LIE GROUPS AND LIE ALGEBRAS

JOSEPH REDEKER

ABSTRACT. Our goal in this paper is to introduce Lie groups and their corresponding Lie algebras, and to give some examples. We begin by providing some background definitions and concepts, then we will define matrix groups and their topologies. Once we have done that we will be able to introduce matrix Lie groups.

Contents

1. Preliminary Discussion 1
2. Groups 2
3. Topology of Matrix Lie Groups 3
4. Lie Algebras and The Matrix Exponential 6
Acknowledgements 12
References 13

1. Preliminary Discussion

Before we begin our discussion on matrix groups, we must go through some basic definitions and provide context for the rest of the paper.

Definition 1.1. Let A be an $n \times n$ matrix with elements in the field $\mathbb{F}$. From this point forward we will denote the set of all $n \times n$ matrix with elements in the field $\mathbb{F}$ as $M_n(\mathbb{F})$. Let X_r and X_c denote X as a row or column matrix and define $R_A : \mathbb{F}^n \to \mathbb{F}^n$ and $L_A : \mathbb{F}^n \to \mathbb{F}^n$ as,

$$R_A(X) = X_r \cdot A \text{ and } L_A(X) = A \cdot X_c$$

It follows that both R_A and L_A are linear for $A \in M_n(\mathbb{F})$, and any linear function $f : \mathbb{F}^n \to \mathbb{F}^n$ is equivalent to R_A for some $A \in M_n(\mathbb{F})$.

Definition 1.2. Let V be a vector space and define a **norm** as a function $f : V \to \mathbb{R}$ such that for all $a \in \mathbb{F}$ and all $u, v \in V$,

(1) $f(a \cdot v) = |a| \cdot f(v)$
(2) $f(u + v) \leq f(u) + f(v)$
(3) If $f(v) = 0$ then v is the zero vector

A norm that we are most familiar with is the Euclidean norm which takes the form,

$$\|v\| := \sqrt{\langle v, v\rangle}$$

where $\langle v, v\rangle = \sum_{i=1}^{n} v_i^2$ if $v \in \mathbb{R}^n$ and $\langle v, v\rangle = \sum_{i=1}^{n} v_i \cdot \overline{v_i}$ if $v \in \mathbb{C}^n$

However, in this paper we will discuss the **operator norm** as well. It is sometimes called the matrix norm, and for some matrix $A \in M_n(\mathbb{F})$, the operator norm of A is defined as,

$$\|A\| := \inf\{c \geq 0 : \|Av\| \leq c\|v\| \text{ for all } v \in \mathbb{F}^n\}$$

2. Groups

We are interested in studying $M_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For example, $M_n(\mathbb{F})$ will denote all $n \times n$ matrices with entries in either $\mathbb{R}$ or $\mathbb{C}$. Though $M_n(\mathbb{F})$ does not form a group under multiplication, the subset of invertible matrices does. Below we will study this group further.

Definition 2.1. The **general linear group** is the group of invertible matrices with entries in $\mathbb{F}$. Formally we write this as,

$$GL_n(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \,|\, \exists A^{-1} \in M_n(\mathbb{F}) \text{ such that } A \cdot A^{-1} = A^{-1} \cdot A = I\}$$

The general linear group is typically denoted as $GL_n(\mathbb{F})$. This group also has some interesting subgroups which we call **matrix groups**.

Definition 2.2. U is called a **matrix group** if it is a subgroup of $GL_n(\mathbb{F})$.

Definition 2.3. The **orthogonal group** is a matrix group that preserves inner products on $\mathbb{F}^n$. To be clear,

$$\mathbb{O}_n(\mathbb{F}) := \{A \in GL_n(\mathbb{F}) \,|\, \langle R_A(X), R_A(Y)\rangle = \langle X, Y\rangle \text{ for all } X, Y \in \mathbb{F}^n\}$$

Notation differs depending on the field. The orthogonal groups that will be useful to us in this paper will be over the real or complex fields, called the **orthogonal group** and the **unitary group**, denoted $O(n)$ and $U(n)$ respectively.

