Pontryagin Duality

Pontryagin duality was mentioned earlier as the existence of an isomorphism between every LCA group G and its double dual $\widehat{\widehat{G}}$, which is the space of characters on $\widehat{G}$. This is a consequence of the **Pontryagin map** $\delta : G \to \widehat{\widehat{G}}$ where $\delta_x(\chi) = \chi(x)$. It is easy to see that each δ_x is a group homomorphism and thus a character, since for any $\chi_1, \chi_1 \in \widehat{G}$ we have

$$\delta_x(\chi_1\chi_2) = (\chi_1 \circ \chi_2)(x) = \chi_1(x)\chi_2(x) = \delta_x(\chi_1)\delta_x(\chi_2).$$

Theorem 7.1. *The Pontryagin map δ is an isomorphism.*

Proof. It suffices to show that δ is an injective group homomorphism and that the image of $\delta(G)$ is dense and closed in $\widehat{\widehat{G}}$ For $x, y \in G$ we have

$$\delta_{xy}(\chi) = \chi(xy) = \chi(x)\chi(y) = \delta_x(\chi)\delta_y(\chi),$$

So δ is a group homomorphism. To show that δ is injective it suffices to show that if for some $x \in G$ we have $\chi(x) = 1$ for all $\chi \in \widehat{G}$, then we must have $x = 1$, since 1 is the multiplicative identity. Assume that there exists $x_0 \in G$ such that $x_0 \neq 1$ and $\chi(x_0) = 1$ for all $\chi \in \widehat{G}$. There exists $g \in C_c(G)$ such that $g(1) = 1$ and $g(x_0^{-1}) = 0$. Then the function $f(x) = g(x_0^{-1}x)$ is not equivalent to $g(x)$, but we can show that for all $\chi \in \widehat{G}$,

$$\hat{f}(x) = \bar{\chi}(x_0)\hat{g}(\chi) = 1 \cdot \hat{g}(\chi) = \hat{g}(\chi).$$

So that the Fourier transforms $\hat{g}$ and $\hat{f}$ are equivalent even though g and f are not, contradicting the injectivity of Fourier transforms. Then no such x_0 exists, and δ is injective.
To show that $\delta(G)$ is dense, we need the following two lemmas:

Lemma 7.2. *Given a function $\varphi \in C_c(\widehat{G})$, there exists a sequence of functions $f_j \in C_0^*(G)$ such that their Fourier transforms $\hat{f}_j$ converge to φ and are supported in the support of φ.*

Lemma 7.3. *Given a function $f \in C_0^*(G)$ that has its Fourier transform in $C_c(\widehat{G})$, for all $x \in G$ we have $f(x) = \hat{\hat{f}}(\delta_{x^{-1}})$. This looks remarkably similar to the Inversion Theorem, which we will prove later about a general function $f \in L^1(G)$.*

Now, $\delta(G)$ is dense in $\widehat{\widehat{G}}$ unless there exists an open subset $U \subset \widehat{\widehat{G}}$ that is disjoint from $\delta(G)$. Assume that such a U exists. Then Lemma **Lemma 7.2** gives us a nonzero function $\psi \in C_0^*(\widehat{G})$ such that $\hat{\psi}$ is supported in U, as well as a sequence f_n in $C_0^*(G)$ such that their Fourier transforms $\hat{f}_n$ converges to ψ. **Lemma 7.3** shows that $f_n(x) = \hat{\hat{f}}_n(\delta_{x^{-1}}) = \hat{\psi}_n(\delta_{x^{-1}})$. But ψ is supported only on U, which is disjoint from the image $\delta(G)$, and so f_n converges to zero for all x, so that $\psi = 0$. But we constructed ψ to be nonzero, and so we have a contradiction, and so the image of δ is dense in $\widehat{\widehat{G}}$.

