SCHUR-WEYL DUALITY

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Contents

Introduction. In this paper, we build up to one of the remarkable results in representation theory called Schur-Weyl Duality. It connects the irreducible representations of the symmetric group to irreducible algebraic representations of the general linear group of a complex vector space. We do so in three sections:

(1) In Section 1, we develop some of the general theory of representations of finite groups. In particular, we have a subsection on character theory. We will see that the simple notion of a character has tremendous consequences that would be very difficult to show otherwise. Also, we introduce the group algebra which will be vital in Section 2.
(2) In Section 2, we narrow our focus down to irreducible representations of the symmetric group. We will show that the irreducible representations of S_n up to isomorphism are in bijection with partitions of n via a construction through certain elements of the group algebra. Finally, we mention the beautiful Robinson-Schensted correspondence. The correspondence upgrades a formula involving the dimension of the irreducible representations of S_n to a bijection which has many combinatorial applications.
(3) In Section 3, we prove the classical case of Schur-Weyl Duality by using the Double Centralizer Theorem. We describe exactly which representations

of the general linear group $\mathrm{GL}(V)$ come from this relationship. And we close the section by showcasing other forms of Schur-Weyl Duality for other matrix groups and Lie algebras.

1. Representation Theory of Finite Groups

1.1. **Preliminaries.** A *representation* of a finite group G on a finite-dimensional complex vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We will often abuse notation by referring to V as the representation of G and write gv for $\rho(g)(v)$ whenever the map ρ is understood from context. We say that a subspace W of a representation V is a *subrepresentation* if W is *invariant* under the the action of G, that is, $gw \in W$ for all $g \in G$ and $w \in W$. Notice that a representation V always contains at least two subrepresentations, namely, 0 and V itself. We say V is *irreducible* if it contains no proper nonzero invariant subspaces, i.e., 0 and V are the only subrepresentations of V. However, note that we do *not* consider $V = 0$ to be an irreducible representation. We will see that this is analogous to the reason 1 is not a prime number.

Examples 1.1. Let G be a finite group. Here are some examples of representations.

(1) The *trivial representation* $\mathbb{C}$ where $gv = v$ for all $g \in G$ and $v \in U$. Notice that the trivial representation is always (trivially) irreducible.
(2) Let G act on a finite set X. The *permutation representation* is the vector space with basis X where $\rho(g)(x) = gx$ for all $g \in G$ and $x \in X$. When G acts on itself by multiplication, we get the *regular representation.*

We can always build up new vector spaces from old using operations such as the tensor product. Similarly, given representations V and W of G, we can form the representations $V \oplus W$, $V \otimes W$, $\mathrm{Sym}^n V$, $\mathrm{Alt}^n V$, V^*, and $\mathrm{Hom}(V, W)$ where each is given their ordinary underlying vector space structure. For the first four, just let G act factorwise, e.g., $g(v \otimes w) = gv \otimes gw$ in $V \otimes W$. For the dual $V^* = \mathrm{Hom}(V, \mathbb{C})$ of V, we require $\rho^*(g)$ be the transpose of $\rho(g^{-1})$. This forces ρ to respect the natural pairing of V^* and V in the following sense: $\langle \rho^*(g)(\lambda), \rho(g)(v)\rangle = \langle \lambda, v\rangle$ for all $g \in G$, $\lambda \in V^*$, and $v \in V$. Having this, the action on $\mathrm{Hom}(V, W)$ is given by the identification $\mathrm{Hom}(V, W) = V^* \otimes W$.

One of the main goals in representation theory is to classify all representations of a given group G. For a general (possibly infinite) group, this is hard. And if we work over a field other than $\mathbb{C}$, this can complicate matters even more. However, we will see that this classification is possible for representations over $\mathbb{C}$ of a finite group. But instead of directly classifying all representations, we can narrow our focus down to *indecomposable* representations, those which cannot be written as a direct sum of others. So we then can build up all other representations from these. At this point, we have introduced two notions of atomicity: irreducible and indecomposable. Clearly, if a representation is irreducible, then it is indecomposable. Luckily, in the case of representations over $\mathbb{C}$ of finite groups, the converse holds.

Maschke's Theorem 1.2. *If W is a subrepresentation of V of a finite group G, then there exists a complementary subrepresentation W' of V so that $V = W \oplus W'$.*

Proof. Pick W'' to be *any* subspace so that $V = W \oplus W''$ and let $\pi' : V \to W$ be the projection along W'' onto W. Now define the projection $\pi : V \to W$ by

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g\pi'(g^{-1}v).$$

Notice that $h\pi(v) = \pi(hv)$ for all $h \in G$. In particular, $W' := \operatorname{Ker} \pi$ is a complementary G-invariant subspace because we have $\pi(hv) = h\pi(w) = 0$ for all $h \in G$ and $v \in W'$. In which case, $V = W \oplus W'$. □

Corollary 1.3. *Every representation over $\mathbb{C}$ of a finite group can be decomposed as a direct sum of irreducible representations.*

Examples 1.4. Corollary 1.3 holds in further generality. If we work over a different scalar field or characteristic zero but still require our group to be finite, then the previous proof of Maschke's Theorem 1.2 still works. For scalar fields with positive characteristic, we cannot have the characteristic of the scalar field divide the order of the group (otherwise, the projection cannot be defined). To see why this generalization of Corollary 1.3 fails if we do not assume this:

(1) Let $V = \mathbb{F}_2^2$ and $G = \mathbb{Z}/2\mathbb{Z}$ where $1 \in \mathbb{Z}/2\mathbb{Z}$ sends $(x, y) \in \mathbb{F}_2^2$ to (y, x). Notice the span of $(1, 1) \in \mathbb{F}_2^2$ is invariant under the action $\mathbb{Z}/2\mathbb{Z}$. However, the remaining complementary subspaces—the span of $(1, 0)$ and the span of $(0, 1)$—are not invariant. So $\mathbb{F}_2^2$ is indecomposable but not irreducible with this action of $\mathbb{Z}/2\mathbb{Z}$.

To see why the generalization of Corollary 1.3 fails if the group is infinite:

(2) Let $V = \mathbb{C}^2$ and $G = \mathbb{C}$ where $z \in \mathbb{C}$ acts on $(x, y) \in \mathbb{C}^2$ by

$$z(x, y) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + zy \\ y \end{pmatrix}.$$

Then the only 1-dimensional invariant subspace is the span of $(1, 0)$. Again, we have $\mathbb{C}^2$ is indecomposable but not irreducible under this action of $\mathbb{C}$.

However, we can adjust the proof of Maschke's Theorem 1.2 by swapping the sum with integration with respect to a Haar measure to show a version of Corollary 1.3 for compact groups, e.g. S^1. More precisely, any finite-dimensional continuous representation of a compact group can be decomposed into irreducible representations.

So the previous corollary tells us that there always *exists* a decomposition of a representation into a sum of irreducible representations. So the natural question is now: in what sense (if any) is this decomposition unique? But we have yet to even describe a way to distinguish representations.

A *G-map* f of representations V and W of G is a linear map $f : V \to W$ that commutes with the action of G, that is, $gf(v) = f(gv)$ for all $g \in G$ and $v \in V$. Notice that $\operatorname{Ker} f$ and $\operatorname{Im} f$ are always subrepresentations of V and W respectively. We say a G-map $f : V \to W$ is an *isomorphism* of representations if it is an isomorphism of vector spaces. In this case, we say that V and W are *isomorphic* and denoted this by $V \simeq W$. Let $\operatorname{Hom}_G(V, W)$ denote the set of all G-maps from V to W and define $\operatorname{End}_G(V) = \operatorname{Hom}_G(V, V)$. The following elementary lemma is perhaps the most useful fact about G-maps.

Schur's Lemma 1.5. *Let $f : V \to W$ be a nonzero G-map.*

(1) If V is irreducible, then f is injective.

(2) If W is irreducible, then f is surjective.
(3) If $V = W$ is irreducible, then $f = tI$ for some nonzero $t \in \mathbb{C}$.

Proof. The first two parts boil down to considering the subrepresentations $\operatorname{Ker} f$ and $\operatorname{Im} f$. The last part comes from taking any $f \in \operatorname{End}_G(V)$ and applying the first parts to show $f - tI \in \operatorname{End}_G(V)$ is the zero map for any eigenvalue t of f. □

Corollary 1.6. *For any representation V of a finite group, we can write*

$$V \simeq V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}$$

where V_i are distinct irreducible representations. The number of factors k and the V_i that occur (up to isomorphism) along with their multiplicities n_i are all unique.