Recall that two vectors $X, Y \in \mathbb{F}^n$ are said to be **orthogonal** if $\langle X, Y\rangle = 0$. It is a basic result that a matrix A is orthogonal if its columns are orthogonal and unit length.

Proposition 2.4. *If $A \in \mathbb{O}(n)$, then having A^* denote the Hermitian transpose, $A \cdot A^* = I$.*

Proof. Let $v_i \in \mathbb{F}^n$ and have

$$A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$$

Then the ij^{th} component of $A \cdot A^*$ is $\langle v_i, v_j\rangle$. Since the columns and rows are orthogonal then $\langle v_i, v_j\rangle = 0$ for all $i \neq j$. When $i = j$, then $\langle v_i, v_j\rangle = 1$ using the fact that all the columns and rows are unit length. This shows that $A \cdot A^* = I$. □

Keep in mind that in the argument above our definition of the inner product changes if we are dealing real or complex numbers. However, the argument will work in both cases.

Matrices that are members of the orthogonal group are important in a number of ways, but before we discuss these we define an **isometry**. Given the Euclidean norm, distance between two vectors is given by the norm of the difference between those vectors. An isometry is a function $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distance. In other words, if A is the matrix of f, then

$$\|R_A(X) - R_A(Y)\| = \|X - Y\|$$

Proposition 2.5. *If $A \in \mathbb{O}_n(\mathbb{F})$, then $R_A : \mathbb{F}^n \to \mathbb{F}^n$ is an isometry.*

Proof. Note,

$$\|R_A(X) - R_A(Y)\| = \|R_A(X - Y)\| = \sqrt{\langle R_A(X - Y), R_A(X - Y)\rangle}$$

Since $A \in \mathbb{O}_n(\mathbb{F})$,

$$\sqrt{\langle R_A(X - Y), R_A(X - Y)\rangle} = \sqrt{\langle (X - Y), (X - Y)\rangle} = \|X - Y\|$$

□

Proposition 2.6. *If $A \in \mathbb{O}(n)$ then $|det(A)| = 1$.*

Proof. To begin, first note that

$$\det(\overline{A}) = \overline{\det(A)}$$

since the determinant is a polynomial of the entries in A. Then this implies

$$\det(A^*) = \det(\overline{A}^T) = \det(\overline{A}) = \overline{\det(A)}$$

using the fact that A^T is the determinant of A. From this, we can conclude that

$$1 = \det(I) = \det(A \cdot A^*) = \det(A) \cdot \det(A^*) = \det(A) \cdot \overline{\det(A)} = |\det(A)|$$

□

Definition 2.7. The **special orthogonal group** is denoted $SO(n)$ and defined as,

$$SO(n) := \{A \in O(n) \,|\, \det(A) = 1\}$$

Definition 2.8. The **special unitary group** is denoted $SU(n)$ and defined as,

$$SU(n) := \{A \in U(n) \,|\, \det(A) = 1\}$$

We mention isometries earlier because one common group of isometries that will be seen throughout this paper is $\mathbb{O}(n)$. This group is typically thought of as the group of distance preserving transformations in a Euclidean space and its subgroup $SO(n)$ has incredible applicability in physics. SO(n) is generally called the rotational group as its elements correspond to a rotation around a point or line (in 2 or 3 dimensions respectively).

3. Topology of Matrix Lie Groups

To begin the discussion of matrix Lie groups, we must build a background for uncovering the topology of the groups mentioned above. We are interested in the topology of these groups because we will arrive at a convenient way to define a matrix Lie group which relies on subgroups of $GL_n(\mathbb{F})$ to be closed. This provides a simple test whether a group is a matrix Lie group. In order to begin our discussion on the topology of these groups we must first define a bijection between $M_n(\mathbb{R})$ and R^{n^2}. We create this bijection by a simple mapping of the elements in the matrix to a component an n^2-tuple vector in R^{n^2}, and we will denote this bijection as σ. For example, if we have a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $\sigma(A) = (a, b, c, d)$. For the real case, the first n elements of our n-tuple is the first row of our matrix. The next n elements being the second row and so on for all n rows giving us our n^2-tuple. Now we can define a similar bijection between $M_n(\mathbb{C})$ and R^{2n^2}. If we look at a complex matrix $A \in M_2(\mathbb{C})$,

