THE WORD AND CONJUGACY PROBLEMS

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ABSTRACT. This paper considers two questions central to geometric group theory, the word and conjugacy problems. We use Dehn functions to derive a sufficient and necessary condition to solve the word problem, and find examples of groups with solvable conjugacy problems.

CONTENTS

1. INTRODUCTION

A little over 100 years ago, Max Dehn posed three questions, the word problem, the conjugacy problem and the isomorphism problem. The first two questions are the main focus of this paper. The word problem asks for a group G given two words w, w', is $w = w'$? Equivalently, given an arbitrary word w, is it the identity element of G? The conjugacy problem asks that given two words $w, w' \in G$, are w and w' conjugate, i.e. is there a word $z \in G$, such that $zwz^{-1} = w'$? It may be clear now that a group with a solvable conjugacy problem will have a solvable word problem but the converse is not necessarily true. The paper is laid out as follows. The introduction establishes many well known group theoretic propositions and definitions that will be used later in the paper. Section 2 will cover Dehn functions, the most important tool in looking at the two decision problems. Section 3 gives a solution to the word problem and Section 4 will provide a partial solution in the case of a hyperbolic group to the conjugacy problem as it is still an open question.

Throughout this paper, we will denote group presentations as $P = \langle A \,|\, R \rangle$ where A is the set of generators and R the relators. The free group on A will be denoted $F(A)$. We will write $=_G$ and $=_F$ for equality in a group G and in the free group $F(A)$ respectively, if there is any confusion about the equality. Unless specified, words will be freely reduced, i.e. redundancies of the type xx^{-1} will be eliminated. We will use 1 to denote the identity element of an arbitrary group. We will denote $xyx^{-1}y^{-1}$ as $[x, y]$.

Given a group G with a presentation $P = \langle A \,|\, R \rangle$, and a word $w = a_1^{\pm 1} a_2^{\pm 1} ... a_n^{\pm 1}$, where $a_i \in A$. Define the *length* of the word, $l(w)$ to be the number n above. This leads us to the definition of the word metric.

Definition 1.1. If G is presented as above, then the *word metric* with respect to the generatating set A is $d_A(w_1, w_2) = l(w_1 w_2^{-1})$.

Remark. Note that this metric corresponds to the distance between two points in the Cayley graph of G with respect to A, represented by $\Gamma(G, A)$, with the usual graph metric.

The proof that this is a metric can be verified in Howie [8]. At first glance, it seems that this definition is dependent on our choice of generators. The following definition will help to clarify that ambiguity.

Definition 1.2. We say the function $f : X \to X'$ is a *quasi-isometry* if there exists $\lambda, \epsilon \in \mathbb{R}$ such that

$$\frac{1}{\lambda} d(x,y) - \epsilon \leq d(f(x), f(y) \leq \lambda d(x,y) + \epsilon.$$

Two metric spaces (X, d) and (X', d') are *quasi-isometric* if there exists f, g such that $f : X \to X'$ and $g : X' \to X$ are both quasi-isometries.

Example 1.3. $\mathbb{Z}$ and $\mathbb{R}$ are quasi-isometric to each other. Note that the embedding of $\mathbb{Z} \to \mathbb{R}$ is a $(1,0)$ quasi-isometry. Now, the round to the nearest integer function $\mathbb{R} \to \mathbb{Z}$ is a $(1, \frac{1}{2})$ quasi-isometry.

In fact, quasi-isometries induce an equivalence relation between metric spaces. For our purposes, the important consequence of this is the following proposition:

Proposition 1.4 (Howie [8])**.** *If A, B are two finite generating sets for a group G, then when they are equipped with the word metric, (G, A) and (G, B) are quasi-isometric metric spaces.*

Proof. Set $\epsilon = 0$ and let λ be defined as follows:

$$\lambda = \max\{d_B(a, 1) | a \in A\}.$$

We claim $id : (G, B) \to (G, A)$ is a quasi-isometry, where id represents the identity function. Let $g, h \in G$. If $d_A(g, h) = k$ then

$$d_A(g,h) = l_A(g^{-1}h) \leq \lambda k \leq \lambda d_B(g,h).$$

A similar proof shows that $d_A(g,h) \geq \frac{k}{\lambda}$. Hence f is a quasi-isometry. We can establish the converse by switching the roles of A, B and finding a $\lambda' = \max\{d_A(b, 1) \,|\, b \in B\}$, which completes the proof. □

Remark. Prop 1.4 allows us to discuss the word metric without specifying a generating set.