Proof. Schur's Lemma 1.5 tells us that a map in $\operatorname{Hom}_G(V, W)$ with the decompositions $V = \bigoplus V_i^{\oplus n_i}$ and $W = \bigoplus W_j^{\oplus m_j}$ must take the factor $V_i^{\oplus n_i}$ into a factor $W_j^{\oplus m_j}$ where $V_i \simeq W_j$. Applying this to the identity map on V with two different decompositions will show the stated uniqueness. □

Example 1.7. Consider an abelian group G and an irreducible representation V of G. For $h \in G$, let $f_h : V \to V$ so that $f_h(v) = hv$. This is a G-map because $gf_h(v) = ghv = hgv = f_h(gv)$ for all $g \in G$. By Schur's Lemma 1.5, for each $h \in G$, there exists a $t_h \in \mathbb{C}$ so that $f_h = t_h I$. Thus, the span of any nonzero vector v is a nonzero invariant subspace of V and, hence, is V. Therefore, all irreducible representations of an abelian group are 1-dimensional.

1.2. **Group Algebra.** We will now introduce a powerful lens though which to view representations. The *group algebra* $\mathbb{C}[G]$ of a group G is the associative $\mathbb{C}$-algebra with basis G and where multiplication is inherited from group multiplication, i.e.,

$$\left(\sum_{p\in G} a_p p\right)\left(\sum_{q\in G} b_q q\right) = \sum_{p,q\in G} a_p b_q pq = \sum_{g\in G} c_g g$$

where c_g is the sum of all $a_p b_q$ where $g = pq$.

We can generalize the notation of a representation onto other algebraic structures. A *representation* of an associative $\mathbb{C}$-algebra A on a complex vector space V is an algebra homomorphism $\rho : A \to \operatorname{End}(V)$. In particular, V is an A-module. We want to stress that throughout this paper, we will *only* be working with associative algebras over the scalar field $\mathbb{C}$.

Notice that given a representation of a finite group G, we can extend it linearly to get a representation of the group algebra $\mathbb{C}[G]$. And conversely, given a representation of $\mathbb{C}[G]$, we can restrict it to G to get back a representation of G. So it may seem that introducing the group algebra has not given us anything new. But it gives us a new language and tools which make constructions and definitions more transparent. For instance, here is a short excerpt of the dictionary between a group and its corresponding group algebra.

G-representation	$\mathbb{C}[G]$-module
subrepresentation	submodule
irreducible representation	simple module
G-map	$\mathbb{C}[G]$-homomorphism

We will not yet prove the following proposition on the structure of $\mathbb{C}[G]$. It actually follows from the main theorem that we will show in the in the next subsection.

Proposition 1.8. *As algebras,*

$$\mathbb{C}[G] = \bigoplus_i \operatorname{End}(V_i)$$

where the sum is over all distinct irreducible representations V_i of G.

But Proposition 1.8 can be proved without the mentioned theorem, using more ring theoretic tools as seen in the appendix. In fact, Proposition 1.8 can be used to prove the mentioned main theorem.

Although the image of $g \in G$ under a representation is a map in $\mathrm{GL}(V) \subseteq \operatorname{End}(V)$, it is not necessarily in $\operatorname{End}_G(V)$. Clearly, the image of g in $\operatorname{End}(V)$ is a G-map if and only if g is in the center $Z(G)$. Similarly, it can be seen that the image of $f \in \mathbb{C}[G]$ in $\operatorname{End}(V)$ is a G-map if and only if f is in the center $Z(\mathbb{C}[G])$. We will now show an alternate characterization of elements in $Z(\mathbb{C}[G])$, which involves functions $\alpha : G \to \mathbb{C}$ so that $\alpha(g) = \alpha(hgh^{-1})$ for all $g, h \in G$. We call such functions *class functions*, and they will appear again later on.

Proposition 1.9. *Let V be a representation of G and $\alpha : G \to \mathbb{C}$ a function. Let*

$$f = \sum_{g \in G} \alpha(g) g \in \mathbb{C}[G].$$

The following are equivalent:

(1) f belongs to $Z(\mathbb{C}[G])$.
(2) the image of f in $\operatorname{End}(V)$ is in $\operatorname{End}_G(V)$.
(3) α is a class function.
(4) α is a sum of indicator functions on conjugacy classes of G.

In particular, here is an important such function.

Lemma 1.10. *Let $V^G = \{v \in V : gv = v \text{ for all } g \in G\}$. Then the image of*

$$\pi = \frac{1}{|G|} \sum_{g \in G} g \in Z(\mathbb{C}[G])$$

in $\operatorname{End}(V)$ is a G-map and a projection onto V^G.

Notice that V^G is the sum of all copies of the trivial representation found in V.

1.3. **Character Theory.** The *character* of a representation V of G, is the map

$$\chi_V : G \to \mathbb{C} \quad \text{where} \quad \chi_V(g) = \operatorname{tr}(g|_V)$$

is the trace of g on V. Although this is a simple notion, we will soon see it is absolutely vital to the study of representations of finite groups.

Since the trace of a linear transformation is the sum of eigenvalues, the identities

$$\chi_{V \oplus W} = \chi_V + \chi_W, \qquad \chi_{V \otimes W} = \chi_V \chi_W, \qquad \chi_{V^*} = \overline{\chi_V} \tag{1.11}$$

can easily be verified. In addition, by the properties of trace, we have that for any character χ of a representation, $\chi(g) = \chi(hgh^{-1})$ for all $g, h \in G$, that is, χ is a class function on G. Let $\mathcal{F}(G)$ denote the set of all class functions $G \to \mathbb{C}$. Then we can endow $\mathcal{F}(G)$ with the inner product

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g) \tag{1.12}$$

for $\alpha, \beta \in \mathcal{F}(G)$. We have the following fundamental theorem of character theory.

Theorem 1.13. *The set of characters χ_V for irreducible V of G form an orthonormal basis for $\mathcal{F}(G)$ with respect to the inner product in* (1.12).

Proof. Let V and W be irreducible representations of G. By the properties (1.11),

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g) = \mathrm{tr}(\pi|_{\mathrm{Hom}(V,W)})$$

where π is the projection from $\mathrm{Hom}(V, W)$ onto $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$ in Lemma 1.10. Therefore, by Schur's Lemma 1.5, we have that

$$\langle \chi_V, \chi_W \rangle = \mathrm{tr}(\pi|_{\mathrm{Hom}(V,W)}) = \dim \mathrm{Hom}_G(V, W) = \begin{cases} 1, & \text{if } V \simeq W \\ 0, & \text{if } V \not\simeq W. \end{cases}$$

So characters of irreducibles are orthonormal and, thus, are linearly independent.

Now we will now show that the linearly independent set of irreducible characters is in fact maximal and, hence, forms a basis for $\mathcal{F}(G)$. Suppose $\alpha \in \mathcal{F}(G)$ such that $\langle \alpha, \chi_V \rangle = 0$ for all irreducible V. It suffices to show $\alpha = 0$. Notice that

$$0 = \langle \alpha, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \chi_V(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\, \mathrm{tr}(g|_V) = \mathrm{tr}(f|_V)$$

where $f = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} g \in \mathrm{End}_G(V)$ by Proposition 1.9. Since $f \in \mathrm{End}_G(V)$, then Schur's Lemma 1.5 tells us that $f = tI$ for some $t \in \mathbb{C}$. Therefore, we have that $t = \mathrm{tr}(f|_V)/\dim V = 0$ and, thus, $\alpha = 0$. □

Here are some important consequences of this which fall out almost immediately.

Corollary 1.14. *We have that $V \simeq W$ if and only if $\chi_V = \chi_W$.*

Example 1.15. This tells us that all representations V of S_n are *self-dual* since

$$\chi_{V^*}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1}) = \chi_V(g)$$

for all $g \in S_n$ because g and g^{-1} have the same cycle type and, thus, are in the same conjugacy class. Hence $V \simeq V^*$. This will be a useful fact later on. Note that for a group G in general, this does not hold.

Corollary 1.16. *A representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Corollary 1.17. *If V and W are representations with V being irreducible, then the multiplicity of V in the decomposition of W is $\langle \chi_V, \chi_W \rangle$.*

Corollary 1.18. *For the regular representation R of G, we have that*

$$R = \bigoplus V_i^{\oplus \dim V_i}$$

where the sum is over all distinct irreducible representations V_i of G.

