A CONVERSATION BETWEEN MODEL THEORY AND GRAPH THEORY

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Abstract. In general terms, Ramsey Theory studies the properties of structures that are preserved under taking partitions, and covers problems in areas including logic, theoretical computer science, number theory, game theory, and beyond. In this paper, we develop the model theoretic machinery to prove Ramsey's Theorem, and then use more classical combinatorial methods to get a finer resolution on these results, thus highlighting the relationship between model theory and combinatorics.

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1. Introduction

Ramsey's Theorem, Theorem 3.1, states that any complete infinite graph whose edges are coloured by k colours will contain an infinite monochromatic complete subgraph. Thus the property of completeness is preserved under partitioning the vertex set. A finite version of this theorem also holds. Finite Ramsey's Theorem, Theorem 3.3, asserts that given any n and k, integers, there is some integer r such that any k-colouring of a complete graph on r vertices will contain a complete monochromatic subgraph on n vertices.

However, model theory offers no insight into bounds on the required sizes of the graphs. For this, we must return to classical combinatorial methods.

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Finally, we use model theoretic intuition and machinery to deduce particular properties of graphs that indicate high disorder, and thus infer better bounds on Ramsey numbers in certain classes of graphs. This illustrates the delicate and fruitful interaction between model theory and combinatorics.

2. A Brief Introduction to Model Theory

For a more complete introduction, see [1]

Model theory is, very generally, the study of the relationship between a formal language and its interpretations, or models.

A *language* $\mathcal{L}$ is a finite collection of finitary relation symbols, $(R_i)_{i<j}$, finitary function symbols $(F_i)_{i<k}$, and constants $(c_i)_{i<l}$, and a *model* $\mathcal{A}$ for a language $\mathcal{L}$ is a pair (A, I), where A is a set (usually called the *universe* of the model) and I is an *interpretation function*, mapping m-ary relation symbols to m-ary relations on A, n-ary function symbols to n-ary functions on A, and constant symbols to elements of A. We often merely write

$$\mathcal{A} = \langle A, I(R_0), ..., I(R_j), I(F_0), ..., I(F_k), I(c_0), ..., I(c_l)\rangle$$

for a model for $\mathcal{L}$.

Example 2.1. Let $\mathcal{L}_G = \{P\}$ where P is a binary relation. Then the *theory of 2-colourings of simple complete graphs* is the smallest consistent set of formulas of $\mathcal{L}_G$ satisfying:

(i) $\neg R(x, x)$
(ii) $R(x, y)$ iff $R(y, x)$

where R is the interpretation of P.

A model of this theory is a complete simple graph with the edge between x and y coloured red if $R(x, y)$ and coloured blue otherwise.

Note that this is equivalent to a graph in which there is an edge between x and y if and only if $R(x, y)$. I shall use these interpretations interchangeably throughout this paper.

2.1. Ultrafilters and Ultraproducts.

Ultrafilters provide a model theoretic framework for discussing size. In particular, an ultrafilter may be thought of as a collection of the "large" subsets of a set. In a similar vein, Tao [2] compares an ultrafilter to a method of determining the outcome of a binary election over the elements of a set; a candidate wins if and only if the set of elements who voted for them is in the ultrafilter.

We can then use an ultrafilter to construct an *ultraproduct* of a collection of models, which we can think of as their limit or average. Because of the properties of the ultrafilter, the structure of the ultraproduct reflects and smooths the properties of the individual models, as illustrated by Theorem 2.6.

Definition 2.2 (Ultrafilter)**.** An *ultrafilter* $\mathcal{D}$ on a set I is a subset of $\mathcal{P}(I)$ (where $\mathcal{P}$ denotes the power set) such that

(i) $I \in \mathcal{D}$
(ii) $\mathcal{D}$ is closed under supersets, i.e. if $X \in \mathcal{D}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{D}$.
(iii) $\mathcal{D}$ is closed under finite intersections, i.e. if $X_0, ..., X_n \in \mathcal{D}$, then $X_0 \cap ... \cap X_n \in \mathcal{D}$.
(iv) For every $X \subseteq I$, exactly one of X, $X \setminus I$ belongs to $\mathcal{D}$.

In addition to the above, a *principal ultrafilter* satisfies:

(v) There is some $i_0 \in I$ such that $X \in \mathcal{D}$ iff $i_0 \in X$.

It is a consequence of Zorn's Lemma that any collection of subsets of I can be extended to an ultrafilter.

We will use the following fact frequently in this paper:

Proposition 2.3. *Let I be a set and let*

$$\mathcal{F} = \{X \subseteq I \big| |I \setminus X| < \infty\}.$$

Then $\mathcal{F}$ is a filter and the non-principal ultrafilters are just those that contain $\mathcal{F}$.