Next, we talk about a certain kind of metric space that will appear often in this paper.

Definition 1.5. Let (X,d) be a metric space and $x, y \in X$ with $d(x,y) = l$. A *geodesic segment* is the image of an isometric mapping $f : [0, l] \to X$ such that $f(0) = x$ and $f(1) = y$.

Informally, we can think of a geodesic segment as the segment of smallest possible length between x and y, as we can see that $d(f(x), f(y)) = l$. Continuing from the previous definition, we can now define a geodesic triangle in an arbitrary metric space.

Definition 1.6. Assume f, X, x, y are defined as above. A segment is called k *local geodesic* if $d(f(a), f(b)) = |a - b|$ for all $a, b \in [0, l]$ and $|a - b| \leq k$.

Basically all segments in the same closed $k-$ball are geodesic.

Definition 1.7. Given three points x, y, z, there is a *geodesic triangle* xyz if there are geodesic segments, xy, yz, xz.

The following definitions extend the notion of a geodesic segment.

Definition 1.8. A segment is *quasi-geodesic* if there are (λ, ϵ) such that $a, b \in [0, l]$ implies

$$\frac{1}{\lambda}|a - b| - \epsilon \leq d(f(a), f(b)) \leq \lambda|a - b| + \epsilon.$$

We now turn to the notion of a hyperbolic group, a particular case which we will consider in the word and conjugacy problems.

Definition 1.9. A triangle, given by vertices xyz is δ thin if any point on the edge xy is in a δ neighborhood of the edges $yz \cup xz$. We say that a group is $\delta-$*hyperbolic* if every geodesic triangle in its Cayley graph is δ thin.

Note that δ is a global constant, that is one value of δ must work for every triangle. In addition, if two groups, G and H are quasi-isometric to each other and G is hyperbolic, then it follows H is also hyperbolic. From Proposition 2.1, it follows that our choice of presentation does not matter in deciding whether or not a group G is hyperbolic. Now we give a few examples of hyperbolic groups.

Example 1.10. Any finite group is trivially hyperbolic as we can choose δ to be larger than the diameter of the Cayley graph. Also, every free group is $\delta-$hyperbolic as every triangle is degenerate, i.e. triangles must have two edges coincide. Note that an easy example of a non-hyperbolic group is $\mathbb{Z}\times\mathbb{Z}$. From example 1.3 we know that $\mathbb{Z}\times\mathbb{Z}$ is quasi-isometric to the Euclidean plane, and hence not hyperbolic.

We conclude this section with a theorem on hyperbolic groups:

Theorem 1.11 (Howie [8])**.** *If a group G is $\delta-$hyperbolic, then it can be finitely presented.*

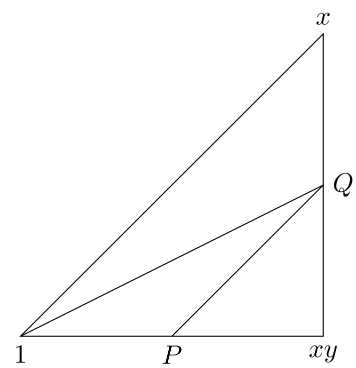
Proof. Choose a finite generating set for G, call it S. Let $d_S = d$ be the word metric with respect to this generating set. Let $X_n = \{d(g,1) \leq n \,|\, g \in G\}$. Choose $R_n = \{xyz \,|\, x, y, z \in X_n, xyz =_G 1\}$. If $G_n = \langle X_n \,|\, R_n \rangle$, we claim that for N large enough, the correspondence $\phi : G_N \to G_{N+1}$ is an isomorphism, where ϕ is the embedding of G_N in G_{N+1}. Choose N such that $N > 2\delta$. Note first that this group homomorphism is surjective. Indeed, choose an element $g \in X_{N+1} \backslash X_N$. Then, in the Caley graph, g is connected to a vertex in X_N by a generator in X_N. We can find $u, v \in X_N$ such that

$$uvg =_{G_{N+1}} 1.$$

Hence, $\phi(G_N)$ contains all the generators of G_{N+1}, and ϕ is surjective.