$$A = \begin{pmatrix} a + bi & c + di \\ e + fi & g + hi \end{pmatrix}$$

then the bijection will be $\sigma(A) = (a, b, c, d, e, f, g, h)$. In the complex case, the first $2n$ elements of the $2n$-tuple are the real parts and the real coefficients of the elements in the first row of our matrix, and so on for all of the rows giving us the components of our $2n^2$-tuple.

From there we can finally discuss what it means for $M_n(\mathbb{R})$ to have topology by considering each element of the $n \times n$ matrix as an element in an n^2-tuple which defines a point in the Euclidean space of n^2 dimensions (similarly for $M_n(\mathbb{C})$. By looking at this set of points in Euclidean space, we can determine what it means for a subset of $M_n(\mathbb{F})$ to be open or closed. However, we must first define what it means to be open and closed, and how that applies to continuity and path connectedness. By considering our bijection σ, all of this discussion corresponds nicely to matrices. At that point, we can define what it means for a group to be a matrix Lie group.

Definition 3.1. For some $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ be defined as,

$$B(x, r) := \{y \in \mathbb{R}^n \mid |x - y| < r\}$$

This "ball" around x is useful to understand the topology of sets containing x, as you will see in the next definition.

Definition 3.2. Let $U \subset R^n$. U is called **open** if for every $x \in U$, there is an $r > 0$ such that for all $y \in B(x, r)$ then $y \in U$. Conversely, a set U is called **closed** if $\mathbb{R} \backslash U$, which is commonly called the complement of U, is open (denoted U^C).

In fact, it is actually possible for a set to be both open and closed. These types of sets are called **clopen**. Some examples of clopen sets would be the empty set and the entire space. The fundamental property of open and closed sets is that the finite intersection of closed sets is closed, and the union of arbitrarily many open sets is open.

Definition 3.3. Let $U \subset R^n$ and $V \subset R^m$. A function $f : U \to V$ is said to be **continuous** if for any open set $O \subset V$ then the preimage of O, $f^{-1}(O)$, is also open.

This definition is equivalent to the $\epsilon - \delta$ definition that we see in Euclidean space.

Definition 3.4. Let $U \subset R^n$. U is said to be **path connected** if for ever pair of elements $x, y \in U$ there exists a continuous function $f : [0, 1] \to U$ such that $f(0) = x$ and $f(1) = y$.

To put it simply, if U is path connected, then it is not made up of separate parts. Now that we have discussed the preliminaries, we are able to prove that some of our previous examples are matrix groups.

Proposition 3.5. *$SL_n(\mathbb{F}), \mathcal{O}(n), SO(n)$, and $SU(n)$ are all examples of matrix groups.*

Proof. To show $SL_n(\mathbb{F})$ is a matrix group we must prove that it contains the identity element, is closed under multiplication, and that the inverse of every element is also in the group.

$$\det(I) = 1$$

So the identity is contained in $SL_n(\mathbb{F})$. Now assume $A, B \in SL_n(\mathbb{F})$ and consider the determinant of their product,

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

Hence $SL_n(\mathbb{F})$ is closed under multiplication. Finally,

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = 1 \cdot \det(A^{-1})$$

Therefore, $\det(A^{-1}) = 1$ and so $A^{-1} \in SL_n(\mathbb{F})$ as required.

Now we will show $\mathcal{O}(n)$ is a matrix group. Clearly, the identity is in $\mathcal{O}(n)$. So consider the complex conjugate of the product of two elements $A, B \in \mathcal{O}(n)$,

$$(AB)^* = B^* A^*$$

which means,

$$AB(AB)^* = ABB^*A^* = AIA^* = AA^* = I$$

Hence $AB \in \mathcal{O}(n)$. By definition of orthogonal matrices, if $A \in \mathcal{O}(n)$ then so must $A^{-1} = A^*$. Both $SO(n)$ and $SU(n)$ follow directly from these first two cases. $\square$

In fact, these matrix groups are actually what are called **matrix Lie groups**. Before we fully understand what that means, we will define a Lie group, albeit roughly.