In particular, we obtain the beautiful formula

$$|G| = \dim R = \sum (\dim V_i)^2. \tag{1.19}$$

Notice since this sum is finite, there must be finitely many irreducible representations. But in fact, since the indicator functions of conjugacy classes also form a basis for $\mathcal{F}(G)$, we get the following corollary.

Corollary 1.20. *The number of distinct irreducible representations of G is equal to the number of conjugacy classes of G.*

Interestingly, in general, there is no natural bijection between irreducible representations and conjugacy classes of a finite group. But we will later see that there is a somewhat natural bijection in the case of the symmetric group S_n.

The orthogonality in Theorem 1.13 is called row orthogonality for reasons we will see when constructing character tables. Similarly, there is a corresponding column orthogonality.

Corollary 1.21. *For $g, h \in G$, we have*

$$\frac{1}{|G|}\sum_i \overline{\chi_{V_i}(g)}\chi_{V_i}(h) = \begin{cases} 1/c(g), & \text{if } g \text{ and } h \text{ are conjugate} \\ 0, & \text{otherwise} \end{cases}$$

where the sum is over all distinct irreducible representations V_i of G and $c(g)$ is the size of the conjugacy class of g.

Proof. Since characters are class functions, we can define $\chi_V : \mathrm{Conj}(G) \to \mathbb{C}$ by evaluating on a representative. Then the row orthogonality says

$$\langle \chi_{V_i}, \chi_{V_j}\rangle = \frac{1}{|G|}\sum_k |C_k|\overline{\chi_{V_i}(C_k)}\chi_{V_j}(C_k) = \delta_{ij} \tag{1.22}$$

where the sum is over all conjugacy classes $C_k \in \mathrm{Conj}(G)$ and δ_{ij} is the Kronecker delta function. Consider the matrix $X = (x_{ij})$ where $x_{ij} = \sqrt{|C_j|/|G|}\chi_{V_i}(C_j)$. Then (1.22) says $XX^H = I$ where X^H is the conjugate transpose of X. Therefore, we have that $X^H = X^{-1}$ and, thus, $X^H X = I$ as well. This says

$$\frac{\sqrt{|C_i||C_j|}}{|G|}\sum_k \overline{\chi_{V_k}(C_i)}\chi_{V_k}(C_j) = \delta_{ij}$$

where the sum is over all distinct irreducible representations V_i of G. □

Example 1.23. It would an absolute sin to have a section on character theory and never mention a *character table*. So we will compute the character table for the alternating group A_4.

In general, if we have a representation $G/N \to \mathrm{GL}(V)$ for some normal subgroup N, we can lift to a representation on G by the composition $G \to G/N \to \mathrm{GL}(V)$. So notice that $N = \langle (12)(34), (14)(23)\rangle$ is a normal subgroup of A_4 with $A_4/N \simeq \mathbb{Z}/3\mathbb{Z}$. Since A_4/N is abelian, then by Example 1.7, $A_4/N \simeq \mathbb{Z}/3\mathbb{Z}$ has three 1-dimensional irreducible representations, call them U, U', and U''. We know that one is the trivial representation. The other two come from naturally sending $1 \in \mathbb{Z}/3\mathbb{Z}$ to one of the two primitive 3rd roots of unity. The lifts of these representations must be irreducible for G as well because they are 1-dimensional. So we have the partial character table

	1	4	4	3
A_4	1	(123)	(132)	(12)(34)
U	1	1	1	1
U'	1	ω	ω^2	1
U''	1	ω^2	ω	1

where the top row indicates sizes, and representatives, for the conjugacy classes of A_4. Now, since there are 4 conjugacy classes of A_4, then by Corollary 1.20, we only need to find one more irreducible representation V to complete the table. By (1.19), we know that $\dim V = 3$ because $|A_4| = 12 = 1^2 + 1^2 + 1^2 + 3^2$. Using this, we can then compute the rest of the table by using column orthogonality (1.21).

	1	4	4	3
A_4	1	(123)	(132)	(12)(34)
U	1	1	1	1
U'	1	ω	ω^2	1
U''	1	ω^2	ω	1
V	3	0	0	-1

The representation V comes from the restriction of a representation of S_4 that we will later call the standard representation. Also, it turns out that V can be realized geometrically as the rotational symmetries of the tetrahedron.

2. Irreducible Representations of the Symmetric Group

2.1. **Specht Modules.** Recall that conjugacy classes of S_n are completely determined by cycle type, which can be encoded as a partition. A *partition* λ of n is an integer sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ so that $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ and $n = \lambda_1 + \cdots + \lambda_k$. Let $p(n)$ denote the number of partitions of n. Since the number of conjugacy classes of S_n is $p(n)$, then by Corollary 1.20, we know that there are also $p(n)$ many distinct irreducible representations of S_n. But, remarkably, we will see that not only does each partition λ uniquely give rise to a conjugacy class, but λ also gives rise to an irreducible representation of S_n.

To a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, we associate a combinatorial object called the *Young diagram* of λ which is a collection of n cells arranged in left-justified rows with λ_i cells on the ith row. Given the Young diagram of λ, we number the cells by the integers $1, \dots, n$ to form a *tableau.* In this case, a picture is more illuminating than the definition. Here is the tableau for $(4, 2, 2, 1)$ with the *canonical labeling.*

1	2	3	4
5	6		
7	8		
9			

There is a natural action of the symmetric group on tableaux by permuting the labels. For a tableau of λ of n, consider the following subgroups of S_n:

$$P_\lambda = \{g \in S_n : g \text{ preserves each row of } \lambda\}$$

and, similarly,

$$Q_\lambda = \{g \in S_n : g \text{ preserves each column of } \lambda\}.$$

To each of these subgroups, we can associate an element in the group algebra $\mathbb{C}[S_n]$:

$$a_\lambda = \sum_{g \in P_\lambda} g \quad \text{and} \quad b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) g.$$

Finally, we call their product $c_\lambda = a_\lambda b_\lambda \in \mathbb{C}[S_n]$ the *Young symmetrizer* of λ.

Theorem 2.1. *The image $V_\lambda = \mathbb{C}[S_n]c_\lambda$ is an irreducible representation of S_n (under left multiplication). Furthermore, every irreducible representation of S_n is isomorphic to V_λ for some unique partition λ of n.*

We call V_λ a *Specht module.* Notice that different labellings of the same partition will give different elements of $\mathbb{C}[S_n]$ for the Young symmetrizer. We will always work with tableaux with the canonical labeling. However, in this case, these different Young symmetrizers will give rise to isomorphic Specht modules.

This Young symmetrizer construction may at first seem to come out of nowhere. But recall that a left module is simple if and only if it is a minimal left ideal, and such ideals are generated from primitive idempotents. So a simple left $\mathbb{C}[S_n]$-module (a irreducible representation of S_n) is of the form $\mathbb{C}[S_n]e$ for a primitive idempotent e. We will later show that these Young symmetrizers are primitive idempotents up to scaling. In particular, recall that from Proposition 1.8, as an algebra

$$\mathbb{C}[S_n] \simeq \bigoplus_{\lambda} \operatorname{End}(V_\lambda)$$

where the sum is over all irreducible representations V_λ of S_n. Fix a partition μ of n and pick a basis for V_μ. Notice that the matrix E_{ii} in $\operatorname{End}(V_\mu)$ of all zeros except for a single one in the (i,i)th position i is a primitive idempotent. Moreover, the image c of E_{ii} under the map $\operatorname{End}(V_\mu) \hookrightarrow \bigoplus_\lambda \operatorname{End}(V_\lambda) \to \mathbb{C}[S_n]$ then has the property $\mathbb{C}[S_n]c \simeq \operatorname{End}(V_\mu)E_{ii} \simeq V_\mu$. The choice of i is analogous to the choice of labeling of our tableau.

Example 2.2. We will compute the Young symmetrizers for all partitions of 3 and, thus, all the distinct irreducible representations of S_3. Here are all the canonical tableau.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

For $\lambda = (3)$, we have that $P_{(3)} = S_3$ and $Q_{(3)} = \{1\}$. Hence

$$c_{(3)} = a_{(3)}b_{(3)} = \sum_{g \in S_3} g.$$

For any $h \in S_3$, we have $hc_{(3)} = c_{(3)}$. Therefore, $V_{(3)} = \mathbb{C}[S_3]c_{(3)}$ is the trivial representation.

Now for $\lambda = (2,1)$, we have that $P_{(2,1)} = \{1, (12)\}$ and $Q_{(2,1)} = \{1, (13)\}$. Hence

$$c_{(2,1)} = a_{(2,1)}b_{(2,1)} = (1 + (12))(1 - (13)) = 1 + (12) - (13) - (132).$$

It turns out that $V_{(2,1)} = \mathbb{C}[S_3]c_{(2,1)}$ is the span of $c_{(2,1)}$ and $(13)c_{(2,1)}$.