$\mathcal{F}$ is called the *Fréchet filter*.

Note that this implies that any infinite set admits a non-principal ultrafilter, and non-principal ultrafilters exist only over infinite sets.

We can also use ultrafilters to take a limit of a collection of models, giving a new model, the *ultraproduct*, that reflects the average properties of the component models.

Definition 2.4 (Ultraproduct)**.** Let $\mathcal{L} = \{R_0, ..., R_j, F_0, ..., F_k, c_0, ..., c_l\}$ be a language. Let I be an index set and let $(\mathcal{A}_i)_{i \in I}$ be a collection of models for $\mathcal{L}$. Let $\mathcal{D}$ be an ultrafilter over I.

Then the *ultraproduct*, $\mathcal{U}$, is a model for $\mathcal{L}$ with universe:

$$U = \prod_{\mathcal{D}} A_i = \{f_{\mathcal{D}} \big| f \in \prod_I A_i\},$$

where $f_{\mathcal{D}}$ represents the equivalence class of f under the following relation:

$$f =_{\mathcal{D}} g \text{ iff } \{i \in I \big| f(i) = g(i)\} \in \mathcal{D},$$

and interpretation function given by:

(i) Let R be an m-ary relation symbol in $\mathcal{L}$ and P_i its interpretation in $\mathcal{A}_i$. Then the interpretation, P, of R in $\mathcal{U}$ is

$$P(f^0_{\mathcal{D}}, ..., f^{m-1}_{\mathcal{D}}) \text{ iff } \{i \in I \big| P_i(f^0_{\mathcal{D}}(i), ..., f^{m-1}_{\mathcal{D}}(i))\} \in \mathcal{D},$$

where $f^0_{\mathcal{D}}, ..., f^{m-1}_{\mathcal{D}} \in U$.

(ii) Let F be an n-ary function symbol in $\mathcal{L}$ and G_i its interpretation in $\mathcal{A}_i$. Then the interpretation, G, of F in $\mathcal{U}$ is

$$G(f^0_{\mathcal{D}}, ..., f^{n-1}_{\mathcal{D}}) = \phi(f^0_{\mathcal{D}}, ..., f^{n-1}_{\mathcal{D}}),$$

where $\{i \big| G_i(f^0_{\mathcal{D}}(i), ..., f^{m-1}_{\mathcal{D}}(i)) = \phi(f^0_{\mathcal{D}}, ..., f^{n-1}_{\mathcal{D}})\} \in \mathcal{D}$, and$f^0_{\mathcal{D}}, ..., f^{n-1}_{\mathcal{D}} \in U$.

(iii) Let c be a constant symbol in $\mathcal{L}$ and a_i its interpretation in $\mathcal{A}_i$. Then the interpretation, a, of c in $\mathcal{U}$ is

$$c = \nu,$$

where $\{i \big| a_i = \nu\} \in \mathcal{D}$.

We write

$$\mathcal{U} = \prod_{\mathcal{D}} \mathcal{A}_i$$

for the ultraproduct.

The following propositions ensure $\mathcal{U}$ is well-defined and serves as a meaningful counterpart to average or limit.

Proposition 2.5. *Let $(\mathcal{A}_i)_{i\in I}$ be a collection of models for some language $\mathcal{L}$, and let $\mathcal{D}$ be an ultrafilter over I. Let $n < \omega$ and $\phi(x_0, ..., x_{n-1})$ be a formula in $\mathcal{L}$. Then, for $f_k, g_k \in \prod_i A_i$ with $f_k =_{\mathcal{D}} g_k$ for every $k < n$, we have that*

$$\prod_{\mathcal{D}} \mathcal{A}_i \vDash \phi(f_0, ..., f_{n-1}) \text{ iff } \prod_{\mathcal{D}} \mathcal{A}_i \vDash \phi(g_0, ..., g_{n-1})$$

Theorem 2.6 (Łoś' Theorem)**.** *Let $\mathcal{D}$ be an ultrafilter over a set I, let $(\mathcal{A}_i)_{i\in I}$ be a collection of models for some language $\mathcal{L}$, and let ϕ be a formula of $\mathcal{L}$. Then*

$$\prod_i \mathcal{A}_i \vDash \phi(f_0, ..., f_{n-1}) \text{ iff } \{i \in I \big| \mathcal{A}_i \vDash \phi(f_0(i), ..., f_{n-1}(i))\} \in \mathcal{D}$$

where $f_0, ..., f_{n-1} \in \prod_i \mathcal{A}_i / =_{\mathcal{D}}$

(Sketch of Proof). (See [3] for more details) The proof proceeds by induction on the complexity of ϕ. We get the logical structure from the definition of an ultrafilter:
(ii) gives us $A, A \implies B \quad \therefore B$.
(iii) gives us $A, B \quad \therefore A \wedge B$.
(iv) gives us $A \vee \neg A$. □

3. Ramsey's Theorem and First Consequences

Ramsey first published what we now call *Ramsey's Theorem*, Theorem 3.1, in 1930, which he used to show a certain class of formulas are decidable in Peano Arithmetic [4]. Ramsey's Theorem is essentially a generalisation of the pigeonhole principle and guarantees that completeness is preserved under partitions, or, in the words of Theodore Motzkin, "complete disorder is impossible." [5].

