THE FUNDAMENTAL GROUP AND CW COMPLEXES

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Abstract. This paper is a quick introduction to some basic concepts in Algebraic Topology. We start by defining homotopy and delving into the Fundamental Group of a topological space and Van Kampen's Theorem. The fundamental group is then generalized into homotopy groups which lets us define a weak homotopy equivalence. Lastly, we introduce CW complexes and two theorems (Whitehead's Theorem and CW Approximations) that suggest the importance of CW complexes in general. This paper assumes some background in Group Theory and some familiarity with Topological Spaces.

Contents

1. Fundamental Group

1.1. **Building Blocks.** In order to talk about the fundamental group, we must first build up the concept of loops and explore what makes them similar to one another. For the following, we will use A and X as topological spaces and I to denote the closed interval $[0,1]$. In addition, we will assume continuity in the topological sense. A majority of this section will derive from the lecture notes of Cavalieri, Renzo [1] and *A Concise Course in Algebraic Topology* by May, Peter [3].

Definition 1.1. A **path** is a continuous function $f : I \to A$. A path is called a **loop** if $f(0) = f(1)$.

Definition 1.2. A **homotopy** between two maps $f, g : X \to Y$ is a continuous function $h : X \times I \to Y$ such that $h(s,0) = f(s)$ and $h(s,1) = g(s)$.

Definition 1.3. Two maps $f, g : X \to Y$ are **homotopic** if there exists a homotopy between the two maps. We denote that two maps are homotopic by writing $f \simeq g$. Two paths f', g' are **path homotopic** if $f' \simeq g'$ with the additional condition that $f'(0) = g'(0)$ and $f'(1) = g'(1)$.

Date: 29 August 2016.

Since it will be clear in the following contexts whether two paths are simply homotopic or path homotopic, we will denote both by the same symbol $\simeq$.

We can use the concept of a homotopic map to now define how topological spaces can be similar to each other.

Definition 1.4. Two topological spaces X, Y are **homotopy equivalent** if there exist two maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$. We denote that two spaces are homotopy equivalent by writing $X \simeq Y$

As an exercise, we demonstrate a homotopy equivalence between S^1 and $\mathbf{R}^2 \setminus \{(0,0)\}$.

Example 1.5. $S^1 \simeq \mathbf{R}^2 \setminus \{(0,0)\}$

Proof. Let us define the function $f : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$ as the inclusion of S^1 in $\mathbf{R}^2 \setminus \{(0,0)\}$ and $g : \mathbf{R}^2 \setminus \{(0,0)\} \to S^1$ as follows:

$$f \circ g\,(x, y) = \left(\frac{x}{\|(x,y)\|}, \frac{y}{\|(x,y)\|} \right)$$

We can easily see that $g \circ f = \mathrm{id}_{S^1}$.

Now, we just need to show that $f \circ g \simeq \mathrm{id}_{\mathbf{R}^2 \setminus \{(0,0)\}}$. Let $h : \mathbf{R}^2 \setminus \{(0,0)\} \times I \to \mathbf{R}^2 \setminus \{(0,0)\}$ such that

$$h((x,y),t) \mapsto \left(x \cdot \frac{1 - t(1 - \|(x,y)\|)}{\|(x,y)\|}, y \cdot \frac{1 - t(1 - \|(x,y)\|)}{\|(x,y)\|} \right)$$

Then, we can see that h is a homotopy between g and $\mathrm{id}_{\mathbf{R}^2 \setminus \{(0,0)\}}$ which lets us conclude that $f \circ g \simeq \mathrm{id}_{\mathbf{R}^2 \setminus \{(0,0)\}}$

Therefore, $S^1 \simeq \mathbf{R}^2 \setminus \{(0,0)\}$ □

In the context of algebraic topology, treating the class of homotopic paths as one equivalence class lets us isolate common traits between paths without having to individually demonstrate specific traits for each path. However, in order to show that we can form such an equivalence class, we must first show that homotopy is, indeed, an equivalence relation.

Proposition 1.6. Homotopy is an equivalence relation.

Proof. In order to demonstrate this proposition, we need to show that homotopy is reflexive, symmetric, and transitive.

Given any map $f : X \to Y$, we can easily see that $f \simeq f$ since we can define a constant homotopy $h : X \times I \to Y$ such that for all $t \in I$, $h(x,t) = f(x)$.