To show that $\delta(G)$ is closed it suffices to show that δ is proper, which means that the inverse of every compact set in $\widehat{\widehat{G}}$ is compact. Since G is closed under inversion, it suffices to show that $\gamma(x) = \delta(x^{-1})$ is proper. Take an arbitrary compact set $K \subset \widehat{\widehat{G}}$. From **Lemma 7.3** we again can find a $\psi \in C_0^*(\widehat{G})$ such that $\hat{\psi}$ is compactly supported, nonnegative on $\widehat{\widehat{G}}$, and larger than or equal to 1 on K. We also have a sequence f_n in $C_0^*(G)$ such that $\hat{f}_n \leq 0$, exists in $C_c(\widehat{G})$, and converges to ψ. We can find a j large enough such that $\|\hat{\hat{f}}_n - \hat{\psi}\| < 1/2$. From **Lemma 7.3** we again have $f_n(x) = \hat{\psi}_n(\delta_{x^{-1}})$ so that we can find a compact set $C \subset G$ where $|f_j| < 1/2$ outside C. But $\hat{\psi}_j \leq 1/2$ on K, so $\gamma^{-1}(K)$ lies in C. Inversion maps closed sets to closed sets, so $\gamma^{-1}(K)$ is closed and bounded and thus compact. Then γ is proper and $\delta(G)$ is proper.

We have thus proved that δ is an injective homomorphism and its image is closed and dense in $\widehat{\widehat{G}}$, and so δ is an isomorphism giving us $G \cong \widehat{\widehat{G}}$. $\square$

We can now prove the **Inversion Formula**.

Theorem 7.4. *$f(x) = \hat{\hat{f}}(\delta_{x^{-1}})$ for all $x \in G$ and $f \in L^1(G)$ that have their Fourier transform $\hat{f} \in L^1(\widehat{G})$.*

Proof. Consider the mapping of the Fourier transform $F : C_0^*(G) \to C_0^*(\widehat{G})$ that takes $f \to \hat{f}$. We show that this is an isometric isomorphism between the spaces. Define the inverse map to be $\widehat{F} : C_0^*(\widehat{G}) \to C_0^*(G)$ where $\widehat{F}(\psi)(x) = \hat{\psi}(\delta_{x^{-1}})$. Let B be the space of $f \in C_0^*(G)$ such that $\hat{f} \in C_0^*(\widehat{G})$. From **Lemma 7.3** in the above proof, we have that $\widehat{F} \circ F(f) = f$. Since $\|f\|_G = \|\hat{\hat{f}}\|_{\widehat{\widehat{G}}}$, we have by our definition of norm on $C_0^*(\widehat{G})$ that $\|f\|_0^* = \|F(f)\|_0^*$. But $C_c(\widehat{G}) = F(G)$ is dense in $C_0^*(\widehat{G})$ as proven in the last section, so that F is a surjective isometry from the closure $\overline{B}$ to $C_0^*(\widehat{G})$. Similarly, the Fourier transform on $C_0^*(\widehat{G})$ is dense in $C_0^*(\widehat{\widehat{G}})$, so that by applying the inverse transform and Pontryagin duality, we have that $\widehat{F}(C_0^*(\widehat{G}))$ is dense in $C_0^*(G)$. Then F is indeed an isometric isomorphism.

For f that meet the criteria of the formula, $\hat{f}$ is vanishing and thus $\hat{f} \in C_0^*(G)$. From above, we can find a $g \in C_0^*(\widehat{G})$ such that $\hat{g} = \hat{f}$ and $g(x) = \hat{f}(\delta_{x^{-1}})$ for all $x \in G$. But the Fourier transform is injective, so that $f(x) = g(x) = \hat{\hat{f}}(\delta_{x^{-1}})$. $\square$

8. Plancharel's Theorem

Not only is the Fourier transform an isomorphism between L^1 spaces, but it can be extended further. We can now prove **Plancharel's Theorem**, which states that there is a surjective isometry and thus a unitary equivalence between the spaces

$L^2(G)$ and $L^2(\widehat{G})$.