3.1. **Ramsey's Theorem.**

We first prove infinite Ramsey's Theorem. Note how we use the various properties of ultrafilters to ensure that we can construct an infinite sequence.

Theorem 3.1 (Ramsey's Theorem)**.** *Let G be an infinite set, and let $[G]^n$ denote the set of n-element subsets of G. Then for any 2-colouring of the elements of $[G]^n$, there is some infinite $H \subseteq G$ such that $[H]^n$ is monochromatic.*

Note this implies that any infinite complete graph (i.e., $n = 2$) whose edges are coloured by two colours will contain an infinite, monochromatic complete subgraph.

Proof. (Based off proof in [1])

First note that we may assume G is countably infinite, as any superset of G will satisfy the hypotheses of the theorem.

Write $[G]^n = A_0 \sqcup A_1$ where A_0 and A_1 are the two colours. We wish to find an infinite subgraph H such that $H \subseteq A_0$ or $H \subseteq A_1$.
Endow G with the order type ω:

$$x_0 < x_1 < x_2 < ...$$

Let $\mathcal{D}$ be a nonprincipal ultrafilter on G.

Note that for any $x \in G$, the set of elements less than x is finite, and hence not in $\mathcal{D}$. We now want to use the partition of $[G]^n$ to obtain a corresponding partition on G, so that we can investigate the structure of G via our ultrafilter.

Define:

$$A_0^n = A_0 \qquad\qquad A_1^n = A_1$$

We then consider the set of $n-1$-element subsets of G such that $\mathcal{D}$-almost all of their n-element supersets are in A_0^n or A_1^n respectively:

$$A_0^{n-1} = \{x_0 < ... < x_{n-2} | \{x \in G | x_{n-2} < x \text{ and } \{x_0, ..., x_{n-2}, x\} \in A_0^n\} \in \mathcal{D}\}$$

$$A_1^{n-1} = \{x_0 < ... < x_{n-2} | \{x \in G | x_{n-2} < x \text{ and } \{x_0, ..., x_{n-2}, x\} \in A_1^n\} \in \mathcal{D}\}.$$

Using the same idea, we inductively define

$$A_0^{n-r-1} = \{x_0 < ... < x_{n-r-2} | \{x \in G | x_{n-r-2} < x \text{ and } \{x_0, ..., x_{n-r-2}, x\} \in A_0^{n-r}\} \in \mathcal{D}\}$$

$$A_1^{n-r-1} = \{x_0 < ... < x_{n-r-2} | \{x \in G | x_{n-r-2} < x \text{ and } \{x_0, ..., x_{n-r-2}, x\} \in A_1^{n-r}\} \in \mathcal{D}\},$$

whence we obtain

$$A_0^1 = \{x_0 | \{x \in G | x_0 < x \text{ and } \{x_0, x\} \in A_0^2\} \in \mathcal{D}\}$$

$$A_1^1 = \{x_0 | \{x \in G | x_0 < x \text{ and } \{x_0, x\} \in A_1^2\} \in \mathcal{D}\}.$$

Since $\mathcal{D}$ is an ultrafilter and thus, for every subset $X \subseteq G$, exactly one of X or $G \setminus X$ is in $\mathcal{D}$, it follows that

$$G = A_0^1 \cup A_1^1$$

and further that either $A_0^1 \in \mathcal{D}$ or $A_1^1 \in \mathcal{D}$.

Without loss of generality, we may assume $A_0^1 \in \mathcal{D}$. We now use the ultrafilter and the structural properties of the A_0^k above to construct the desired homogeneous infinite subset.

Let $j_0 \in A_0^1$, and suppose we have defined $H_m = j_0 < j_1 < ... < j_{m-1}$ such that for any $0 \leq r < m$ (in the canonical ordering of the natural numbers),

$$y_0 < ... < y_{r-1} \in H_m \text{ implies } y_0 < ... < y_{r-1} \in A_0^r.$$

Then, corresponding to each sequence $y_0 < ... < y_{r-1}$, we get the associated set of elements x greater than y_{r-1} for which the set $\{y_0, ..., y_{r-1}, x\} \in A_0^{r+1}$:

$$X_{y_0,...,y_{r-1}} = \{x \in G | y_{r-1} < x \text{ and } \{y_0, ..., y_{r-1}, x\} \in A_0^{r+1}\}.$$

We know $X_{y_0,...,y_{r-1}}$ is non-empty (and indeed infinite) since, by construction, $X_{y_0,...,y_{r-1}} \in \mathcal{D}$.