If $f \simeq g$ and h is a homotopy between f and g, we can define h^{-1} as $h^{-1}(x,t) = h(x, 1-t)$. Then, h^{-1} is a homotopy from g to f, so $g \simeq f$.

If $f \simeq g$ where h is a homotopy from f to g and $g \simeq m$ where h' is a homotopy from g to m, we can define a function H as the piecewise function:

$$H(x,t) = \begin{cases} h(x, 2t) & \text{if } t \leq \frac{1}{2} \\ h'(x, 2t-1) & \text{if } t > \frac{1}{2} \end{cases}$$

We can see that H is continuous since $h(x,1) = g(s) = h'(x,0)$ by definition. Thus, $f \simeq m$.

Therefore, $\simeq$ is an equivalence relation. □

Given a path f in a topological space, we define the equivalence class of all paths that are homotopic to f with the notation $[f]$.

Now, we can finally introduce fundamental groups.

1.2. **Fundamental Group.**

Definition 1.7. Given a topological space X and a point $x \in X$, we define $\pi_1(X, x)$ as the set of all equivalence classes of loops $[f]$ such that $f(0) = x$.

$\pi_1(X, x)$ is the set that underlies the fundamental group. The final step to completing our construction of the fundamental group is to show that $\pi_1(X, x)$ is a group. However, we have yet to introduce an operation on paths that can act as the group operator. Hence, we define concatenation.

Definition 1.8. For two paths f, g, we define concatenation $\cdot : F \times F \to F$ (F is the set of all paths) as follows:

$$(f \cdot g)(t) = \begin{cases} f(2t) & \text{if } t \leq \frac{1}{2} \\ g(2t-1) & \text{if } t > \frac{1}{2} \end{cases}$$

Something to consider is that $f \cdot g$ is a path if and only if $f(1) = g(0)$ since otherwise concatenation would not be well-defined. We do not have to worry about this fact since we will be working with loops which all share a common starting and ending point.

For equivalence classes, we define $[f]\cdot[g] = [f\cdot g]$. By using piecewise functions, we can see that this definition is well-defined. For example, if $f, f' \in [f]$ and $g, g' \in [g]$ and $[f'] \cdot [g'] = [f' \cdot g']$ and $[f] \cdot [g] = [f \cdot g]$, we know there exist homotopies h, h' from $f \to f'$ and $g \to g'$ respectively, so we can create a homotopy between $f \cdot g$ and $f' \cdot g'$ by concatenating h and h'. Thus, $[f \cdot g] = [f' \cdot g']$

Now, we finalize our characterization of the Fundamental Group.

Proposition 1.9. $\pi_1(X, x)$ is a group under concatenation.

Proof. To demonstrate this, we need to show that $\pi_1(X, x)$ under concatenation satisfies the group axioms. To continue, let $[f], [g], [q] \in \pi_1(X, x)$.

- **Closure** The concatenation of two paths is a path and since each path begins and ends on x, the concatenated path starts and ends on x. Thus, $[f] \cdot [g] = [f \cdot g] \in \pi_1(X, x)$.
- **Identity** Let $c_x : I \to X$ such that $c_x(t) = x$ (the constant function). Then, we can define the function $h : I \times I \to A$ such that:

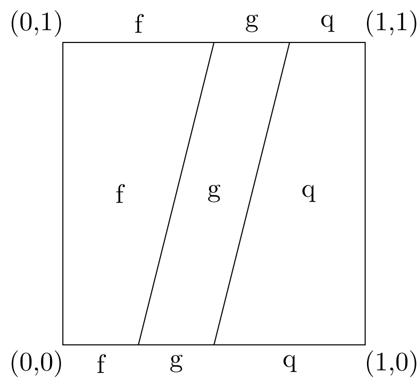
  $$h(i, t) = \begin{cases} f(t(1+i)) & \text{if } t(1+i) < 1 \\ x & \text{if } t(1+i) \geq 1 \end{cases}$$