For $\lambda = (1,1,1)$, we have that $P_{(1,1,1)} = \{1\}$ and $Q_{(1,1,1)} = S_3$. Hence

$$c_{(3)} = a_{(3)}b_{(3)} = \sum_{g \in S_3} \operatorname{sgn}(g)g.$$

For any $h \in S_3$, we have $hc_{(1,1,1)} = \operatorname{sgn}(h)c_{(1,1,1)}$. Therefore, $V_{(1,1,1)} = \mathbb{C}[S_3]c_{(1,1,1)}$ is the alternating representation.

The general case for S_n is that $V_{(n)}$ is the trivial representation, $V_{(1,\dots,1)}$ is the alternating representation, and $V_{(n-1,1)}$ is the *standard representation.* Consider the permutation representation of the natural action of S_n on $\{1, 2, \dots, n\}$. The image of the element of S_n are the familiar permutation matrices. This representation is *not* irreducible because it contains of the proper S_n-invariant subspace U spanned by $1 + 2 + \cdots + n$. But the complementary invariant subspace of U is irreducible.

We will now prove Theorem 2.1 though a sequence of lemmas.

Lemma 2.3. *There exists $t_\lambda \in \mathbb{C}[S_n]^*$ such that $a_\lambda g b_\lambda = t_\lambda(g)c_\lambda$ for all $g \in \mathbb{C}[S_n]$.*

Proof. It suffices to show this for $g \in S_n$. Notice if $g \in P_\lambda Q_\lambda$, we can uniquely write $g = pq$ with $p \in P_\lambda$ and $q \in Q_\lambda$. Thus,

$$a_\lambda g b_\lambda = (a_\lambda p)(q b_\lambda) = a_\lambda(\operatorname{sgn}(q) b_\lambda) = \operatorname{sgn}(q) c_\lambda.$$

Now if $g \notin P_\lambda Q_\lambda$, then notice that the coefficient for the identity 1 in $a_\lambda g b_\lambda$ is 0 because if $pgq = 1$ for some $p \in P_\lambda$ and $q \in Q_\lambda$, then $g = p^{-1}q^{-1} \in P_\lambda Q_\lambda$. Thus, we must show that $\ell_\lambda(g) = 0$. So it suffices to show that there exists a transposition $q \in Q_\lambda$ so that $p = gqg^{-1} \in P_\lambda$ because then

$$a_\lambda g b_\lambda = (a_\lambda p) g(\operatorname{sgn}(q) q b_\lambda) = -a_\lambda(gqg^{-1})gqb_\lambda = -a_\lambda g b_\lambda$$

and, hence, $a_\lambda g b_\lambda = 0$. Consider the tableau $T' = gT$ where T is the original tableau and T' is a tableau of the same shape where each entry i in T is replaced by $g(i)$. Equivalently, we want to show that there exists two distinct integers which lie in the same row in T and in the same column in T'. (This is because p is a row-preserving transposition in T because $p \in P_\lambda$, and $p = gqg^{-1}$ is a column-preserving transposition in T' because $q \in Q_\lambda$.)

So suppose there were not two such integers. We could then find a row-preserving permutation $p_1 \in P_\lambda$ of T and a column-preserving permutation $q_1' \in gQ_\lambda g^{-1}$ of T' so that $p_1 T$ and $q_1' T'$ have the same first row. Continuing this, we could find $p \in P_\lambda$ and $q' = gqg^{-1} \in gQ_\lambda g^{-1}$ so that $pT = q'T' = q'gT = gqT$. But then $p = gq$ and, hence, $g = pq^{-1} \in P_\lambda Q_\lambda$. □

Notice that under the lexicographical order, partitions are totally ordered.

Lemma 2.4. *If $\lambda > \mu$, then $a_\lambda g b_\mu = 0$ for all $g \in \mathbb{C}[S_n]$.*

Proof. Again, it suffices to show this for $g \in S_n$. And similarly, it suffices to find a transposition $q \in Q_\mu$ so that $p = gqg^{-1} \in P_\lambda$ because then

$$a_\lambda g b_\mu = (a_\lambda p) g(\operatorname{sgn}(q) q b_\mu) = -a_\lambda(gqg^{-1})gqb_\mu = -a_\lambda g b_\mu$$

and, hence, $a_\lambda g b_\mu = 0$. Let T be the tableau used to construct a_λ and T' be the tableau used to construct b_μ. Thus, we want to show there exists two distinct integers which lie in the same row of T and in the same column of gT'.

Notice if $\lambda_1 > \mu_1$, then this must be true because of the pigeonhole argument that the λ_1 many entries of the first row of T have μ_1 many possible columns in gT' to lie in. So if $\lambda_1 = \mu_1$ and two such integers did not exist, we could find a row-preserving permutation $p_1 \in P_\lambda$ of T and a column-preserving permutation $q_1' \in gQ_\mu g^{-1}$ of gT' so that $p_1 T$ and $q_1' gT'$ have the same first row.

We can continue this for each consecutive row until we have $\lambda_i > \mu_i$. By the pigeonhole argument, there exists two distinct integers which lie in the same row of $p_{i-1} \cdots p_1 T$ and in the same column of $q_{i-1}' \cdots q_1' gT'$. □

Lemma 2.5. *We have that $c_\lambda c_\lambda = n_\lambda c_\lambda$ where $n_\lambda = n!/\dim V_\lambda$.*

Proof. It follows from Lemma 2.3 that $c_\lambda c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$. Now, consider the map $F : \mathbb{C}[S_n] \to V_\lambda$ where $x \mapsto x c_\lambda$. Notice that F multiplies by n_λ on V_λ, while F multiplies by 0 on $\operatorname{Ker} F$. Therefore, $\operatorname{tr} F = n_\lambda \dim V_\lambda$.

On the other hand, since the coefficient of $1 \in S_n$ in c_λ is 1, the coefficient of g in $F(g) = g c_\lambda$ is also 1. Thus, we also have that $\operatorname{tr} F = \dim \mathbb{C}[S_n] = n!$. □

Lemma 2.6. *For each partition λ of n, V_λ is an irreducible representation of S_n.*

Proof. Notice that $c_\lambda V_\lambda = \mathbb{C}c_\lambda$ by Lemma 2.3 and because $c_\lambda c_\lambda = n_\lambda c_\lambda \neq 0$ by Lemma 2.5 where $c_\lambda c_\lambda \in c_\lambda V_\lambda$. Let $W \subseteq V_\lambda$ be a subrepresentation. There are two cases. If $c_\lambda W = 0$, then $W \cdot W \subseteq V_\lambda \cdot W = 0$ and, hence, $W = 0$. But if $c_\lambda W = \mathbb{C}c_\lambda$, then

$$V_\lambda = \mathbb{C}[S_n]c_\lambda = \mathbb{C}[S_n](c_\lambda W) = (\mathbb{C}[S_n]c_\lambda)W \subseteq W$$

because W is a representation of S_n. Therefore, V_λ is irreducible. □

Lemma 2.7. *If $\lambda \neq \mu$, then $V_\lambda \not\cong V_\mu$.*

Proof. Without loss of generality, assume $\lambda > \mu$. Then by the previous proof along with Lemma 2.4, we have that $c_\lambda V_\lambda = \mathbb{C}c_\lambda$ but $c_\lambda V_\mu = c_\lambda \mathbb{C}[S_n]c_\mu = 0$. □

Now the proof of Theorem 2.1 follows from the previous two lemmas and the fact that partitions of n enumerate all conjugacy classes of S_n, which by Corollary 1.20, enumerate all irreducible representations of S_n.

2.2. **Dimension Formulas.** Although we have given an explicit way to compute all the irreducible representations of S_n, it still takes some work to really see how these representations behave. It is not even clear what the dimension of the irreducibles are. As we saw in the previous section, the most powerful tool for studying representations of finite groups was characters. It turns out that we can compute the characters $\chi_\lambda := \chi_{V_\lambda}$, but this still takes some work.

Some of the proofs of the statements in this subsection are quite involved and not important for our purposes. So we will omit proofs and just focus on the results. All proofs in this subsection can be found in [5].