Definition 3.6. A **Lie group** G is a group that is a smooth manifold with the criterion that the group operation $G \times G \to G$ and the inverse map $g \to g^{-1}$ are differentiable.

Intuitively, a smooth manifold is a space M that locally resembles Euclidean space and is regular enough for us to be able to define the concept of smooth functions on M. We will not describe them rigorously here. They require thorough consideration which will take away from the purpose and aim of this paper. We state this definition for completion because we cannot define a matrix Lie group without first providing the formal definition of a Lie group. However, we will not go into that discussion further. What matters is that the following definition of a matrix Lie group rests on a criterion, specifically closedness, to define what a matrix Lie group is. We will assume this criterion is sufficient to show the stated matrix groups are Lie groups.

Definition 3.7. A **matrix Lie group** is any subgroup G of $GL_n(\mathbb{C})$ such that the corresponding subset of R^{n^2} or R^{2n^2} is closed.

Proposition 3.8. *$SL_n(\mathbb{F}), \mathcal{O}(n), SO(n)$, and $SU(n)$ are all examples of matrix Lie groups.*

Proof. We have already shown that these are all subgroups. What remains to be shown is that they are closed. First we will show that $\mathbb{O}(n)$ is a matrix Lie group. Let $A \in M_n(\mathbb{F})$ and construct a function $f : M_n(\mathbb{F}) \to \mathbb{F}$ such that,

$$f(A) := A \cdot A^*$$

It is clear to see that this function will be continuous since each $f(A)_{ij}$ is just a polynomial in terms of the entries of A, all of which are members of $\mathbb{F}$. Now if we consider the identity element, $\{I\}$ which is a closed subset of $M_n(\mathbb{F})$, then we know that $f^{-1}(A)$ must also be closed. Here we used the basic fact that if a function is continuous, then for any closed subset of the codomain, the preimage of the subset is also closed. However, note that

$$f^{-1}(I) = \mathbb{O}(n)$$

Therefore $\mathbb{O}(n)$ is a matrix Lie group.

Now we must show that $SL_n(\mathbb{F})$ is a matrix Lie group and the other two will follow directly. Consider the determinant function, $\det : M_n(\mathbb{F}) \to \mathbb{R}$. Again, this function is continuous since the determinant function is merely the sums and products of entries in A. Now note that the set $\{1\} \subset \mathbb{R}$ is closed, implying that $\det^{-1}(1)$ is also closed, but

$$\det^{-1}(1) = SL_n(\mathbb{F})$$

and so $SL_n(\mathbb{F})$ is a matrix group.

We conclude the proof by noting

$$SU(n) = U(n) \cap SL_n(\mathbb{F})$$

and

$$SO(n) = O(n) \cap SL_n(\mathbb{F})$$

Since the intersection of closed sets is closed, then both $SU(n)$ and $SO(n)$ are closed. □

4. Lie Algebras and The Matrix Exponential

Definition 4.1. If $G \subset R^n$ and $x \in G$, then the **tangent space** of G at the point x is

$$T_xG := \{f'(0) \mid f : I \to G \text{ is differentiable with } f(0) = x\}$$

Now that we have defined the tangent space, understanding a Lie Algebra is straight forward.

Definition 4.2. The **Lie algebra** of a matrix Lie group G is the tangent space at I and is commonly written as $\mathfrak{g} = T_IG$.

It is important to note here that our matrix group is being viewed as a subset of R^{n^2} or R^{2n^2} and this is why we can apply this definition to discuss matrix Lie groups in regard to this definition.