  We can see that h is a homotopy between f and $f \cdot c_x$ since h is a continuous function and $h(s, 0) = f$ and $h(s, 1) = f \cdot c_x$. Therefore, $f \simeq f \cdot c_x$, so $[f] \cdot [c_x] = [f]$. Similarly, $[c_x] \cdot [f] = [f]$, so we can conclude that $[c_x]$ is the identity element.
- **Inverse** We define f^{-1} as $f^{-1}(s) = f(1-s)$. Then, we can see that $f \cdot f^{-1}$ is a loop. We define another function as follows:

  $$h(i, t) = \begin{cases} f(2it) & \text{if } t < \frac{1}{2} \\ f^{-1}(2i(1-t)) & \text{if } t \geq \frac{1}{2} \end{cases}$$

h takes the midpoint of the loop $f \cdot f^{-1}$ and traces it back along f until it reaches $f(0)$. Since h is a continuous function and $h(i, 0) = c_x$ and $h(i, 1) = f \cdot f^{-1}(i)$, $[f] \cdot [f^{-1}] = [c_x]$.

- **Associativity** We can use a similar method as above to find an explicit homotopy between $(f \cdot g) \cdot q$ and $f \cdot (g \cdot q)$ to show that $([f] \cdot [g]) \cdot [q] \simeq [f] \cdot ([g] \cdot [q])$, but that process is rather tedious. Instead, we can draw the domain square of our homotopy to illustrate the process:



The bottom of the domain square corresponds to the loop $(f \cdot g) \cdot q$ and the top of the domain square corresponds to the loop $f \cdot (g \cdot q)$. Thus, the homotopy h with the above domain square shows that $\pi_1(X, x)$ is associative under concatenation.

□

As an exercise, let us compute the fundamental group of a disc.

Example 1.10. The fundamental group of a disc is isomorphic to the trivial group.

Proof. This fact is almost trivial since given any loop within a disc, we can continuously contract the loop into a single point. In fact, we can see that any loop f is homotopic to c_x by $h(i, t) = f(t)(1 - i) + c_x(t)i$ which is called the **straight-line homotopy**. Therefore, the fundamental group is $\pi_1(D^2, x) = \{[c_x]\}$ which is isomorphic to the trivial group. □

Although we could try to find the fundamental group for any given topological space via brute force, this method becomes impractical very quickly. Luckily, if we are able to divide the given space into overlapping, simpler spaces, we can use the simpler spaces' fundamental groups to build the original space's fundamental group. However, to approach the theorem at hand, we first need to explore some properties of the fundamental group; namely, the induced homomorphism.

The rest of this paper will derive heavily from *Algebraic Topology* by Hatcher, Allen [2].

Lemma 1.11. If (X, x_0) and (Y, y_0) are topological spaces with a continuous map $\phi : (X, x_0) \to (Y, y_0)$, then $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $\phi_*([f]) = [\phi \circ f]$ is a homomorphism.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Notice that for any continuous map, $\phi \circ (f \cdot g) = (\phi \circ f) \cdot (\phi \circ g)$. Thus, we can see that

$$\phi_*([f]\cdot[g]) = \phi_*([f\cdot g]) = [\phi\circ(f\cdot g)] = [(\phi\circ f)\cdot(\phi\circ g)] = [\phi\circ f]\cdot[\phi\circ g] = \phi_*([f])\cdot\phi_*([g])$$

□

In addition, note that $\phi_*([f])$ is well-defined since if $f, f' \in [f]$ where h is a homotopy between f and f', then $\phi \circ h(s)$ is a homotopy between $\phi \circ f$ and $\phi \circ f'$. Thus, $\phi_*([f]) = \phi_*([f'])$.

From now on, we will mostly be working with spaces with a fixed basepoint $x_0 \in X$, so any reference to the homotopy group as $\pi_1(X)$ should be understood as $\pi_1(X, x_0)$.

For the following proof, $i_{\alpha\beta}$ is the homomorphism $\pi_1(X_\alpha \cap X_\beta) \to \pi_1(X_\alpha)$ induced by the inclusion map $X_\alpha \cap X_\beta \to X_\alpha$. $*$ refers to the free product and $*_i$ is the free product of the sets indexed by i.