Next, we show injectivity by proving that $\ker(\phi) = 1$. In order to do this, we note that if we have $xyz \in R_{N+1}$, then we can get this relation from those in R_N. For all $x \in X_{N+1} \backslash X_N$, add x to X_N and the relation $x_1 x_2 x^{-1}$ to R_N. We can find x_1, x_2 with the following properties. Let $l(x_1) > \delta, l(x_2) > \delta$ and $l(x) = l(x_1) + l(x_2)$.

We break up the rest of the proof into three cases. First, assume that $z \notin X_N$, and $x, y \in X_N$. Split z into z_1, z_2 as above, and choose P to be the point where the splitting occurs. Note that this is within a δ neighborhood of a point Q which is on the triangle $1, x, xy$. The line PQ, and the line adjoining Q and the opposite vertex, divide the original triangle into 3 smaller triangles, all of which have side lengths smaller than N. We know that the side from 1 to P is less than $\frac{N+1}{2}$ and the side P to Q is less than $\delta < \frac{N}{2}$. By the triangle inequality, we know the third side has a length less than N. We give the following picture:



Hence we have $xyz_1z_2 = 1$, and by our additional relation, we are done. Now, take the case where $y, z \notin X_N, x \in X_N$. Split x into x_1x_2. We know both x_1 and x_2 have lengths smaller than $N + 1$. Now, divide the triangle as in case 1. Only one side can have a length of $N + 1$, all the others have lengths less than or equal to N. Applying case 1 finishes the proof. If we have $x, y, z \notin X_N$, then we split one of the sides and proceed as in case 2. □

2. Dehn Functions

The Dehn function is the crucial tool we will use to solve the word problem. In this section, we present key properties of Dehn functions and provide a solution to the word problem. From Theorem 16.1 in Armstrong [2], we note that if a group $G = \langle A \,|\, R \rangle$, then $G \cong F(A)/N(R)$ where $N(R)$ is the normal closure of R in $F(A)$. Hence, it follows that a word is the identity in G if and only if it can be written as a product of conjugates of the relators in the free group, i.e.

$$\Pi_{i=1}^{N} x_i r_i x_i^{-1} =_F w$$

where x_i is a word and $r_i \in R$. Using this fact, we can now define the area of a word, as

$$Area(w) = \min\{N | \Pi_{i=1}^{N} x_i r_i x_i^{-1} =_F w\}.$$

That is, Area(w) is the minimum number of conjugates we need to express a word that is the identity.

Example 2.1. A good example from Howie is if we are given the presentation $\langle a, b \,|\, [a, b]\rangle$ then we see that $[a^2, b^2] =_F a[a, b]a^{-1}[a, b]ba[a, b]a^{-1}b^{-1}b[a, b]b^{-1}$, hence $\text{Area}([a^2, b^2]) = 4$.

Now we can define the Dehn function.

Definition 2.2. The *Dehn function* $\delta(n) : \mathbb{N} \to \mathbb{N}$ is given by

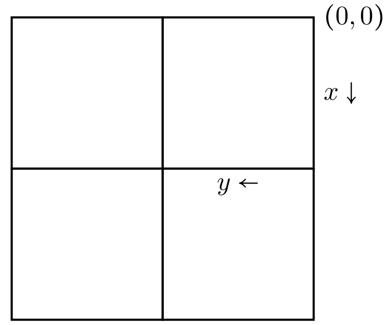
$$\delta(n) = \max\{Area(w)|l(w) \leq n\}.$$

The Dehn function is closely related to the Van Kampen diagram. The diagram, K is a 2-complex which can be constructed in the following way: Given a word w and a group $G = \langle A \,|\, R\rangle$, find an expression for w in terms of conjugates of the relators, so that $w = \Pi_{i=1}^{N} x_i r_i x_i^{-1}$. Then, construct a 2-complex based on the expression $\Pi_{i=1}^{N} x_i r_i x_i^{-1}$ using the usual rules of a Cayley graph to connect edges. If the diagram K has the property that its boundary $\partial K = w$, then we say K is the Van Kampen diagram for w.