Hence, since there are only finitely many sequences $y_0 < ... < y_{r-1}$,

$$X_m = \bigcap_{y_0 < ... < y_{r-1} \in H_m} X_{y_0,...,y_{r-1}} \quad \in \mathcal{D}.$$

Thus, in particular, $X_{y_0,...,y_{r-1}}$ is nonempty, and so we may choose $j_m \in X_m \cap A_0^1$.

Since X_m corresponds to those elements x greater than j_{m-1} for which every set containing elements of H_m and x is in A_0^r for some $1 \leq r \leq m+1$, it follows that $H_{m+1} = \{j_0 < ... < j_{m-1} < j_m\}$ satisfies (1).

Hence we may construct an infinite set $H = \bigcup_m H_m$ satisfying the conclusion of the theorem. □

The use of ultrafilters in the preceding proof ensured that the sets we were working with were always "big", and, in particular, infinite. This guaranteed that we could construct an infinite H of one colour. However, because of the fundamental reliance on ultrafilters, the proof is very non-constructive, and does not give either a way to build this infinite subgraph, nor any information on its relative size. Indeed, different ultrafilters may pick out very different monochromatic subgraphs!

Furthermore, we threw away a lot of information at the start of the proof when we assumed G was countably infinite, which implies there may be a subtler relationship between the cardinality of the graph and its homogeneous subset than this proof can see.

Note that one easily obtains the following generalisation:

Corollary 3.2. *Let n, $m \in \omega$. Then for every colouring of elements of $[G]^n$ with m colours, there is an infinite $H \subseteq G$ such that $[H]^n$ is monochromatic.*

Proof. This follows by repeatedly applying Ramsey's Theorem. □

3.2. **Finite Ramsey's Theorem.**
We can use Ramsey's Theorem above to obtain the following more familiar result.

Theorem 3.3 (Finite Ramsey's Theorem)**.** *For every $n < \omega$, there exists some $r(n) < \omega$ such that every 2-colouring of the complete graph on $r(n)$ vertices has a monochromatic complete subgraph of size n.*

Proof. Let $n < \omega$. Assume, in order to derive a contradiction, that for every $m < \omega$, there is some graph G_m on m vertices with no monochromatic complete subgraph of size n. Note that in $\mathcal{L}_G$, the non-existence of a homogenous n-subset can be expressed as a first order formula as follows:

$$\neg((\exists x_0)...(\exists x_{n-1}))\left(\left(\bigwedge_{0\leq i<j\leq n-1} R(x_i,x_j)\right)\vee\left(\bigwedge_{0\leq i<j\leq n-1} \neg R(x_i,x_j)\right)\right)$$

Thus, by Theorem 2.6, this property will be preserved in the ultraproduct.

Let $\mathcal{D}$ be a non-principal ultrafilter over ω, and consider $\prod_{\mathcal{D}} G_m$.

Claim: $\prod_{\mathcal{D}} G_m$ is infinite.

Proof of Claim: It suffices to show $\prod_{\mathcal{D}} G_m$ is larger than M for any $M < \omega$.

Let $M < \omega$ be arbitrary. Then $\{m : |G_m| \leq M\}$ is finite, and hence $\{m : |G_m| > M\} \in \mathcal{D}$.

Note that the property of having size at least M is expressible in first order logic as follows:

$$(\exists x_0)\ ...\ (\exists x_{M-1})\left(\bigwedge_{0\leq i<j\leq M-1} x_i \neq x_j\right)$$

Hence, applying Theorem 2.6 again, we get $|\prod_{\mathcal{D}} G_m| > M$, as required.

Thus $\prod_{\mathcal{D}} G_m$ is an infinite graph with no homogeneous n-subsets, contradicting Ramsey's Theorem. □

We see that the ultraproduct construction gives us a neat, quick connection between the infinite and the finite. However, because non-principal ultrafilters do not see finite sets, this proof gives no bounds on $r(n)$, which is of particular interest in applications. We will discuss this in the next section.

4. Elucidating Detail

As noted above, the proofs in the previous section give no information on the sizes of the graphs with monochromatic complete subgraphs. In order to deduce some bounds on Ramsey numbers, we must exploit the power of more classical combinatorial methods. Note this highlights the symbiotic relationship between model theory and combinatorics; we used model theory to provide an elegant and compelling proof of a general theorem, but we require the finesse of combinatorics to elucidate greater detail from these results.