Theorem 1.12 (Van Kampen's Theorem). Given a topological space $X = \bigcup X_\alpha$ where each X_α is a path connected open set containing $x_0 \in X$ and each $X_\alpha \cap X_\beta$ is path connected, the homomorphism $\Phi : *_\alpha\, \pi_1(X_\alpha) \to \pi_1(X)$ is surjective. If each $X_\alpha \cap X_\beta \cap X_\gamma$ is also path connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ where $\omega \in \pi_1(X_\alpha \cap X_\beta)$ which means that Φ induces an isomorphism $\pi_1(X) \approx *_\alpha\, \pi_1(X_\alpha)/N$.

Proof. First, let f be a loop in X starting at x_0 which is chosen to be in $\bigcap X_\alpha$. Then, we assert that there exists a finite partition of I, $0 = s_0 < s_1 < \cdots < s_k = 1$, such that for all $m \in [k]$, $\exists\, \alpha$ such that $f([s_{m-1}, s_m]) \subset X_\alpha$. This fact becomes clear once we note that for all $i \in I$, there exists an open interval $I_i \subset I$ such that $\exists\, \alpha$ where $f(I_i) \subset X_\alpha$. Since I is compact, there is a finite cover of I composed of I_i, so our partition can be acquired by getting the limit points of every open interval.

Let $S_i = [s_{i-1}, s_i]$ and X_i be the subspace that S_i maps to. Then, thanks to our path-connected condition, there exists a path from $f(s_i)$ to x_0 that lies within $X_i \cap X_{i+1}$ which we will call g_i. g'_i will denote $g(1 - i)$. Then, we can concatenate f in the following manner to get a series of loops:

$$(f(S_1) \cdot g_1) \cdot (g'_1 \cdot f(S_2) \cdot g_2) \cdot \cdots \cdot (g'_{k-1} \cdot f(S_k))$$

This expansion of f is a homotopy of f to a word in $*_\alpha\, \pi_1(X_\alpha)$, so we have proven that Φ is surjective.

The second portion of the proof is completed by considering factorizations of $[f] \in \pi_1(X)$. We define a factorization of $[f]$ as some concatenation of loops $[f_1] \cdot [f_2] \cdot \cdots \cdot [f_n]$ where f_i is a loop in some X_α and f is homotopic to $f_1 \cdot f_2 \cdot \cdots \cdot f_n$. We say that two factorizations are equivalent if they can be made equal by the following two transformations and their inverses:

- Combine two adjacent terms if they both belong to the fundamental group of the same X_α
- Regard a term $[f_i]$ in the fundamental group of X_α as an element in the fundamental group of X_β if it belongs in the fundamental group of $X_\alpha \cap X_\beta$

A series of these transformations may change the element's place in the free product of the subsets, but not within the quotient group with the normal subset. In fact,

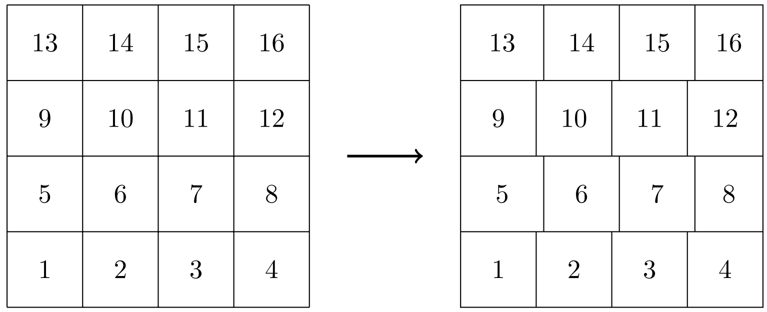


Figure 1. Altering of the domain square if the square were originally partitioned into 16 rectangles

if we can show that two factorizations of $[f]$ are equivalent, then the second half of our proof will be complete.

Let $p = [f_1] \cdot [f_2] \cdot \cdots \cdot [f_n]$ and $q = [f_1'] \cdot [f_2'] \cdot \cdots \cdot [f_m']$ be factorizations of $f \in \pi_1(X, x_0)$. Since p and q are both in the same homotopy class $[f]$, there exists a homotopy $h : I \times I \to X$ between p and q. Let us draw the domain of h as a unit square and, much like what we did in the first part of this proof, partition the square such that each rectangle maps into one X_α by h. Currently, all the corners of the rectangles that are not on the boundary of the square lie in the intersection of four rectangles. These points could be mapped to four separate X_α whose intersection might not be path-connected. Our proof relies on each point being mapped into three rectangles at most, so we must slightly alter the partition of our domain square. Note that since each X_α is an open set, the set $X_\alpha \cap X_\beta$ is also open. We can thus shift any rectangle's dimension by some ϵ amount and still have each rectangle's image lie within some X_α. We use this method and increase the horizontal dimension of every other row of rectangles as depicted in Figure 1.