Proposition 4.3. *The Lie algebra $\mathfrak{g}$ of the matrix Lie group G is a real subspace of $M_n(\mathbb{F})$.*

Proof. To prove the Lie algebra $\mathfrak{g}$ of the matrix Lie group G to be a real subspace of $M_n(\mathbb{F})$, we must show that $\mathfrak{g}$ is closed under addition and scalar multiplication. Begin by defining $\lambda \in \mathbb{R}$ and $A \in \mathfrak{g}$. There is a path $f : (-\epsilon, \epsilon) \to G$ defined as $f(t)$ such that $f(0) = I$ and $f'(0) = A$. Now we can consider an alternative path

$$h(t) = f(\lambda t)$$

In this case, $h(0) = I$ and $h'(0) = \lambda \cdot A$ which shows that $\lambda \cdot A \in \mathfrak{g}$. Now let $A, B \in M_n(\mathbb{F})$ then there are paths $f(t)$ and $h(t)$ such that $f(0) = I$ and $h(0) = I$ with $f'(0) = A$ and $h'(0) = B$. Now consider a path

$$p(t) = f(t) \cdot h(t)$$

This means that $p(0) = I$ and by the product rule,

$$p'(0) = f'(0) + h'(0)$$

Therefore, $p'(0) = A + B$, implying $A + B \in \mathfrak{g}$ and this shows that $\mathfrak{g}$ is a real subspace of $M_n(\mathbb{F})$.

Proposition 4.4. $\mathfrak{gl_n}(\mathbb{F}) = M_n(\mathbb{F})$*, where* $\mathfrak{gl_n}(\mathbb{F})$ *is the Lie algebra of* $Gl_n(\mathbb{F})$

Proof. Let $A \in M_n(\mathbb{F})$ and consider the path $f : (-\epsilon, \epsilon) \to Gl_n(\mathbb{F})$ such that

$$f = I + tA$$

This satisfies $f(0) = I$ and $f'(0) = A$. Note that the $\det(f(0)) = 1$, which is non-zero. Using the fact that the determinant function is continuous, we can choose ϵ small enough so we can be certain that our determinant on our path is sufficiently close to one, i.e. non-zero, over the whole domain. This implies that $A \in \mathfrak{gl_n}(\mathbb{F})$ and since A was an arbitrary matrix in $M_n(\mathbb{F})$ then $\mathfrak{gl_n}(\mathbb{F}) = M_n(\mathbb{F})$. □

Proposition 4.5. *The Lie algebra of* $\mathbb{O}_n(\mathbb{F})$ *is,*

$$\mathfrak{o_n}(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid A + A^* = 0\}$$

We do not quite have the definitions and background to tackle this proposition since it requires a different technique than the one used in the last proof. However, we will come back to this proposition later in this section.

Definition 4.6. Let $A \in M_n(\mathbb{F})$. Then the exponential of A is given by the power series,

$$e^A := I + A + (1/2!)A^2 + (1/3!)A^3 + \dots \tag{1}$$

Proposition 4.7. *For any* $A \in M_n(\mathbb{F})$*, equation (1) converges to some matrix* $B \in M_n(\mathbb{F})$*.*

Proof. In this proof we will use the basic fact that for matrices $A, B \in M_n(\mathbb{F})$,

$$\|AB\| \leq \|A\|\|B\|$$

It is important to notice that we are using the operator norm that is defined above. From this we conclude that

$$\|A^n\| \leq \|A\|^n$$

Now,

$$\sum_{n=1}^{\infty} \|\frac{A^m}{m!}\| \leq \sum_{n=1}^{\infty} \frac{\|A\|^n}{n!} = e^{\|A\|}$$

Since the norm of any $n \times n$ matrix is bounded, then we can conclude that the sum will converge absolutely for all $A \in M_n(\mathbb{F})$ which implies it will converge. $\square$

The exponential map proves useful in a number of ways, it defines a map $f : M_n(\mathbb{F}) \to GL_n(\mathbb{F})$ with $f(0) = I$ and $f'(0) = A$. We can use this mapping to determine the properties of Lie algebras of a group, and it has important algebraic properties that we can take advantage of when solving problems. Here are some examples.