4.1. **The Infinite Case.**

Ramsey's Theorem only guarantees that every infinite graph will have an infinite subset. It is natural to ask, in the spirit of Finite Ramsey's Theorem, Theorem 3.3, whether given a set of size κ, and a colouring of n element subsets of κ using m colours will always give a homogeneous subset of size λ.

Observe how we use combinatorial and set theoretic perspectives in the proofs that have a finer ability to distinguish between sizes of sets.

Definition 4.1 (Arrow Notation)**.** Let κ, λ, and m be (finite or infinite) cardinals, and let n be a natural number. Then, write:

$$\kappa \rightarrow (\lambda)^n_m$$

to denote *"every colouring of n-element subsets of κ with m colours gives a homogeneous subset of size λ"*

Proposition 4.2. *If*

$$\kappa \rightarrow (\lambda)^n_m$$

then

$$\kappa' \rightarrow (\lambda')^{n'}_{m'}$$

where $\kappa < \kappa'$, $\lambda > \lambda'$, $m < m'$, and $n < n'$.

Example 4.3. Ramsey's Theorem, Theorem 3.1, implies

$$\aleph_0 \rightarrow (\aleph_0)^n_m$$

Example 4.4. Finite Ramsey's Theorem, Theorem 3.3, implies that for any $n < \omega$, there is some $r(n) < \omega$ such that

$$r(n) \rightarrow (n)^2_2$$

The following result, however, gives that the most obvious generalisation of Ramsey's Theorem, namely $\aleph_1 \rightarrow (\aleph_1)^n_m$, does not hold.

Theorem 4.5.

$$2^\kappa \nrightarrow (\kappa^+)^2_2$$

Proof. (See [6]) One shows that $\{0,1\}^\kappa$ with the lexicographic ordering has no κ^+-length monotone sequences using a pigeonhole argument. From that, one can construct a partition of 2^κ into two sets such that any κ^+ monochromatic subgraph would constitute a κ^+-length monotone sequence. Thus there is some 2-colouring of 2^κ with no κ^+ monochromatic subgraph. □

However, the following theorem gives that a graph with more than $2^{\aleph_0}$ vertices will admit a monochromatic subgraph of size $\aleph_1$. So, assuming the Continuum Hypothesis, $\aleph_1$ is the least cardinal for which this does not hold.

Definition 4.6 ($\beth_\alpha$)**.**

$$\beth_0 = \aleph_0$$
$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$

For a limit ordinal λ,

$$\beth_\lambda = \sup\{\beth_\alpha \big| \alpha < \lambda\}$$

Thus, the *Continuum Hypothesis* states that $\beth_1 = \aleph_1$ and the *Generalised Continuum Hypothesis* states that $\beth_\alpha = \aleph_\alpha$ for every ordinal α.

Theorem 4.7 (Erdős-Rado)**.**

$$\beth_n^+ \to (\aleph_1)_{\aleph_0}^{n+1}$$

where κ^+ denotes the successor cardinal of κ, or the cardinality of the least ordinal that κ maps into but not onto [1].

Proof. In this proof, we make use of the following small lemma regarding cardinal arithmetic:

Lemma: Let κ and λ be cardinals with $2 \leq \lambda < \kappa$ and κ infinite. Then

$$\lambda^\kappa = 2^\kappa.$$

Proof:

$$2^\kappa \leq \lambda^\kappa \leq \kappa^\kappa \leq 2^{\kappa\times\kappa} = 2^\kappa$$

The first two inequalities follows from $2 \leq \lambda \leq \kappa$, the third inequality follows from the fact that a function from κ to κ is a subset of $\kappa \times \kappa$, and the final equality is a standard result in set theory. □

Now to the proof of the theorem. We proceed by induction on n.

Base Case ($n = 1$): We want to show

$$\beth_1^+ \to (\aleph_1)_{\aleph_0}^2.$$

This is equivalent to asserting that any graph of size $2^{\aleph_0}$ with edges coloured in $\aleph_0$ colours has a homogeneous subset of size $\aleph_1$.

Let $\kappa = (2^{\aleph_0})^+$ and let $F : [\kappa]^2 \to \aleph_0$ be the function assigning a two element subset (or equivalently, an edge) to one of $\aleph_0$ colours.

For each $a \in \kappa$, let $F_a : \kappa \setminus \{a\} \to \aleph_0$ be given by $F_a(x) = F(a, x)$, the function that given a vertex distinct from a returns the colour of the edge between it and a.

Claim: There exists a set $A \subseteq \kappa$ such that $|A| = 2^{\aleph_0}$ and for every countable subset $C \subseteq A$ and every $u \in \kappa \setminus A$, there is a $v \in A \setminus C$ such that

$$F_u(x) = F_v(x)$$

for every $x \in C$.

[1]i.e., $\kappa^+ = |\inf\{\lambda \text{ an ordinal } \big| \kappa < |\lambda|\}|$. See [7] for more information.