Next, we define γ_k to be a path on $I \times I$ from the left edge to the right edge such that the first k rectangles (where we begin counting from the bottom left rectangle to the right then up) are divided from the rest of the rectangles by γ_k. Observe that $h|_{\gamma_k}$ is a loop in X since the left and right edge both map to x_0. The last step to setting up this proof is to homotope to x_0 each vertex that is not mapped to x_0 by using a similar fashion to the first half of this proof: by concatenating paths to x_0 and back. Then, we obtain a factorization of $[h|_{\gamma_k}]$. In addition, we can homotope each $h|_{\gamma_k}$ to $h|_{\gamma_{k+1}}$ by "pushing" $h|_{\gamma_k}$ across the $k+1$-th rectangle. In fact, this shows that the factorization we get for $[h|_{\gamma_k}]$ and $[h|_{\gamma_{k+1}}]$. Since equivalence of factorizations are transitive and symmetrical, we can see that γ_0 is equivalent to γ_t where t is the total number of rectangles. Lastly, we note that γ_0 is equal to p and γ_t is equal to q. Thus, our proof is complete. $\square$

2. Higher Homotopy Groups

Until now, we have been only analyzing loops using fundamental groups. However, the fundamental group is insufficient when analyzing higher dimensional features of a topological space. By defining homotopy groups in general, we gain access to higher dimensions.

Definition 2.1. The homotopy group of $n \geq 1$ dimensions, $\pi_n(X, x_0)$, is composed of based homotopy classes of maps $f : (S^n, s_0) \to (X, x_0)$ where s_0 is the basepoint in S^n and is mapped to x_0.

A complication arises when defining the group operation for higher homotopy groups. Since we are no longer dealing with a one-dimensional domain and instead working with unit n-dimensional cubes, concatenation is not well-defined for higher homotopy groups. We circumvent this problem by "attaching" two faces of n-dimensional cubes together. For example, if $f, g \in \pi_n(X)$, then

$$f \cdot g(t_1, t_2, \cdots, t_n) = \begin{cases} f(2t_1, t_2, t_3, \cdots, t_n) & \text{if } t_1 < \frac{1}{2} \\ g(2t_1 - 1, t_2, t_3, \cdots, t_n) & \text{if } t_1 \geq \frac{1}{2} \end{cases}$$

This definition of higher homotopy classes preserves our definition of the fundamental group $\pi_1(X, x_0)$. We have merely changed our perspective of loops into maps $(S^1, s_0) \to (X, x_0)$.

Ultimately, most of the properties of fundamental groups, such as induced homomorphisms, are mirrored by higher homotopy groups. However, Van Kampen's Theorem is not mirrored by higher homotopy groups. We cannot calculate higher homotopy groups by subdividing our topological space.

Although more difficult to calculate, the concept of higher dimensional homotopy groups provides another way to categorize topological spaces, namely, by weak homotopy equivalence.

Definition 2.2. Two topological spaces X, Y are **weak homotopy equivalent** if there exists a map $f : X \to Y$ such that for all $n \in \mathbf{N}$, f induces an isomorphism $\pi_n(X, x) \to \pi_n(Y, f(x))$ for all $x \in X$.

This definition of a weak homotopy equivalence is not quite complete in that the definition is not sufficient for weak homotopy to be an equivalence relation. However, for the purposes of this paper, the definition above is sufficient.

Note that a homotopy equivalence is stronger than a weak homotopy equivalence, as one may conclude simply by their respective names. In fact, we can prove this claim.