Theorem 2.3 (Neumann [9])**.** *The Van Kampen diagram for a word w exists if and only if $w =_G 1$. The minimum number of two cells in the Van Kampen diagram is $Area(w)$.*

Proof. The Van Kampen diagram must start and end at the same point, which is exactly when $w =_G 1$. Next, note that each conjugate of the form $x_i r_i x_i^{-1}$ represents a closed 2 cell in the Van Kampen diagram, as we can follow the path x_i to a point, then r_i will be a cycle in the diagram starting at the end of x_i. Finally, by taking x_i^{-1} we follow the path back to the original point. Hence, the minimum number of two cells is exactly $Area(w)$. □

Example 2.4. Continuing from Example 2.1, we now know the area of $[a^2, b^2]$. From Theorem 2.3, we can now construct a Van Kampen diagram of the word.



We start at the point labeled $(0, 0)$, and then follow the expression for the area of the word. Everytime there is an x, we traverse one side down, while x^{-1} means we traverse one side up. For y, we traverse one side left, and y^{-1} is one side right.

The one ambiguity that must be addressed here is the fact that at first glance, it seems that $\delta(n)$ depends on the presentation of G. Again, by defining a new equivalence relation, we will see that the Dehn function is independent of presentation.

Definition 2.5 (Bridson [3])**.** Let $f, g : \mathbb{N} \to \mathbb{N}$. We say that $f \preceq g$ if there exists $A, B, C, D, E > 0$ such that $f(n) \leq Ag(Bn + C) + Dn + E$. We say that f, g are equivalent $(f \simeq g)$ if $f \preceq g$ and $g \preceq f$.

Example 2.6. If a group has Dehn function n^2 and another group has a Dehn function $n^2 + n + 1$, we see that we can really think of these Dehn functions as equivalent for our purposes. Note that all polynomials of the same degree are in the same equivalence class.

The next theorem shows this equivalence class is well defined. The proof requires two Lemmas.

Lemma 2.7 (Alonso [1]). *If w' and w'' are words in G, and $w = w'w''$ then $Area(w) \leq Area(w') + Area(w'')$.*

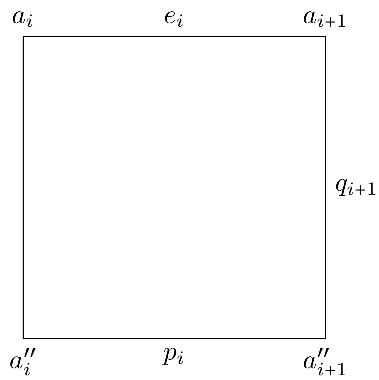
Proof. Note that if w' can be represented as k conjugates and w'' can be represented as m conjugates, then concatenating the two sets of conjugates, we have an expression for w in terms of conjugates. Since $Area(w)$ is the minimum of all such expressions, we are done. □

Lemma 2.8 (Alonso [1]). *Assume G, G' are simply connected groups (that is, their associated Cayley complex is a simply connected space) and G is finitely presented such that it has only k two cells in its Caley graph. Suppose there is a function $f : G \to G'$ that satisfies $d'(f(x), f(y)) \leq \lambda d(x, y) + \epsilon$. Define w' as follows. If $w = x_1...x_n$, then $x_i' = f(x_i)$. Let p_i be the geodesic segment between x_i' and x_{i+1}', and then let $w' = p_1...p_n$. The conclusion is that $Area_G'(w') \leq Area_G(w)\delta_{G'}(k(\lambda + \epsilon))$.*

Proof. Note that if x_i, x_{i+1} are adjacent in G, then the distance between x_i', x_{i+1}'' is $\lambda + \epsilon$. Hence, the boundary of a Van Kampen diagram for w' which is of length n will be $k(\lambda + \epsilon)$. The lemma follows from these two observations. □