Proposition 4.8. *(Proposition 2.3) [1] If $A, B \in M_n(\mathbb{F})$ then,*
(1) e^A has an inverse, $(e^A)^{-1} = e^{-A}$
(2) for $\alpha, \beta \in \mathbb{F}$ then $e^{(\alpha+\beta)A} = e^{\alpha A}e^{\beta A}$
(3) If A and B commute, then $e^{A+B} = e^A e^B = e^B e^A$
(4) $\|e^A\| \leq e^{\|A\|}$
(5) If B is an invertible matrix, then $e^{BAB^{-1}} = Be^A B^{-1}$
(6) $(e^X)^* = e^{X^*}$

The exponential map has even more use than having important algebraic properties. In fact, we can provide an alternative but equivalent definition for a Lie algebra which involves the exponential. The definition provided at the beginning of the section is more intuitive in regards to the topology of the Lie algebra, while this equivalent definition will show the connection that the Lie algebra has with its corresponding matrix Lie group.

Proposition 4.9. *If G is a matrix Lie group, then the **Lie algebra** of G, denoted $\mathfrak{g}$, is the set of all matrices A such that e^{tA} is in G for all real numbers t.*

Note: Even if G is a group on $\mathbb{C}$, we do not require that t be complex, only real. In fact, we can define an exponential mapping,

$$exp : \mathfrak{g} \to G$$

However, this mapping is generally neither injective nor surjective. Nonetheless it is incredibly useful for passing information between the Lie algebra and the matrix Lie group. This is an equivalent definition as the one provided earlier for the Lie algebra, but the equivalence will not be proven here in this paper. However, for those interested in the proof, it can be found in Tapp's *Matrix Groups for Undergraduates*[2] under theorem 7.1.

Proposition 4.10.

$$det(e^A) = e^{Tr(A)}$$

Here we let $Tr(A)$ represent $trace(A)$.

Proof. For this proof, we will assume the following lemma.

If $f : (-\epsilon, \epsilon) \to M_n(\mathbb{F})$ is differentiable with $f(0) = I$, then

$$\left.\frac{d}{dt}\right|_{t=0} \det(f(t)) = \text{trace}(f'(t))$$

The proof of the Lemma will be omitted, but the proof of it can be found in [4] (Lemma 5.10). Let $f(t) = \det(e^{tA})$. Then,

$$\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{\det(e^{(t+h)A}) - \det(e^{tA})}{h} \\
&= \lim_{h\to 0} \frac{\det(e^{tA}e^{hA}) - \det(e^{tA})}{h} \\
&= \lim_{h\to 0} \frac{\det(e^{tA})\det(e^{hA}) - \det(e^{tA})}{h} \\
&= \det(e^{tA}) \lim_{h\to 0} \frac{\det(e^{hA}) - 1}{h} \\
&= f(t) \frac{d}{dt}\bigg|_{t=0} \det(e^{tA})
\end{aligned}$$

and since,

$$\frac{d}{dt}\bigg|_{t=0} (e^{tA}) = A$$

then by the lemma,

$$f'(t) = f(t) \cdot \mathrm{Tr}\left(\frac{d}{dt}\bigg|_{t=0} e^{tA} \right)$$

All that remains to be shown is that $\frac{d}{dt}\Big|_{t=0} e^{tA} = A$. This can be seen by differentiating each term of the power series and evaluating at $t = 0$. Therefore,

$$f'(t) = f(t) \cdot \mathrm{Tr}(A)$$

and the solution to this differential equation is of the form $f(t) = e^{t\cdot \mathrm{Tr}(A)}$. Plugging in $t = 1$ we see,

$$\det(e^A) = e^{\mathrm{Tr}(A)}$$

□

Proposition 4.11. *The Lie algebra of $SL_n(\mathbb{F})$ is,*

$$\mathfrak{sl}_\mathfrak{n}(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid Tr(A) = 0\}$$