Theorem 2.3. If $f : X \to Y$ is a homotopy equivalence, then f is a weak homotopy equivalence.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be corresponding homotopy equivalences. Then, we know that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. We can see that the induce homomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ and $g_* : \pi_n(Y, y_0) \to \pi_n(X, g(y_0))$ compose into isomorphisms $f_* \circ g_* : \pi_n(Y, y_0) \to \pi_n(Y, f \circ g(y_0))$ and $g_* \circ f_* : \pi_n(X, x_0) \to \pi_n(X, g \circ f(x_0))$. We can then observe from the chain below that f_* and g_* are both isomorphisms:

$$\pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \to \pi_n(X, (g \circ f)(x_0)) \to \pi_n(Y, (f \circ g \circ f)(x_0))$$

Thus, the homomorphisms induced by f for homotopy groups are isomorphisms, so f is a weak homotopy equivalence. $\square$

However, weak homotopy equivalence has even more gaps since weak homotopy equivalence is not even symmetric. In other words, the existence of a weak homotopy equivalence $X \to Y$ does not guarantee a corresponding weak homotopy equivalence from $Y \to X$. Thus, a weak homotopy equivalence is usually not a good method of categorization. However, CW complexes are an exception to this weakness.

3. CW Complex

A CW Complex is a way of building up a topological space by attaching the boundary of n-dimensional discs onto k-dimensional spheres where $k < n$. The structures are defined inductively as below:

Definition 3.1. A **CW complex** X is constructed inductively starting from X^0 which is a collection of points. X^{n+1} is constructed by using a collection of attaching maps of the form $j_\alpha : S^n \to X^n$ and producing the quotient space of the disjoint union $X^n \sqcup_\alpha D_\alpha^{n+1}$ where $x \in \partial D_\alpha^n$ is identified with $j_\alpha(x)$.

Definition 3.2. An open disc $D^n \setminus \partial D^n$ that has been attached to X^{n-1} is called an **n-cell**. $X^n \subset X$ is called an **n-skeleton**.

We can start getting a better mental grasp of CW complexes by building simple CW complexes from scratch.

Example 3.3 (S^2 as a CW complex)**.** Although there are many ways to express S^2 as a CW complex, the simplest construction begins with a single point $X^0 = \{x\}$. We do not need any 1-cells, so instead, we go straight to building our 2-cells. Note that since we have no 1-cells, $X^0 = X^1$. Now, we use a map $j : \partial D^2 \to X^1$ where for all $d \in \partial D^2$, $j : d \mapsto x$. This completes our construction of S^2.

In fact, from this example, we can extrapolate and see that S^n can be represented as a CW complex with a single point and one n-cell. Another relatively simple example is the construction of a torus.

Example 3.4 (Torus as a CW complex)**.** In order to build up a torus, we can, once again, start off with just one vertex x. Then, we use two maps $j_1, j_2 : \partial D^1 \to X^0$ to attach two D^1 (which is just a line segment) onto the point x. Visually, this looks like two rings that share a point. Let us call name the rings S_1 and S_2. We then use one last attaching map $j_3 : \partial D^2 \to X^1$ where the loop that corresponds to the attaching map is $S_1 \cdot S_2 \cdot S_1^{-1} \cdot S_2^{-1}$. This completes our construction of a torus.

It is important to note while considering CW complexes that one revolution around the boundary of the attaching disc does not necessarily correspond to one revolution in the image of the loop we are attaching the disc to. For example, we can attach D^2 onto S^1 such that one revolution around ∂D^2 corresponds to two revolutions around S^1 to create a CW complex X. Then, our topological space is completely different from just simply attaching D^2 onto S^1 (which we'll call Y) since one loop around the edge of Y belongs to the same homotopy class as the constant loop in Y while a similar loop in X does not share the same equivalence class as the constant loop. In fact, when building a CW complex, attaching a disc to a loop l in some dimension k provides a continuous space for l to be mapped to the constant loop, thereby merging $[l]$ and $[c]$ in $\pi_k(X)$. Another name for X is RP^2 or the real projective plane.

Definition 3.5. A **CW pair**, (X, A) refers to a CW complex X and a subcomplex A. A **subcomplex** $A \subset X$ is a closed subspace that is the union of a collection of cells of X.

The notion of CW pairs allows us to talk about stronger maps between two CW complexes. CW complexes are inductive by nature, so the construction of a CW complex involves several subcomplexes which act as skeletons or subspaces of skeletons. Thus, given two CW complexes and corresponding subcomplexes $A \subset X, B \subset Y$, it would be useful to talk about maps that not only map $X \to Y$, but also map $A \to B$. Thus, when we have a map $f : (X, A) \to (Y, B)$, the implication is that $f(A) \subset B$.