Theorem 2.9 (Alonso [1]). *If we have two groups, $G = \langle A \,|\, R \rangle$ and $G' = \langle A' \,|\, R' \rangle$ that are quasi-isometric to each other, then $\delta_G(n) \simeq \delta_{G'}(n)$.*

Proof. Suppose that we have a mapping $f : G \to G'$ that is a (λ, ϵ) quasi-isometry and $f' : G' \to G$ is a quasi isometry from G' to G. Choose a word w in G such that $l(w) \leq n$ and $Area(w) = \delta_G(n)$. We can write w as $a_1...a_n$ where $a_i \in A$, and $1 \leq i \leq n$. Let e_i be the edge that joins a_i to a_{i+1} in the Cayley graph of G, A. Set $a_i' = f(a_i)$ and $a_i'' = f'(a_i')$. We can define w' and w'' from the a_i' and the a_i'' respectively. For $1 \leq i \leq n$, join a_i to a_i'' with a geodesic segment, call it q_i. Each q_i has finite length, since Cayley graphs are connected. Let e_i be the word that joins a_i to a_{i+1} and p_i the corresponding word joining a_i'' to a_{i+1}''. We now note that $l(p_i) \leq \lambda(\lambda + \epsilon) + \epsilon$. This inequality follows from the fact that the distance between x_i' and x_{i+1}' is $\lambda + \epsilon$. Mapping these points back, by properties of quasi-isometries we have the desired inequality. Next, we see that p_i has finite length, call its length D. We present the following picture:



From the picture, we see we can rewrite w in terms of w'' as follows,

$$w = w'' + q_0^{-1}e_0q_1p_0^{-1} + ... + q_m^{-1}e_mq_0p_m^{-1}.$$

The idea here is to take w'' then go in a loop through the edges connecting w and w''. From Lemma 3.1, we see that $Area(w) \leq Area(w'') + C$ for some constant C. Let k be the maximal number of 2 cells in G' for all words of length less than or equal to n. Now, we can apply lemma 3.2 to get that

$$Area(w) \leq Area(w')\delta_G(k(\lambda + \epsilon)) + C.$$

But now we see that $Area(w') \leq \delta_{G'}(n)$ and $Area(w) = \delta_G(n)$ by our choice of G. We have that $\delta_G(w) \leq \delta_{G'}(n)$. By using a similar construction, we can prove the reverse inequality. We conclude $\delta_G(w) \simeq \delta_{G'}(n)$ as desired. □

It follows from Proposition 2.2.1 the Dehn function is independent of presentation. Dehn functions also give us an alternative way to define hyperbolic groups.

Definition 2.10. A group G is hyperbolic if there is a $f : \mathbb{N} \to \mathbb{N}$ such that $\delta(n) \leq f(n), \forall n$ and $f(n)$ is a linear function.

We have assumed that this definition is equivalent to the δ thin triangles definition. The proof of this is quite technical and long; the curious reader is advised to read [11] section 2 for the full proof. Short gives the additional characterization that a group is hyperbolic if

$$\lim_{n\to\infty} \frac{\delta(n)}{n^2} = 0.$$

This seemingly innocous fact is actually quite remarkable. Since hyperbolic groups must have linear Dehn functions, it states that it is in fact impossible to find a group with Dehn function n^k for $k \in (1, 2)$ or to find a Dehn function of the form $n \log n$. But, we can find groups with Dehn function $n^{\frac{5}{2}}$. In fact, Bridson provides us a much stronger statement. Call a number, $x \in \mathbb{R}$ isoperimetric if there is a group with Dehn function $\delta(n) = n^x$. Then, if A is the set of all isoperimetric numbers, its closure $\overline{A} = \{1\} \cup [2, \infty)$. The proof of this fact can be found in [3]. Hence, there is a certain uniqueness of hyperbolic groups with respect to the word problem, a connection that will be further explored later in the paper.