Proof of Claim: Construct a sequence of sets $A_0 \subseteq A_1 \subseteq ... \subseteq A_\alpha \subseteq ...$ as follows:

Let $A_0 \subseteq \kappa$ be an arbitrary set of size $2^{\aleph_0}$ and suppose we have picked $A_0 \subseteq A_1 \subseteq ... \subseteq A_\alpha$. Then let $A_{\alpha+1}$ be such that $|A_{\alpha+1}| = 2^{\aleph_0}$ and for every countable $C \subseteq A_\alpha$ and every $u \in \kappa \setminus C$, there is a $v \in A_{\alpha+1}$ such that

$$F_u(x) = F_v(x)$$

for every $x \in C$.

Note that we can always do this as, given a countable set C and a $u \in \kappa \setminus C$, there are at most $|C|^{\aleph_0} = 2^{\aleph_0}$ (by lemma) possible colourings of the edges between u and elements of C, and so one can always extend A to contain v's for all such colourings.

At each limit ordinal β, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$.

Then, setting $A = \bigcup_{\alpha<\omega_1} A_\alpha$, we get the required properties.

This proves the claim.

We now construct a sequence $(x_\alpha)_{\alpha<\omega_1} \subseteq A$ which we will use to demonstrate the existence of a homogeneous subset.

Let $a \in \kappa \setminus A$ be arbitrary.

Let $x_0 \in A$ be arbitrary, and suppose we have chosen $(x_\alpha)_{\alpha<\beta}$. Then set $C = \{x_\alpha | \alpha < \beta\}$ and let $x_\beta \in A$ be such that $F_a(x) = F_{x_\beta}(x)$ for every $x \in C$. Let $X = \{x_\alpha | \alpha < \omega_1\}$.

Note that if $\alpha < \beta$, $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\alpha)$ by construction of X, and $F_a : X \to \aleph_0$. Thus, by the pigeonhole principle, since $\mathrm{im} F_a$ is countable, there is a subset $H \subseteq X$ of size $\aleph_1$ such that F_a is constant on H. On other words, H is an $\aleph_1$-set of vertices such that for every $x_\alpha \in H$, $\{a, x_\alpha\}$, $\{x_\beta, x_\alpha\}$ all have the same colour, for any $\beta < \alpha$. Thus, $[H]^2$ is the required homogeneous subset of size $\aleph_1$.

Inductive Step: Now suppose

$$\beth_{n-1}^+ \to (\aleph_1)^n_{\aleph_0}.$$

The argument follows similarly as for the $n = 1$ case, so we will only provide an outline.

Let $\kappa = \beth_{n-1}^+$ and $F : [\kappa]^{n+1} \to \aleph_0$ denote a colouring of $n+1$-element subsets of κ by $\aleph_0$ colours.

Again, we let $F_a(\{x_0, ...x_{n-1}\})$ denote the colouring of the $n+1$-subset $\{a, x_0, ..., x_{n-1}\}$. Then we may similarly construct an $A \subseteq \kappa$ of size $2^{\aleph_0}$ such that for every countable subset $C \subseteq A$ and every $u \in \kappa \setminus A$ there is some $v \in A \setminus C$ such that $F_u(\{x_0, ..., x_{n-1}\}) = F_v(\{x_0, ..., x_{n-1}\})$ for every $\{x_0, ..., x_{n-1}\} \in [C]^n$.
As before, we let $a \in [\kappa \setminus A]^n$ arbitrary and construct $X = \{x_\alpha \big| \alpha < \omega_1\}$ such that $F_a(\{x_0, ...x_{n-1}\}) = F_{x_\beta}(\{x_0, ..., x_{n-1}\})$ for every $\{x_0, ..., x_{n-1}\} \in [\{x_\alpha \big| \alpha < \beta\}]^n$. As before, since F_a maps an $\aleph_1$ set into an $\aleph_0$ set, it is constant on a set H of size $\aleph_1$, and $[H]^{n+1}$ is our required homogenous subset. □

4.2. The Finite Case.

Ramsey's Theorem has a number of applications in complexity and computer science, where having explicit bounds on $r(n)$, or some notion of how quickly it grows, is particularly important. For example, if one can guarantee that given a sufficiently large domain, one can solve a problem, it is highly pertinent to be able to give an idea of how large that domain ought to be so that it can be applied in practice. (See, for example [8].)

One of the earliest results, due to Erdős and Szekeres [9] [10], is that $r(n)$ grows exponentially.

Theorem 4.8.

$$2^{n/2} < r(n) < \binom{2k-2}{k-1} < 4^{k-1}$$

for $n \geq 3$*.*

Proof. **($2^{n/2} < r(n)$):**

Let $N \leq 2^{n/2}$ and let K_N be the complete graph on N vertices.