As intuitive as CW complexes may be initially, their powerful nature becomes clear with the consideration of two major theorems about CW complexes called Whitehead's Theorem and CW Approximations.

4. Whitehead's Theorem

As mentioned above, a weak homotopy equivalence's "weakness" does not apply to CW complexes. In fact, a weak homotopy equivalence is powerful enough to be a homotopy equivalence for CW complexes which is what Whitehead's Theorem states. However, before getting to the proof of Whitehead's Theorem, we need to define a few concepts; unsurprisingly, as we are initially working with weak homotopy equivalences, the concepts we introduce will be weaker forms of past concepts.

Definition 4.1. A **homotopy rel** A, $h : X \times I \to Y$, where $A \subset X$, refers to a homotopy where $h|_A$ is independent of $t \in I$.

Example 4.2. Referring back to our homotopy equivalence between $\mathbf{R}^2 \setminus \{(0,0)\}$ and S^1, our definition of the function $g : \mathbf{R}^2 \setminus \{(0,0)\} \to S^1$ can be altered to a function $h : \mathbf{R}^2 \setminus \{(0,0)\} \times I \to \mathbf{R}^2 \setminus \{(0,0)\}$ to be a straight-line homotopy such that $h(x, 0)$ is the identity map and $h(x, 1) = g(x)$.

Then, h is a homotopy rel S^1 of $\mathbf{R}^2 \setminus \{(0,0)\}$. Furthermore, since $S^1 \subset \mathbf{R}^2 \setminus \{(0,0)\}$, h is called a **deformation retract**.

Definition 4.3. We define the mapping cylinder, M_f, of a map $f : X \to Y$ as the quotient space of $(X \times I) \sqcup Y$ where $\forall x \in X$, we identify $(x, 1)$ with $f(x) \in Y$.

Notice that Y is a deformation retract of M_f. This observation will be crucial in proving Whitehead's Theorem.

Definition 4.4. A **relative homotopy group** $\pi_n(X, A, x_0)$ where $x_0 \in A \subset X$ is the set of homotopy classes of maps $(I^n, \partial I^n, \partial I^{n-1} \times I \cup I^{n-1} \times \{0\}) \to (X, A, x_0)$.

The relative homotopy group is much like a regular homotopy group with a weaker restriction. In fact, one can observe that the relative homotopy group $\pi_n(X, x_0, x_0)$ is equivalent to the homotopy group $\pi_n(X, x_0)$.

With these definitions, we can now prove Whitehead's Theorem.

Theorem 4.5 (Whitehead's Theorem). For CW complexes X, Y, if f is a weak homotopy equivalence $X \to Y$, then f is a homotopy equivalence $X \to Y$.

To prove Whitehead's Theorem, we can first prove the compression lemma to simplify our work.

Lemma 4.6 (Compression Lemma). Let (X, A) be a CW pair and let (Y, B) be any pair where $B \neq \emptyset$. For all $y_0 \in B$, if $\pi_n(Y, B, y_0) = 0$ for each n such that $X \setminus A$ has an n-cell, then every map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.

Proof of Compression Lemma. In order to prove this lemma, we will use induction on n and show that every n-cell must follow this lemma.

If $n = 0$, our condition that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$ means that $\exists\, y \in Y \setminus B$ and $\forall y \in Y \setminus B$, y is path-connected to B. Thus, any 0-cell that is mapped to Y can be homotoped by a path to some point in B.

Suppose for induction that f can be homotoped so that the $k-1$ skeleton of X is mapped into B. Let g be an attaching map for a k-cell in $X \setminus A$ referred to as e^k. Then, consider $f \circ g : (D^k, \partial D^k) \to (Y, B)$. We can observe that the homotopy class $[f \circ g]$ is an element of $\pi_k(Y, B)$. Thus, with our assumption that $\pi_k(Y, B) = 0$, we know that $f \circ g$ can be homotoped into B, so we have a homotopy rel S^{k-1} of $X^{k-1} \cup e^k$ to B. We can find such a homotopy for every single k-cell at once and map all k-cells into B. We will take for granted the homotopy extension property for CW pairs which allows us to choose a homotopy of $X \to Y$ that preserves our homotopy of $X^{k-1} \cup e^k \to B$. □

Now, with the compression lemma, we can prove Whitehead's Theorem.