3. The Word Problem

Theorem 3.1 (Neumann [9]). *The word problem for a group G is solvable if and only if G has a recursive Dehn function.*

Remark. A recursive function is one which can be computed by an algorithm.

Proof. First, assume that G has a recursive Dehn function. Given a word w of length n, we know that the maximum possible area for w is given by $\delta(n)$. There are only finitely many of combinations of conjugates of the form $x_i^{-1}r_ix_i$, since we know that each of the x_i is a word of finite size, $l(x_i) \leq l(w)$. Hence, by trying all these combinations, we can see if $w =_G 1$. Conversely, assume that G has a solvable word problem. For a given value of n, we can find all words equal to 1 and of length $\leq n$. If we are given a word equal to the identity, we can calculate its area by trying combinations of conjugates as before. Hence we can calculate the area of every word, which gives us Dehn's function. □

Remark. We can loosen the conditions. We only used the recursive function criterion to find a bound for the area of each word. Hence, we could have stated the previous theorem by replacing "recursive Dehn function" with "sub-recursive Dehn function." This weakens the conditions for the reverse direction of the proof, which is the direction of primary interest. Moreover, Theorem 3.2.1 shows that having a solvable word problem is a quasi-isometric invariant.

While the proof gives us sufficient criteria to see if a group has a solvable word problem, it does not give us an efficient solution to the problem. Checking all the combinations of conjugates is wildly impractical in most cases. For example, some Dehn functions grow extremely rapidly. Neumann in [9] notes that the seemingly simple group $H = \langle x, y \,|\, x * (x * y) = x^2\rangle$ has a Dehn function with a very rapid growth rate. In fact define the function $f_k(n) = 2^{2^{\cdot^{\cdot^{n}}}}$, where k gives the number of exponents. The Dehn function $\delta_H(n) \geq f_k(n)$ for all $k \in \mathbb{N}$. Trying to list all the conjugates is not feasible in this case. There are other subtleties with the word problem. Not only do some finitely presented, seemingly simple groups have some very large Dehn functions, but a group's Dehn function might not be recursive.

Theorem 3.2. *Novikov-Boone Theorem. There exists a group G that is finitely presented but does not have a recursive Dehn function.*

Proof. Proof due to Rotman [10] We give a sketch of the proof, without a few difficult lemmas, the details of which can be found in [10]. For this proof, a knowledge of some logic, e.g. familiarity with Turing machines is assumed to complete the technical details of the proof. First, we fix a Turing machine T such that there is $F \subset \mathbb{N}$ that is not recursive, but F is recursively enumerable. We can choose such a T by Theorem 12.1 in Rotman. We can associate a semi-group, $\Gamma(T)$ with each Turing machine T defined by

$$\Gamma(T) = \langle q, h, s_0, .., s_n, q_0, ..q_m \,|\, R(T)\rangle$$

where $s_0, ..., s_n$ is the alphabet of T, $q_0, ..., q_n$ are the states, and q, h are new letters. The list of relations are long, but can be found in Rotman. We now prove that there is a finitely presented semi-group with an unsolvable word problem. Suppose that T is a Turing machine with alphabet given by $A = \{s_0, ..., s_n\}$. Let $\overline{\Omega}$ be the set of positive words (i.e. words without inverse elements) created from

$A \cup \{q, h, q_0, .., q_m\}$. Set Ω to be the positive words on A. Let E be the set of all words of Ω that T can compute. Let

$$\overline{E} = \{w \in \overline{\Omega} | w = q \in \Gamma(T)\}.$$

Define a function $\phi : \Omega \to \overline{\Omega}$ to be $\phi(w) = hq_1wh$, and let $\Omega_1 = \phi(\Omega)$, the image of ϕ. Now we can say that E_1 is the image of E under ϕ. Clearly, E_1 being recursive is equivalent to E being recursive. We know from Lemma 12.4 in Rotman that $E_1 = \Omega \cap \overline{E}$. Now, note that E and hence E_1 are not recursive, since we chose T to not enumerate E at the beginning. Thus, $\Gamma(T)$ has an unsolvable word problem. Now, we need to translate this reulst from the semi-group into an actual group. To this end, construct a new group

$$G(T) = \langle q_0, .., q_m, s_0, ..., s_n, r_i(i \in \mathbb{N}), x, t, k \,|\, R'(T)\rangle.$$