Frobenius Formula 2.8. *Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n. Let $c = (c_1, \ldots, c_n)$ represent the cycle type of $g \in S_n$, that is, c_i is the number of i-cycles in the disjoint cycle representation of g. Set $t = (t_1, \ldots, t_k)$ where $t_i = \lambda_i + k - i$. Then*

$$\chi_\lambda(g) = [x^t]\left(\Delta(x) \cdot \prod_j p_j(x)^{c_i}\right)$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant, $p_j(x) = \sum_{i=1}^k x_i^j$ are the power sums, and $[x^t]f(x)$ is the coefficient of $x_1^{t_1} \cdots x_k^{t_k}$ in f.

Notice that we can specialize the Frobenius formula to the cycle type $c = (n, 0, \ldots, 0)$ of the identity, and–with some reinterpretation–extract some formulas for $\chi_\lambda(1) = \dim V_\lambda$. The hook length formula is the easiest dimension formula to compute. The *hook length* of a cell in a Young diagram is the number of boxes that are either directly to the right or directly below the cell, counting the cell itself only once. Here are all the hook lengths for $\lambda = (4, 2, 2, 1)$.

7	5	2	1
4	2		
3	1		
1			

Hook Length Formula 2.9. *Let λ be a partition of n. Then*

$$\dim V_\lambda = \frac{n!}{\prod h(i,j)}$$

where the product is over all cells (i, j) in the Young diagram of λ, (i, j) denotes the cell in the ith row and jth column, and $h(i, j)$ is the hook length of cell (i, j).

So for $\lambda = (4, 2, 2, 1)$, we see that $\dim V_\lambda = 9!/(7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2) = 216$.

The next dimensional formula involves the concept of standard Young tableau. We will see that it has great combinatorial significance, at the cost of being more difficult to compute. A *standard* Young tableau has a labeling that is strictly increasing to the right across the rows and strictly increasing down the columns. Here are all five of the standard Young tableaux for $\lambda = (3, 2)$.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$$

Also, by the Hook Length Formula 2.9, we have $\dim V_\lambda = 5$. This is no coincidence.

Proposition 2.10. *Let λ be a partition of n. Then*

$$\dim V_\lambda = |\operatorname{SYT}(\lambda)|$$

where $\operatorname{SYT}(\lambda)$ is the set of standard Young tableaux for λ.

Define $f_\lambda = |\operatorname{SYT}(\lambda)|$. We have seen that $|G| = \sum (\dim V_i)^2$ where the sum is over all irreducible representations V_i of G. For $G = S_n$, this says

$$n! = \sum_{|\lambda|=n} f_\lambda^2. \tag{2.11}$$

2.3. **The RSK-Correspondence.** In this section will describe a remarkable combinatorial interpretation for (2.11) known as the *Robinson-Schensted correspondence*. Recall we have defined f_λ to count the number of standard Young tableau of λ. Incredibly, we can promote the equation (2.11) to a bijection

$$S_n \xrightarrow{\text{RSK}} \coprod_{|\lambda|=n} \operatorname{SYT}(\lambda) \times \operatorname{SYT}(\lambda).$$

But permutations and pairs of standard Young tableau are very different objects, so the fact that a nice bijection exists is far from obvious.

The map RSK is an algorithm that takes in permutations $g = g_1 g_2 \cdots g_n$ in one-line notation and iteratively builds a pair of tableau via a process called row-insertion. An *incomplete standard Young tableau* of n is a Young diagram of m for some $m \leq n$ that is filled by a m-subset of $\{1, 2, \ldots, n\}$ that is increasing to the right across the rows and increasing down the columns. As input, *row-insertion* takes in a positive integer k and an incomplete standard Young tableau P not containing k. It works as follows:

First, look at the first row and find the smallest j (first entry in row) so that $k < P(1, j)$. If no such j exists, then add cell containing k at the end of the row and terminate. Otherwise, if j does exist, take $k' = P(1, j)$ and then replace $P(1, j)$ with k and insert k' into the second row using the same rules. And so on, continue until termination.

For $g = g_1 g_2 \cdots g_n \in S_n$, find $\operatorname{RSK}(g)$ as follows: Start with $(P_0, Q_0) = (\emptyset, \emptyset)$. Given a pair of incomplete standard Young tableau (P_{i-1}, Q_{i-1}). Row-insert g_i into P_{i-1} to get P_i, and add the same newly formed cell from P_i to Q_{i-1} and label it i. In the end, we have $\operatorname{RSK}(g) = (P, Q) := (P_n, Q_n)$.

Example 2.12. Here is the algorithm for $g = 351462 \in S_6$.

$$P: \quad \emptyset \xrightarrow{3} \begin{array}{|c|}\hline 3\\\hline\end{array} \xrightarrow{5} \begin{array}{|c|c|}\hline 3&5\\\hline\end{array} \xrightarrow{1} \begin{array}{|c|c|}\hline 1&5\\\hline 3\\\cline{1-1}\end{array} \xrightarrow{4} \begin{array}{|c|c|}\hline 1&4\\\hline 3&5\\\hline\end{array} \xrightarrow{6} \begin{array}{|c|c|c|}\hline 1&4&6\\\hline 3&5\\\cline{1-2}\end{array} \xrightarrow{2} \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array}$$

$$Q: \quad \emptyset \longrightarrow \begin{array}{|c|}\hline 1\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

Example 2.13. Here are the images for all permutations of S_3 under RSK.

123: $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$

132: $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$

213: $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

231: $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$

312: $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

321: $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$

It turns out that RSK exhibits some (non-evident) marvelous symmetry, that has some unexpected consequences.

Proposition 2.14. *If $g \mapsto (P, Q)$, then $g^{-1} \mapsto (Q, P)$.*

So if g is a involution in S_n, then $g \mapsto (P, P)$. So we obtain the following formula.

Corollary 2.15. *We have that*

$$\# \text{ involutions of } S_n = \sum_{|\lambda|=n} f_\lambda.$$

The next fact, will be about the *longest decreasing subsequence* and *longest increasing subsequence* of a permutation g in one-line notation, denoted $\text{lds}(g)$ and $\text{lis}(g)$ respectively.

Proposition 2.16. *Let $g \mapsto (P, Q)$ and λ be the shape of P (or, equivalently, Q). Then $\text{lis}(g) = \lambda_1$ which is the length of first row of P and, similarly, $\text{lds}(g) = \ell(\lambda)$ which is the length of first column of P.*

Corollary 2.17 (Erdös-Szekeres)**.** *If $g \in S_{pq+1}$, then $\text{lis}(g) > p$ or $\text{lds}(g) > q$.*

Proof. Otherwise, the shape of the image of g would fit in a $p \times q$ square. □

3. Schur-Weyl Duality

3.1. **Representations of Lie Groups and Lie Algebras.** Before we can discuss the main content of this section, we will first have to cover some needed Lie theory.

Recall that a *Lie group* G is a group that is also a smooth manifold where the multiplication map $(g, h) \mapsto gh$ and inverse map $g \mapsto g^{-1}$ are both smooth. A *Lie group homomorphism* is group homomorphism that is also smooth. We will really only need the case $G = \text{GL}(V)$ for a finite dimensional complex vector space V.

A *Lie algebra* $\mathfrak{g}$ is a vector space with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* with following two properties: $[X, X] = 0$ for all $X \in \mathfrak{g}$ and

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Note that $\mathfrak{g}$ along with multiplication given by $[\cdot,\cdot]$ forms a nonassociative algebra. A *Lie algebra homomorphism* $\rho : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map that respects the Lie bracket, i.e.,

$$\rho([X,Y]_{\mathfrak{g}}) = [\rho(X), \rho(Y)]_{\mathfrak{h}}$$

for all $X, Y \in \mathfrak{g}$. Like for Lie groups, we are really only interested in the case of $\mathfrak{gl}(V)$ where V is a finite dimensional complex vector space, which we can identify as $\mathrm{End}(V)$ under the Lie bracket given by the commutator $[X,Y] = XY - YX$.

There is a beautiful connection between Lie groups and Lie algebras. However, we do not have time to fully explain it, and it will not be necessary for later in the section. For details, see Chapter 8 of [3]. For any Lie group G, there is an associated Lie algebra $\mathrm{Lie}(G)$ where as sets, $\mathrm{Lie}(G)$ is the tangent space T_eG at the identity. And for any Lie algebra $\mathfrak{g}$, there is a unique simply connected Lie group G where $\mathrm{Lie}(G) = \mathfrak{g}$. In fact, for any discrete subgroup $\Gamma \leq G$, we have that $\mathrm{Lie}(G/\Gamma) = \mathfrak{g}$. And remarkably, if G is simply connected, then if $\rho : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a Lie group homomorphism, then there exists a unique Lie group homomorphism $f : G \to H$ such that $\rho = (df)_e$.