Note that it suffices to show that fewer than half of the colourings of K_N vertices contain a complete graph on n vertices:

Let χ_N denote the number of 2-colourings of K_N, and suppose there are X_n ways to colour K_N such that there is a complete monochromatic subgraph of size n. If every colouring of K_N were to admit a monochromatic complete graph on n vertices, then there would be at most $2X_n$ colourings of K_N. But we are assuming $K_N > 2X_n$, so this is a contradiction.

Choosing a 2-colouring for K_N amounts to picking a subset of the edges to be, say, red, and the remainder will be blue, for example. Thus there are $|\mathcal{P}(E)|$, or $2^{\binom{N}{2}}$ possible colourings, where E is the edge set of K_N.

Given a particular complete subgraph, K_n of K_N, there are $2^{[\binom{N}{2}-\binom{n}{2}]}$ 2-colourings of K_N which leave K_n monochromatic, as there are $\binom{N}{2}-\binom{n}{2}$ edges not in K_n, and we are colouring some subset of those. Hence there are $\binom{N}{n}2^{[\binom{N}{2}-\binom{n}{2}]}$ 2-colourings of K_N with a complete monochromatic subgraph of size n.

Thus, we want to show

$$\binom{N}{n}2^{[\binom{N}{2}-\binom{n}{2}]} < \frac{2^{\binom{N}{2}}}{2}.$$

We have:

$$\binom{N}{n}2^{[\binom{N}{2}-\binom{n}{2}]} = \frac{N!}{(N-n)!n!}\frac{2^{N(N-1)/2}}{2^{n(n-1)/2)}} \leq \frac{N^n}{n!}\frac{2^{N(N-1)/2}}{2^{n(n-1)/2}}$$
$$< (\frac{2^{n/2}}{2^{(n-1)/2}})^n\frac{2^{N(N-1)/2}}{n!} = \frac{2^{n/2}}{n!}2^{N(N-1)/2}.$$

Thus it remains to show that $\frac{2^{n/2}}{n!} \leq \frac{1}{2}$ for $n \geq 3$. But this follows from $\frac{2^{3/2}}{3!} = \frac{\sqrt{2}}{3} < \frac{1}{2}$ and $\frac{2^{n/2}}{n!}$ being a decreasing function.

($r(n) < \binom{2k-2}{k-1}$):

In [10], Erdős and Szekeres use an induction argument to deduce bounds on the minimum number of points on the plane in general position required to guarantee a convex k-gon. This can be used to infer bounds on $r(n)$. □

The proof of the lower bound is an early example of Erdős' *probabilistic method*, which has since been applied to show the existence of a number of combinatorial objects with certain properties. Essentially, one takes a random combinatorial object and shows that the probability of it satisfying our properties is strictly positive. Then there must exist some object with these properties.

However, note that this proof is still non-constructive; while it does give us bounds on the size of $r(n)$, it does not construct an explicit graph of size N satisfying the conditions. More recent bounds have been demonstrated constructively [11], [12].

See [13] for a survey on the known bounds on Ramsey numbers as of 2014.

5. A Model Theoretic Approach to Finding Bounds

Model theory also provides a new light in which to view classical combinatorial results, and improve bounds on Ramsey's Theorem. For example, a long *half graph* is indicative of *instability* of a model. In this vein, call a graph *k-stable* if it has no *k-length half graph* as an induced subgraph. Then, using the previous observation from model theory, Malliaris and Shelah [14] show that, for any finite k, a k-stable finite graph G is guaranteed to contain a monochromatic subgraph of size $|G|^c$, where c is some constant depending only on k. This is significantly better than can be proved in general.

Definition 5.1 (Non-k-Order Property)**.** A graph G has the *k-order property* if and only if for every $(a_i)_{i<k}$ and $(b_i)_{i<k}$, sequences of vertices of G, $i < j < k$ implies $R(a_i, b_j) \wedge \neg R(a_j, b_i)$.

G has the *order property* if and only if G has the k-order property for all k.

G has the *non-k-order property* if no such $(a_i)_i$ and $(b_i)_i$ exist.

Definition 5.2 (Half graph)**.** A *half graph of length k* is a graph on $2k$ vertices divided into two sets $A = \{a_0, ..., a_{k-1}\}$ and $B = \{b_0, ..., b_{k-1}\}$ such that $R(a_i, b_j)$ iff $i < j$.

Note this definition says nothing about the edges between elements of the same partition.

Example 5.3. G has the non-k-order property if and only if it does not admit a monochromatic half graph of length k.

Definition 5.4 (Stability)**.** A graph G is *stable* if and only if G does not have the order property.