The new relators $R'(T)$ are again numerous but can be found in Rotman. Next, we state Boone's Lemma, without proof. If $X = s_0^e ... s_n^e$ is a word in a group G, let $X' = s_0^{-e} ... s_n^{-e}$. If $w = X'q_jY$ for some words X, Y composed only of the alphabet A, then $kw^{-1}tw =_G w^{-1}twk$ if and only if $w =_\Gamma q$. With this lemma, the completion of the proof follows if we choose the same Turing machine that we chose to show that the semi-group has no solvable word problem. □

Example 3.3. An example of a finitely presented group that does not have a recursive Dehn function is

$$\Gamma_k = \langle a_1, .., a_k, t, p \,|\, t^{-1}a_1t = a_1, ..., t^{-1}a_it = a_ia_{i-1}(i > 1), [p, a_i * t](i > 0)\rangle.$$

Dison and Riley [5] show that this group has a Dehn function that is not recursive.

Now we turn to finding a more efficient solution to the word problem.

Definition 3.4 (Brisdon [3])**.** Suppose we have a group $G = \langle a_1, ..., a_n \,|\, u_1v_1^{-1}, ..., u_nv_n^{-1}\rangle$ where u, v are words such that $l(v_i) < l(u_i)$ and $u_i =_G v_i$. Further suppose that every word $w =_G 1$ contains at least one u_i. If $w' \in G$, we can reduce it by taking every instance of a u_i in w' and replacing it with v_i. This process terminates when we have the empty word, or when we have a word that is not the identity. The above process is called a *Dehn algorithm* and the initial presentation is called a *Dehn presentation.*

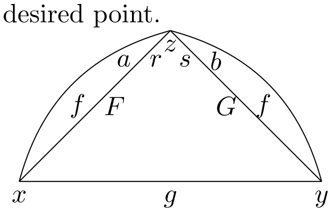
We can see that those groups with Dehn algorithms will have word problems that can be solved quickly. In fact, by examining the above algorithm, we see that searching a word for instances of u_i bounds the number of possible relators that can be applied to w linearly. This leads us to the following proposition.

Proposition 3.5. *A group G that has a Dehn presentation is hyperbolic.*

Surprisingly the converse is also true. To prove this fact we need the following lemma.

Lemma 3.6 (Short [11])**.** *Suppose X is a δ−hyperbolic metric space. Given two points $x, y \in X$, let $f : [0, l] \to X$ be a k local geodesic segment, and $g : [0, l] \to X$ be a geodesic segment such that $g(0) = f(0) = x$ and $g(l) = f(l) = y$. Then the maximum distance between any two points on f, g is 3δ.*

Proof. Choose a point z on f that is of distance at least 2δ from g. If no such point exists, then we are done. Let F be a geodesic from x to z and let G be a geodesic from z to y. Choose a to be the point that is 2δ away from z on f that is closer to x. Let b be the corresponding point closer to y. Choose r to be 2δ away from z on F and closer to the point x. Let s be 2δ away from z on G and closer to the point y. Now, a, r, z form a geodesic triangle, since f is k local geodesic. Likewise for b, s, z. So the distances between a, r and b, s are at most δ. Next, note that the geodesic segments F, G, g form a geodesic triangle. So we know that the distance between r and g is at most δ. So the distance between z and g is at most 3δ. The following picture may help understand the proof, all labelings are done below the desired point.



□

Proposition 3.7 (Duchin [6]). *Every δ−hyperbolic group has a Dehn presentation.*

Proof. Suppose we have a δ−hyperbolic group G with generating set A. Choose $N > 8\delta$, and consider all the words with length at most 8δ. Let u_i be a non-geodesic spelling of a word, and let v_i be the corresponding geodesic spelling. Let R be all such $r_i = u_i v_i^{-1}$. We claim that $G = \langle A \mid R \rangle$ is our desired Dehn presentation. All that remains is to show that in every word, w such that $w =_G 1$ contains at least one u_i. We know, by Lemma 3.6 that there cannot be an 8δ local geodesic loop, because that loop would not stay within a 3δ distance of the identity. Hence if we have $w_G = 1$, then there is a non-geodesic segment of length less than N, which is a u_i. □

Remark. Note that 8δ was chosen somewhat arbitrarily. There is a more complex theorem that an 8δ local geodesic segment stays within 2δ of the geodesic segment. However, since we used a 3δ bound, we did not need 8δ. We could have chosen 4δ or 10δ and the proof would have worked just as well.