A *representation* of a Lie group G on a complex vector space W is a Lie group homomorphism $\rho : G \to \mathrm{GL}(W)$. Similarly, a *representation* of a Lie algebra $\mathfrak{g}$ on a complex vector space W is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$. So in the case where G is simply connected, there is an equivalence between representations of G and of $\mathrm{Lie}(G)$.

Recall that for a finite group G, we had an associative algebra $\mathbb{C}[G]$ where representations of $\mathbb{C}[G]$ *are* representations of G and vice versa. More generally, the group algebra of G has the property that for any associative $\mathbb{C}$-algebra A,

$$\mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[G], A) \simeq \mathrm{Hom}_{\mathrm{Grp}}(G, A^{\times})$$

where $A^{\times}$ is the group of units of A. Analogously, for a Lie algebra $\mathfrak{g}$, we have an associative algebra $\mathcal{U}\mathfrak{g}$ called the *universal enveloping algebra* with the property that for any associative $\mathbb{C}$-algebra A,

$$\mathrm{Hom}_{\mathrm{Alg}}(\mathcal{U}\mathfrak{g}, A) \simeq \mathrm{Hom}_{\mathrm{LieAlg}}(\mathfrak{g}, \mathcal{L}A)$$

where $\mathcal{L}A$ is the Lie algebra of the set A with the commutator as the Lie bracket.

We can actually construct $\mathcal{U}\mathfrak{g}$ as a quotient of the tensor algebra $\mathcal{T}\mathfrak{g}$ of which $\mathfrak{g}$ is a subset. The *tensor algebra* $\mathcal{T}\mathfrak{g}$ is the set $\bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$ where $\mathfrak{g}^{\otimes 0} := \mathbb{C}$ with multiplication given by the canonical isomorphism $\mathfrak{g}^{\otimes n} \otimes \mathfrak{g}^{\otimes m} \to \mathfrak{g}^{\otimes (n+m)}$. Then $\mathcal{U}\mathfrak{g} = \mathcal{T}\mathfrak{g}/I$ where I is the ideal generated by elements $X \otimes Y - Y \otimes X - [X,Y]$ for all $X, Y \in \mathfrak{g} \subseteq \mathcal{T}\mathfrak{g}$.

The isomorphism of the above specialized to $A = \mathrm{End}(V)$ says that studying representations of $\mathfrak{g}$ is the same as studying representations of $\mathcal{U}\mathfrak{g}$. For if we have a Lie algebra representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we can extend to an algebra representation $\rho' : \mathcal{U}\mathfrak{g} \to \mathrm{End}(V)$ by setting $\rho'(X_1 \otimes \cdots \otimes X_n) = \rho(X_1) \cdots \rho(X_n)$ and then extending by linearity. Similarly, given an algebra representation $\rho' : \mathcal{U}\mathfrak{g} \to \mathrm{End}(V)$, we can restrict to $\mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$ to get a Lie algebra representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ because

$$\begin{aligned}\rho([X,Y]_{\mathfrak{g}}) &= \rho'([X,Y]_{\mathfrak{g}})\\ &= \rho'(X \otimes Y - Y \otimes X)\\ &= \rho'(X)\rho'(Y) - \rho'(Y)\rho'(X)\\ &= \rho(X)\rho(Y) - \rho(Y)\rho(X)\\ &= [\rho(X), \rho(Y)]_{\mathfrak{gl}(V)}\end{aligned}$$

for all $X, Y \in \mathfrak{g}$.

3.2. **Schur-Weyl Duality for** $\mathrm{GL}(V)$**.** Consider the space $V^{\otimes n}$ for some complex vector space V. We have two natural actions on this space. Since there are n factors, there is the natural (right) action of S_n of permuting the factors, that is, where for all $\pi \in S_n$, we have

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\pi = v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(n)}.$$

And since each factor is V, we have the natural factorwise action of $\mathrm{GL}(V)$ where

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = g(v_1) \otimes g(v_2) \otimes \cdots \otimes g(v_n)$$

for all $g \in \mathrm{GL}(V)$. Notice that these two actions commute with each other. However, these actions have a stronger connection that we will show. Namely, the spans of the images of S_n and $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$ are centralizers of each other. We will see that this fact remarkably connects the representation theory of these two groups. In order to show this connection, we must prove some lemmas.

Lemma 3.1. *The image of* $\mathcal{U}(\mathfrak{gl}(V))$ *in* $\mathrm{End}(V^{\otimes n})$ *is* $B = \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$*.*

Proof. The action of $X \in \mathfrak{gl}(V)$ on $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ is

$$X(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes Xv_i \otimes \cdots \otimes v_n$$

and, thus, the image of X in $\mathrm{End}(V^{\otimes n})$ is

$$\Pi_n(X) := X \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \mathrm{id} \otimes X \otimes \cdots \otimes \mathrm{id} + \cdots + \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes X.$$

Clearly, the image of $\mathfrak{gl}(V)$ and, thus the image of $\mathcal{U}(\mathfrak{gl}(V))$, is contained in B.

Now since the elementary symmetric polynomial $e_n(x) = x_1 x_2 \cdots x_n$ is expressible as $P(p_1(x), p_2(x), \ldots, p_n(x))$ for some polynomial P in the power sum symmetric polynomials $p_j(x) = x_1^j + x_2^j + \cdots + x_n^j$, we get

$$X \otimes X \otimes \cdots \otimes X = P(\Pi_n(X), \Pi_n(X^2), \ldots, \Pi_n(X^n)).$$

Thus, elements of the form $X^{\otimes n}$ for $X \in \mathrm{End}(V)$ are generated by the images of elements in $\mathcal{U}(\mathfrak{gl}(V))$. And since elements of the form $X^{\otimes n}$ for $X \in \mathrm{End}(V)$ span

$$\mathrm{Sym}^n \mathrm{End}(V) \simeq (\mathrm{End}(V)^{\otimes n})^{S_n} \simeq (\mathrm{End}(V^{\otimes n}))^{S_n} = \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n}),$$

then the image of $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$ is B. □

Proposition 3.2. *The images of* $\mathbb{C}[S_n]$ *and* $\mathcal{U}(\mathfrak{gl}(V))$ *in* $\mathrm{End}(V^{\otimes n})$ *are centralizers of each other.*

Proof. Let A be the image of $\mathbb{C}[S_n]$ in $\mathrm{End}(V^{\otimes n})$. By Maschke's Theorem 1.2 (and the fact that quotient of semisimple ring is semisimple), A is semisimple. The rest follows from the Double Centralizer Theorem A.3. □

Lemma 3.3. *The span of the image of* $\mathrm{GL}(V)$ *in* $\mathrm{End}(V^{\otimes n})$ *is* B.

Proof. Since $\mathrm{GL}(V)$ commutes with S_n, then the image of $\mathrm{GL}(V)$, and thus its span, is contained in $B = \mathrm{End}_A(V^{\otimes n})$.

Conversely, let $X \in \mathrm{End}(V)$ and B' be the span of the image of the elements $g^{\otimes n}$ for $g \in G$. Since $X + tI$ is invertible for all but finitely many $t \in \mathbb{C}$, by interpolation, the polynomial $(X + tI)^{\otimes n}$ is in the subalgebra B' for all but finitely many t and, hence, for all t. In particular, for $t = 0$, we have that $X^{\otimes n} \in B'$. But again, we know that these in fact span $B = \mathrm{End}_A(V^{\otimes n})$. □

Therefore, by the Double Centralizer Theorem A.3, we have the following.

Schur-Weyl Duality 3.4. *We have the decomposition*

$$V^{\otimes n} \simeq \bigoplus_{|\lambda|=n} V_\lambda \otimes \mathbb{S}_\lambda V.$$

as a representation of $S_n \times \mathrm{GL}(V)$ *where* V_λ *runs through all the irreducible representations of* S_n *and each* $\mathbb{S}_\lambda V := \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ *is an irreducible representation of* $\mathrm{GL}(V)$ *or is zero.*

In fact, $\mathbb{S}_\lambda V$ is zero when $\lambda_{d+1} \neq 0$ where $d = \dim V$, that is, the number of parts of λ is greater than d.