Definition 5.5 (Indiscernible)**.** Let $\mathcal{A}$ be a model for a language $\mathcal{L}$. A sequence $(a_i)_{i<\alpha} \subseteq M$, α an ordinal, of Γ-*indiscernible* elements of $\mathcal{A}$ is a sequence $(a_i)_{i<\alpha}$, α an ordinal, such that for any $n < \omega$, any increasing sequences $i_0 < i_1 < ... < i_n < \alpha$ and $j_0 < j_1 < ... < j_n < \alpha$, and any formula $\phi[x_0, ..., x_{n-1}]$ in Γ,

$$M \vDash \phi[a_{i_0}, a_{i_1}, ..., a_{i_n}] \text{ iff } M \vDash \phi[a_{j_0}, a_{j_1}, ..., a_{j_n}].$$

Example 5.6. Ramsey's theorem guarantees the existence of arbitrarily long indiscernible sequences for any set of formulas in the language of graphs, $\mathcal{L}_G$, given a sufficiently large model.

These definitions highlight that the model theoretic notion of stability is intricately intertwined with the graph theoretic construction of a half graph, while indiscernible sets in model theory correspond to monochromatic subgraphs in graph theory. In particular, the existence of long half graphs are indicators that the graph is unstable. Malliaris and Shelah show in [14] Theorem 3.5 (Theorem 5.11 below) that when we assume "good behaviour" and, in particular, require monochromatic half graphs to be bounded, we can guarantee much larger monochromatic complete subgraphs, or indiscernible sets, than predicted by Ramsey's Theorem. Thus, we see that these half graphs are a major cause of disorder in graphs.

However, in order to state the theorem, we need the following definitions and result. From now on, all languages will be $\mathcal{L}_G$ (see Example 2.1) unless specified otherwise.

Definition 5.7 (Modified Arrow Notation)**.** Let Γ be a collection of formulas, n_1 a cardinal, and n_2 an ordinal. Then we will write

$$(n_1) \rightharpoonup (n_2)_\Gamma$$

to mean that for every n_1-length sequence of vertices in our universe G, these exists a non-constant Γ-indiscernible subsequence of order type n_2.

Definition 5.8 (The set Δ_k)**.** ([14], Definition 2.8) Δ_k is the set of formulas $\{R(x_0, x_1)\} \cup \{\phi^i_{k,m} | m \leq k, i = 1, 2\}$, where

$$\phi^i_{k,m} = \exists y \Big(\bigwedge_{l<m} (R(x_l, y))^{\text{if}(i=1)} \wedge \bigwedge_{m \leq l < k} (R(x_l, y))^{\text{if}(i=2)} \Big)$$

and we use the notation $\phi^\psi = \phi$ if ψ holds, and $\neg\phi$ otherwise.

Example 5.9. ([14], Claim 3.4) If $(n_1) \rightarrow (n_2)^k_{2^{|\Delta_k|}}$, then $(n_1) \rightharpoonup (n_2)_{\Delta_k}$

Proposition 5.10 ([14], Observation 2.9)**.** *Let G be a finite graph and let $A = (a_i)_{i<\alpha}$, $\alpha > 2k$ be a Δ_k-indiscernible sequence of vertices of G. Suppose for some sequence $i_0 < i_1 < ... < i_{2k-1} < \alpha$ of indices, and some $v \in G$, the following holds:*

- *for all l with $0 \leq l \leq k-1$, $R(b, a_{i_l})$*
- *for all l with $0 \leq l \leq 2k$, $\neg R(b, a_{i_l})$*

Then G has the k-order property.

This is our first indication of the connection between monochromatic subgraphs and half-graphs.

Proof. For each m such that $0 < m < k-1$, let $C_m = (c_{m_j})_{0 \leq j \leq k-1}$, where $c_{m_j} = a_{m+j}$. Then b witnesses that $\phi^1_{k,m}(c_0, ..., c_{k-1})$ holds in G. This is sufficient to establish the k-order property as two increasing sequences of A satisfy the same Δ_k-formulas.

If we assume the inverse, a similar argument, replacing ϕ^1 with ϕ^2, again establishes the k-order property. ☐

We now state the main theorem of this section, which highlights the utility of model theoretic techniques in classical combinatorics.

Theorem 5.11 ([14], Theorem 3.5)**.** *For each Δ_k, we have $n_1 \rightharpoonup (n_2)_{\Delta_k}$ for any $n_1 > (cn_2)^{(2tr)^k}$, where c is a constant depending on k.*

A proof of this theorem is beyond the scope of this paper (see [14] for full details).

Theorem 5.11 implies that showing a 2-colouring of a finite complete graph has no monochromatic half-graph of length k guarantees it has a monochromatic subgraph of a size growing polynomially in the size of the graph. This is considerably better than earlier bounds and is one of many examples of the productive partnership between combinatorics and model theory.

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