We now have a third definition of hyperbolic group. Next, we turn our focus to a brief overview of the conjugacy problem.

4. The Conjugacy Problem

The conjugacy problem has not been solved in the same sense as the word problem; we still do not have necessary and sufficient conditions for solving the conjugacy problem like we do in the word problem. However, M. Gromov has stated the sufficient conditions to solve the conjugacy problem.

Proposition 4.1 (Gromov [7]). *Let $G = \langle A \mid R \rangle$ be a group with a solvable word problem. Assume that if g, g' are conjugate in G, then there is a sequence $g = g_1, ..., g_n = g$ and a $C > 0$ such that $g_{i+1} = a_i g_i a_i^{-1}$ and $l(a_i) \leq C$ and $C(l(g) + l(g')) \geq l(g_i)$ for all $1 \leq i \leq n$. Then G has a solvable conjugacy problem.*

Proof. Take two elements, $x, y \in G$. We want to see if $zxz^{-1}y^{-1} = 1$ is the identity, for any word $z \in F(A)$. Since G has a solvable word problem this reduces to finding a finite list of z to check. But we know that if x, y are conjugate, we can find an x_1 conjugate to x, that is $x = a_1 x_1 a_1^{-1}$, and we know that there is a bound on the length of a_1 and x_1. We can continue constructing such a sequence until we either see it is impossible, or we get to y. So we can decide if the conjugacy problem is solvable. □

The conditions of this proposition are quite specific and hard to work with practically. It is hard to decide whether or not an arbitrary group G satisfies these conditions. However, we have the following theorem for hyperbolic groups:

Proposition 4.2 (Bridson and Haefliger [4]). *If G is hyperbolic, it has a solvable conjugacy problem*

Proof. We prove the following sufficient condition. Suppose $G = \langle A \mid R \rangle$ is δ−hyperbolic. If w, w' are conjugate elements, either $\max\{l(w), l(w')\} \le 8\delta + 1$ or there are cyclic permutations of w and w', call them w_s, w'_s, and a word u with $l(u) \le 2\delta + 1$ such that $uw_su^{-1} = w$.

To do this, consider a geodesic quadrilateral with sides w, w', u, u^{-1}. We can cyclically permute the diagram such that any vertex on the side labeled w is at least $l(u)$ away from w'. As in the proof of Lemma 3.6, note that the midpoint of w, call it p, is at most 2δ away from any other side. In particular, we know that it is at most 2δ away from the sides defined by u, u^{-1}, if $l(u) < 2\delta + 1$. Now, let q be the point of one of these sides within 2δ of p, and x, y the end vertices of this edge, with x closer to p. It follows that

$$l(u) - \frac{1}{2} \le d(p, y) \le 2\delta + d(q, y)$$

and that

$$d(q, y) \le l(u) - d(x, q).$$

By rearranging the two previous inequalities, we get $d(x, q) \le 2\delta + \frac{1}{2}$. Thus

$$2d(x, p) \le d(q, p) + d(x, q) \le 8\delta + 1.$$

But $2d(x, p) = l(w)$, which shows $l(w) \le 8\delta + 1$. A similar argument holds for $l(w')$. □

However, the converse of this theorem does not hold true in general. For example, take the group suggested by Dison and Riley [5],

$$G = \langle a, t, p \mid [a, p], [a, t] \rangle.$$

They show that this has a Dehn function that is equivalent to n^2 and it is known to have a solvable conjugacy problem. It is an open question as to whether or not every group with a Dehn function bounded above by n^2 has a solvable conjugacy problem.

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