Theorem 8.1. *For a locally compact abelian group G, $L^2(G) \cong L^2(\widehat{G})$.*

Proof. For a function $f \in L^1(G) \cap L^2(G)$ we have that $f * f^* \in C_0^*(G)$. Using properties of C* algebras and the inversion formula, it can be shown that $\|f\|_2^* = \|(\hat{f})\|_2^2$. The Fourier transform from $L^1(G) \cap L^2(G)$ into $L^2(\widehat{G})$ can then be extended to an isometric linear map from $L^2(G)$ to $L^2(\widehat{G})$. Furthermore, the space $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$ so that the image of $L^2(G)$ is dense in $L^2(\widehat{G})$ and therefore surjective. The Fourier transform thus gives us a unitary equivalence between $L^2(G)$ and $L^2(\widehat{G})$. □

9. Examples

We previously gave the dual groups of three common LCA groups. We now prove that these are indeed the duals.
The additive group of reals $\mathbb{R}$ is its own dual: clearly every function $\chi_r(x) = e^{\pi i r x}$, $r \in \mathbb{R}$ is a character of $\mathbb{R}$. We show that every character of $\mathbb{R}$ lies in the set $\{\chi_r\}$. Take an arbitrary $\chi \in \mathbb{R}$. There exists a $\delta > 0$ and $\alpha \neq 0$ such that

$$\int_0^\delta \chi(t)dt = \alpha$$

Since χ is a homomorphism, we have

$$\chi(x) \cdot \alpha = \chi(x) \int_0^\delta \chi(t)dt = \int_0^\delta \chi(x)\chi(t)dt = \int_0^\delta \chi(x+t)dt = \int_x^{x+\delta} \chi(t)dt$$

This is clearly differentiable, and χ is continuous, so χ must be continuously differentiable. Then we have for all x that

$$\begin{aligned} \chi(x+t) &= \chi(x)\chi(t). \text{ Differentiating with respect to } t \text{ gives us} \\ \chi'(x+t) &= \chi(x)\chi'(t). \text{ Letting } t = 0, \\ \chi'(x) &= \chi(x)\chi'(0) \end{aligned}$$

So that the derivative of of $\chi(x)$ is $\chi(x)$ times a constant. Since units are mapped to units we have $\chi(0) = 1$. We also have that $|\chi(x)| = 1$ for all x. Thus χ must be of the form $e^{\pi i r x}$ for some $r \in \mathbb{R}$. □

The additive group of the torus $\mathbb{T}$ has the same restrictions on its dual that we proved on the reals, except we also have $\chi(x + 2\pi) = \chi(x)$ for all x. Then unique characters only exist for each integer, so that the dual group of $\mathbb{T}$ is $\mathbb{Z}$. Then from the Pontryagin duality that we proved of LCA groups, the dual group of $\mathbb{Z}$ must be $\mathbb{T}$. □

10. Conclusion

We can now see that the Fourier transform does not operate arbitrarily, but obtains even its most basic properties from the structure of LCA groups and their duals. For example, the common Fourier transform

$$\int_R f(x)e^{-2\pi i x}$$

just so happens to map functions on $\mathbb{R}$ to another function on $\mathbb{R}$ because $\mathbb{R}$ is its own dual group. The inversion formula

$$f(x) = \hat{\hat{f}}(\delta_{x^{-1}})$$

is what allows us to represent any $L^1(\mathbb{R})$ function as a Fourier transform on the reals, again utilizing the duality between $\mathbb{R}$ and itself. Of course, the real line is only one of the applications of the Fourier transform. The extension to Plancharel's theorem in particular gives us countless properties that extend far beyond the scope of this paper. For example, duality between the circle $\mathbb{T}$ and the countable space $\mathbb{Z}$ means $L^2(\mathbb{T})$ has a countable basis. We can also extend the inversion formula to many other functions. Take any $f, g \in L^1(G) \cap L^2(G)$, and we can apply the inversion formula to $h = f * g$ because $\hat{h} \in L^1(\widehat{G})$ and $h \in L^1(G)$.

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