3.3. **Schur Functors and Algebraic Representations.** We call the covariant functor $\mathbb{S}_\lambda : \mathrm{FinVect} \to \mathrm{FinVect}$ the *Schur functor* of λ. If $f : V \to W$ is a linear map, we define $\mathbb{S}_\lambda f : \mathbb{S}_\lambda V \to \mathbb{S}_\lambda W$ by $(\mathbb{S}_\lambda f)(\Psi) = f^{\otimes n} \circ \Psi$. It can be verified that

$$\mathbb{S}_\lambda(f \circ g) = (\mathbb{S}_\lambda f) \circ (\mathbb{S}_\lambda g) \qquad \text{and} \qquad \mathbb{S}_\lambda \, \mathrm{id}_V = \mathrm{id}_{\mathbb{S}_\lambda}.$$

Since representations of S_n are self-dual by Example 1.15, we get the following more constructive descriptions of the Schur functor.

$$\begin{aligned}
\mathbb{S}_\lambda V &= \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n}) \\
&\simeq (V_\lambda)^* \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \\
&\simeq V_\lambda \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \\
&= \mathbb{C}[S_n] c_\lambda \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \\
&= \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} V^{\otimes n} c_\lambda \\
&\simeq V^{\otimes n} c_\lambda
\end{aligned}$$

Example 3.5. For $\lambda = (n)$, we have that $\mathbb{S}_{(n)} V \simeq \mathrm{Sym}^n V$. Similiarly, for $\lambda = (1, \ldots, 1)$, we have that $\mathbb{S}_{(1,\ldots,1)} V \simeq \mathrm{Alt}^n V$. Note that $\mathrm{Alt}^n V$ is zero if $\dim V < n$. So Schur-Weyl Duality tells us that

$$V^{\otimes n} \simeq \bigoplus_{|\lambda|=n} (\mathbb{S}_\lambda V)^{\oplus f_\lambda}$$

as a representation of $\mathrm{GL}(V)$ where $f_\lambda = \dim V_\lambda$. So, in particular, we have

$$V^{\otimes 2} \simeq \mathrm{Sym}^2 V \oplus \mathrm{Alt}^2 V \qquad \text{and} \qquad V^{\otimes 3} \simeq \mathrm{Sym}^3 V \oplus (\mathbb{S}_{(2,1)} V)^{\oplus 2} \oplus \mathrm{Alt}^3 V.$$

It turns out that $\mathbb{S}_{(2,1)} V$ does not have a description as nice as $\mathrm{Sym}^n V$ or $\mathrm{Alt}^n V$. But it can be shown using Example 2.2, that

$$\mathbb{S}_{(2,1)} V \simeq V^{\otimes 3} c_{(2,1)} \simeq \mathrm{Ker}(V \otimes \mathrm{Alt}^2 V \to \mathrm{Alt}^3 V)$$

where the map $V \otimes \mathrm{Alt}^2 V \to \mathrm{Alt}^3 V$ the canonical map $v_1 \otimes (v_2 \wedge v_3) \mapsto v_1 \wedge v_2 \wedge v_3$.

Recall that any representation of S_n could be decomposed into the Specht modules V_λ. However, it is *not* true that any representation of $\mathrm{GL}(V)$ can be decomposed into the images $\mathbb{S}_\lambda V$ of V under the Schur functors. For example, we cannot get the dual of $\mathbb{S}_\lambda V$. However, we do essentially get the whole class of algebraic representations. Proofs and details of the following can be found in section 5.23 of [2] and in section 15.5 of [3].

A finite dimensional representation W of $\mathrm{GL}(V)$ is *algebraic* if the corresponding map $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$ is a morphism of algebraic varieties. Concretely, this says that if we choose bases, then the coordinates of $\rho(A)$ are rational functions of the coordinates of $A \in \mathrm{GL}(V)$. Any such rational function is, in fact, in $\mathbb{C}[a_{ij}][1/\det A]$.

It turns out that the Schur functors only give us representations which are *polynomial*, that is, the coordinates of $\rho(A)$ are polynomials of the coordinates of $A \in \mathrm{GL}(V)$. Thus, we could never hope to get the algebraic representation $\det^{-1} : \mathrm{GL}(V) \to \mathrm{GL}(\mathbb{C})$ where $\det^{-1}(A) = 1/\det A$. But this is essentially it.

Proposition 3.6. *Let W be an algebraic representation of $\mathrm{GL}(V)$. Then for all $k > 0$ and λ such that $\mathbb{S}_\lambda V \neq 0$, we have that $\mathbb{S}_\lambda V \otimes \det^{-k}$ is an irreducible algebraic representation of $\mathrm{GL}(V)$. Moreover, W decomposes as a direct sum of copies of $\mathbb{S}_\lambda V \otimes \det^{-k}$.*

3.4. **Other Cases of Schur-Weyl Duality.** Note that the key to showing Schur-Weyl Duality for $\mathrm{GL}(V)$ was showing that the images of $\mathbb{C}[S_n]$ and $\mathbb{C}[\mathrm{GL}(V)]$ were centralizers of each other in $\mathrm{End}(V^{\otimes n})$. But there are many other groups that naturally act on $V^{\otimes n}$. So if we can realize the centralizer in $\mathrm{End}(V^{\otimes n})$ of such a group, we can learn more about the representations of that group and its corresponding centralizer. In this section, we will describe a few particularly nice examples of these. As before, we will focus on the results. The proofs of the following material can be found in [1] and [4].

Recall that when we showed Schur-Weyl Duality for $\mathrm{GL}(V)$, we first showed it for $\mathfrak{gl}(V)$ in Proposition 3.2. That is, as a representation of $S_n \times \mathfrak{gl}(V)$,

$$V^{\otimes n} \simeq \bigoplus_{|\lambda|=n} V_\lambda \otimes \mathbb{S}_\lambda V.$$

where V_λ are all the irreducible representations of S_n and each $\mathbb{S}_\lambda V := \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ is an irreducible $\mathfrak{gl}(V)$-representation or zero.

In both these cases, the resulting centralizers have all been the same, the image of $\mathbb{C}[S_n]$. In fact, if we equipped V with an inner product, then we could show that $\mathbb{C}[S_n]$ is still the centralizer of both the image of $\mathfrak{u}(V)$ and the image of $\mathrm{U}(V)$ in $\mathrm{End}(V^{\otimes n})$. But now, we will show some examples where this is not the case. For the remainder of the section, we will often be working with a basis for our underlying vector space. So let $V = \mathbb{C}^d$ with the standard basis.

Let $\mathrm{S}_d(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ denote the group of $d \times d$ permutation matrices. This group is isomorphic to S_d, but we give it different notation because it has a different action on $V^{\otimes n}$. It turns out that the corresponding centralizer for $\mathrm{S}_d(\mathbb{C})$ in $\mathrm{End}(V^{\otimes n})$ can be realized as the image of an algebra called the partition algebra $\mathbb{C}[P_n(d)]$. But we will first have to do a little work to describe this algebra.

First, we will define the *partition monoid* P_n. As a set,

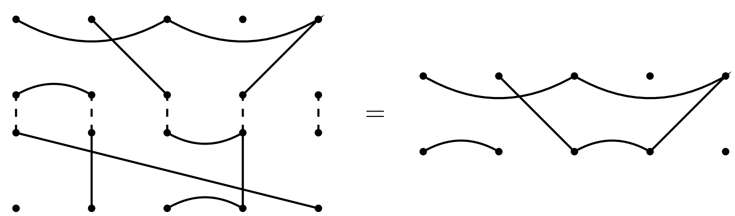
$$P_n = \{\text{set partitions of } \{1, 2, \ldots, n, 1', 2', \ldots, n'\}\}.$$

We will represent a set partition $S \in P_n$ by a graph written as a top row of n vertices corresponding to $1, 2, \ldots, n$ and bottom row of n vertices corresponding to $1', 2', \ldots, n'$ with the property that the collection of path connected components is S, that is, the vertices i, j are connected by a path if and only if i, j are in a same block in S. Clearly, such a graph is not unique for a given set partition S, but any representative will work. This freedom will allow us to draw less cluttered diagrams.

We will define the composition $S_1 \circ S_2$ of two set partitions $S_1, S_2 \in P_n$ as the set partition corresponding to the diagram formed by stacking a diagram of S_1 on top of a diagram of S_2 and identifying each vertex on the bottom row of S_1 with the corresponding vertex on the top row of S_2. The composition will have the diagram with the top row as the top row of S_1 and the bottom row as the bottom row of S_2, ignoring the middle vertices but not the paths through them.