Proof of Whitehead's Theorem. We will be taking two theorems as having been proven in this proof. The first is the existence of a long exact sequence for the relative homotopy group as follows:

$$\cdots \to \pi_n(X) \to \pi_n(Y) \to \pi_n(Y, X) \to \pi_{n-1}(X) \to \cdots$$

The second is that every map between two CW complexes $X \to Y$ is homotopic to a cellular map, which is map that takes the n-skeleton of X to the n-skeleton of Y for all n. This second theorem is also known as the Cellular Approximation Theorem.

We can simplify our work by first using our observation about M_f to see that f is equivalent to the inclusion map of X onto the top face of M_f composed within the deformation retraction of M_f into Y. Thus, for all cases, Y can be replaced with M_f, so we just need to show that M_f can be deformed into X to prove Whitehead's Theorem.

If f is the inclusion of a subcomplex, we can observe from the long exact sequence that $\pi_n(Y, X)$ is isomorphic to the trivial group directly follows from the fact that f induces isomorphisms between the homotopy groups of X and Y. This conclusion allows us to make use of the compression lemma on the identity map of $(Y, X) \to (Y, X)$ to see that Y can be retracted into X.

If f is not the inclusion of a subcomplex, we can replace Y with M_f in the preceding paragraph to come to the same conclusion since f is homotopic to a cellular map by the Cellular Approximation Theorem which makes (M_f, X) a CW pair. □

Whitehead's Theorem opens up a whole new world of possibilities we can make use of. As we saw before, verifying whether two spaces are going to be homotopy equivalent can get very complicated since every relation needs two maps. However, by restricting ourselves to CW complexes, the work has been reduced to finding just one map that is a weak homotopy equivalence. In addition, a weak homotopy equivalence is fundamentally an algebraic concept which we can apply past knowledge to. Unlike some topological concepts, CW complexes seem to fall within the grasp of well-understood algebra. If we could extend this algebraic quality to all topological spaces, it allows a whole category of algebraic analysis that we did not have access to on the purely topological level. The following section will do just that.

5. CW Approximation

Although there are many other ways to build up or breakdown a topological space, each method has its own shortcomings. Although it may seem to also be the case for CW complexes, we can actually see that for every topological space, there exists a CW complex that is a weak homotopy equivalent.

Theorem 5.1. For every topological space X, there exists a CW complex A such that A is weak homotopy equivalent to X.

Proof. Let X^0 be the set of path-connected class of points in X. We construct A^0, the 0-cells of our CW complex, such that there exists a bijection $f : A^0 \to X^0$. Then, f is a map between our 0-skeleton and X.

From this point on, we use an inductive algorithm to construct our CW complex. Our inductive hypothesis is that we have a k-skeleton A^k and a map $f : A^k \to X$ that induces injective homomorphisms $\pi_n(A, a) \to \pi_n(X, f(a))$ for all $a \in A^0$ and $n < k$ and surjective homomorphisms $\pi_k(A, a) \to \pi_k(X, f(a))$. For each $a \in A^0$, we identify the generators of the kernel of the induced homomorphisms $\pi_{k-1}(A, a) \to \pi_{k+1}(X, f(a))$ and attach D^k to these generators. We extend f to these cells by inducing nullhomotopies of the kernel generators and their respective k-cells.

Then, $\forall a \in A$, we attach a S^k to a for each generator in $\pi_k(X, f(a))$. We extend f's domain to include the newly attached spheres by mapping each S^k to its correponding generator in X's homotopy group.

We will take for granted the cellular approximation theorem which allows us to observe that the two steps above do not affect any homotopy groups lower than $k - 1$. Thus, following the procedure above will produce a CW complex A and a weak homotopy equivalence $f : A \to X$. $\square$

Acknowledgements I would like to thank my mentor, Zev Chonoles, for helping me understand the material above well enough to write this paper. I would also like to thank Peter May and Henry Chan for their course on Topology during the 2016 UChicago Math REU. I would also like to especially thank Peter May for organizing the 2016 UChicago Math REU. Lastly, I would like to extend my thanks to Agnes Beaudry and John Wilmes for sparking my interest in pure mathematics during the 2016-2017 academic school year.

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