For example, if $S_1 = \{1354', 23', 4, 1'2', 5'\}$ and $S_2 = \{15', 22', 343'4', 5, 1'\}$, then $S_1 \circ S_2 = \{12353'4', 4, 1'2', 5'\}$ because

$S_1 \circ S_2 =$



Now we define the *partition algebra* $\mathbb{C}[P_n(d)]$ as the set $\mathbb{C}[P_n]$ with multiplication on the basis given by $S_1 S_2 = d^{\ell} \cdot (S_1 \circ S_2)$ for $S_1, S_2 \in P_n$ where ℓ is the number of middle-only blocks of the $3n$ set partition (top, middle, and bottom vertices) induced from the composition. In our previous example, we have $\ell = 1$ because $5'_1 = 5_2$ (identification of $5'$ from S_1 with 5 from S_2) is the only block containing only middle vertices (namely, just itself). So $S_1 S_2 = d \cdot (S_1 \circ S_2)$.

Let $S \in P_n$. For convenience, now label i' as $n + i$ for $1 \leq i \leq n$ in the set partition S. For a sequence $i_1, i_2, \ldots, i_{2n}$ with $1 \leq i_j \leq d$ define

$$\delta(S)^{i_1,\ldots,i_n}_{i_{n+1},\ldots,i_{2n}} = \begin{cases} 1, & \text{if } i_j = i_k \text{ whenever } j \text{ and } k \\ & \text{are in the same block in } S, \\ 0, & \text{otherwise} \end{cases}$$

Let $v_1, \ldots, v_d$ be the standard basis for $V = \mathbb{C}^d$. We can then define the action of S on the basis elements $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$ of $V^{\otimes n}$ where $1 \leq i_j \leq d$ as

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n})S = \sum_{i_{d+1},\ldots,i_{2n}} \delta(S)^{i_1,\ldots,i_n}_{i_{n+1},\ldots,i_{2n}} v_{i_{n+1}} \otimes v_{i_{n+2}} \otimes \cdots \otimes v_{i_{2n}}$$

where the sum is over all sequences $i_{n+1}, \ldots, i_{2n}$ with $1 \leq i_j \leq d$. This action of P_n on $V^{\otimes n}$ actually extends to an action of $\mathbb{C}[P_n(d)]$.

Proposition 3.7. *The spans of the images of* $\mathbb{C}[P_n(d)]$ *and* $\mathrm{S}_d(\mathbb{C})$ *in* $\mathrm{End}(V^{\otimes n})$ *are centralizers of each other.*

Therefore, we can get a version of Schur-Weyl Duality between the algebra $\mathbb{C}[P_n(d)]$ and the group $\mathrm{S}_d(\mathbb{C}) \simeq S_d$. It turns out that P_n has some subsets whose generated subalgebras give other interesting cases of Schur-Weyl Duality.

Take $S_n \subseteq P_n$ to be the set partitions where each block of is the form $\{i, j'\}$ where $i, j \in \{1, 2, \ldots, n\}$. These partitions are identified with bijections on $\{1, 2, \ldots, n\}$. Moreover, partition composition is exactly function composition and the induced action is same as that of the symmetric group. So as the notation suggests, we have that the subalgebra of $\mathbb{C}[P_n(d)]$ generated by S_n is isomorphic to the group algebra $\mathbb{C}[S_n]$ of the symmetric group. Note d does not matter since $\ell = 0$ for any composition. So in this case, we recover the original Schur-Weyl Duality for $\mathrm{GL}(V)$.

Now take $B_n \subseteq P_n$ to be the set partitions where each block is of size two (not necessarily of the previous form). Then B_n is closed under composition. And, we call the the subalgebra $\mathbb{C}[B_n(d)]$ of $\mathbb{C}[P_n(d)]$ generated by B_n the *Brauer algebra*. This gives rise to two interesting cases of Schur-Weyl Duality.

Proposition 3.8. *The spans of the images of* $\mathbb{C}[B_n(d)]$ *and* $\mathrm{O}_d(\mathbb{C})$ *in* $\mathrm{End}(V^{\otimes n})$ *are centralizers of each other where* $\mathrm{O}_d(\mathbb{C})$ *is the orthogonal group.*

Proposition 3.9. *The spans of the images of* $\mathbb{C}[B_n(-2d)]$ *and* $\mathrm{Sp}_{2d}(\mathbb{C})$ *in* $\mathrm{End}(V^{\otimes n})$ *are centralizers of each other where* $\mathrm{Sp}_{2d}(\mathbb{C})$ *is the symplectic group.*

Appendix A. Semisimple Algebras and Double Centralizer Theorem

Recall that a module is said to be *simple* if it contains no proper nonzero submodules. And a module is said to be *semisimple* if it can be decomposed as a direct sum of simple submodules. Note that any module of an algebra over a field k is a vector space over k, and hence it makes to talk about the dimension of the module. We say that an algebra is *semisimple* if all of its finite dimensional modules are semisimple. The following theorem is Maschke's Theorem 1.2 reformulated in the language of the group algebra.

Maschke's Theorem A.1. *Let G be a finite group. Then $\mathbb{C}[G]$ is semisimple.*

The following is an important theorem on the structure of semisimple algebras. However, the proof is quite involved, so we will have to omit it. For full details, see Theorem 3.5.4. of [2].

Theorem A.2. *Let A be a finite dimensional algebra. Then A has finitely many simple modules U_i up to isomorphism. These simple modules are finite dimensional. Moreover, A is semisimple if and only if as an algebra,*

$$A \simeq \bigoplus_i \mathrm{End}(U_i).$$

where U_i are simple A-modules.

As a corollary of the previous two theorems, we get Proposition 1.8. Now, the main result of this section.

Double Centralizer Theorem A.3. *Let V be a finite dimensional vector space, A be a semisimple subalgebra of* $\mathrm{End}(V)$*, and* $B = \mathrm{End}_A(V)$*. Then:*

(1) B is semisimple.
(2) $A = \mathrm{End}_B(V)$.
(3) As a module of $A \otimes B$, we have the decomposition

$$V \simeq \bigoplus_i U_i \otimes W_i$$

where U_i are all the simple modules of A and $W_i := \mathrm{Hom}_A(U_i, V)$ are all the simple modules of B.

Proof. Since A is semisimple, we have the A-module decomposition

$$V \simeq \bigoplus_i U_i \otimes \mathrm{Hom}_A(U_i, V) \tag{A.4}$$

where $a \in A$ acts on $U_i \otimes \mathrm{Hom}_A(U_i, V)$ by $a(u \otimes f) = au \otimes f$. The space $W_i := \mathrm{Hom}_A(U_i, V)$ is the *multiplicity space* of U_i. Then as algebras, we have the natural isomorphisms

$$A \simeq \bigoplus_i \mathrm{End}(U_i)$$

and by properties of module homomorphisms,

$$\begin{aligned} B &= \mathrm{End}_A(V) \\ &\simeq \mathrm{Hom}_A\left(\bigoplus_i U_i \otimes W_i, V\right) \\ &\simeq \bigoplus_i \mathrm{Hom}_A(U_i \otimes W_i, V) \\ &\simeq \bigoplus_i \mathrm{Hom}_A(W_i \otimes U_i, V) \\ &\simeq \bigoplus_i \mathrm{Hom}(W_i, \mathrm{Hom}_A(U_i, V)) \\ &= \bigoplus_i \mathrm{End}(W_i). \end{aligned}$$

We now just need to show that W_i are simple B-modules. We will show that B acts transitively on the nonzero maps in $\mathrm{Hom}_A(U, V)$ where U is a simple A-module. Fix a nonzero $u \in U$. Since U is simple, any map $f \in \mathrm{Hom}_A(U, V)$ is determined by where it takes u because Au is a nonzero submodule of U and hence $Au = U$. So take $f, f' \in \mathrm{Hom}_A(U, V)$ where $f(u) = v$ and $f'(u) = v'$. Since Av is an invariant subspace of V, we can write $V = (Av) \oplus W$ for a complementary invariant subspace W by Maschke's Theorem. Define $T : V \to V$ by $T(av) = av'$ for $av \in Av$ and $T(w) = w$ for $w \in W$. This is an A-homomorphism where $T \circ f = f'$.

By Theorem A.2, we see that B is semisimple, and so we have (1). And we can now view (A.4) as a decomposition of V into simple B-modules W_i where U_i are the corresponding multiplicity spaces. Then, as B-modules,

$$V \simeq \bigoplus_i W_i \otimes \mathrm{Hom}_B(W_i, V) \simeq \bigoplus_i W_i \otimes U_i$$

where the sums run over all simple B-modules W_i. This then implies (3). Also, we have that $U_i \simeq \mathrm{Hom}_B(W_i, V)$, and hence (2) follows. □

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