Proof. Using the above fact that,

$$\det(e^{tA}) = e^{t\cdot \mathrm{Tr}(A)}$$

then $\mathrm{Tr}(A) = 0$ implies $\det(e^A) = 1$. Conversely, if the $\det(e^{tA}) = 1$ for all t then

$$\det(e^{tA}) = 1 = e^{t\cdot \mathrm{Tr}(A)}$$

Hence $t \cdot \mathrm{Tr}(A) = 2\pi i \cdot k$ for all t and for some integer k. This can only be true if $\mathrm{Tr}(A) = 0$. Therefore,

$$\mathfrak{sl}_\mathfrak{n}(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid \mathrm{Tr}(A) = 0\}$$

□

Proposition 4.12. *The Lie algebra of $U(n)$ is,*

$$\mathfrak{u}(\mathfrak{n}) := \{A \in M_n(\mathbb{C}) \mid A + A^* = 0\}$$

Proof. A matrix A is unitary if and only if $A^* = A^{-1}$. Therefore e^{tA} is unitary if and only if

$$(e^{tA})^* = (e^{tA})^{-1} = (e^{-tA})$$

and by proposition 4.8,

$$(e^{tA})^* = (e^{tA^*})$$

Since this must hold for all t, we see by differentiating and evaluating at $t = 0$ that $-A = A^*$ is necessary. So e^{tA} is unitary if and only if $-A = A^*$. This gives the equality

$$\mathfrak{u}(\mathfrak{n}) := \{A \in M_n(\mathbb{C}) \,|\, A + A^* = 0\}$$

□

Proposition 4.13. *The Lie algebra of $SU(n)$ is,*

$$\mathfrak{su}(n) := \{A \in M_n(\mathbb{C}) \,|\, A + A^* = 0 \textit{ and } Tr(A) = 0\}$$

Proof. By Proposition 4.9, the Lie algebra of SU(n) is all matrices A such that $e^{tA} \in$ SU(n). Now, since SU(n)$= SL_n(\mathbb{C}) \cap U(n)$ then the Lie algebra will be given by all matrices A such that $e^{tA} \in SL_n(\mathbb{C}) \cap U(n)$. Then clearly, the Lie algebra must be $\mathfrak{u}(\mathfrak{n}) \cap \mathfrak{sl}_\mathfrak{n}(\mathbb{C})$ which is

$$\{A \in M_n(\mathbb{C}) \,|\, A + A^* = 0 \text{ and } \mathrm{Tr}(A) = 0\}$$

□

Proposition 4.14. *The Lie algebra of $O(n)$ is,*

$$\mathfrak{o}(\mathfrak{n}) := \{A \in M_n(\mathbb{C}) \,|\, A + A^T = 0\}$$

Proof. A matrix A is orthogonal if and only if $A^T = A^{-1}$. By the same reasoning as the prior proof, this implies that for e^{tA} to be in $\mathfrak{o}(\mathfrak{n})$ then $(e^{-tA^T}) = (e^{tA^{-1}})$. Therefore,

$$\mathfrak{o}(\mathfrak{n}) := \{A \in M_n(\mathbb{C}) \,|\, A + A^T = 0\}$$

□

Here we actually found the Lie algebra of $SO(n)$ as well since the identity component of $O(n)$ is $SO(n)$. Therefore, $\mathfrak{o}(\mathfrak{n}) = \mathfrak{so}(\mathfrak{n})$.

Example 4.15. Using the fact that

$$\mathfrak{su}(n) := \{A \in M_n(\mathbb{C}) \,|\, A + A^* = 0 \text{ and trace}(A) = 0\}$$

we can see that $\mathfrak{su}(2)$ is the set of all matrices A such that $\mathrm{Tr}(A) = 0$ and $-A = A^*$ 2×2 matrices. To be clear,

$$\mathfrak{su}(n) := \left\{ \begin{pmatrix} ia & \overline{-b} \\ b & -ia \end{pmatrix} : a \in \mathbb{R}, b \in \mathbb{C} \right\}$$

One can check that it is spanned by the three matrices,

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

These matrices, $\sigma_1, \sigma_2, \sigma_3$, are generators of the Lie algebra.

While we have discussed some examples of Lie algebras in some detail, yet we have not actually shown that it is an **algebra**. In proposition 4.3, we showed that the Lie algebra of a matrix Lie group is a real subspace of $M_n(\mathbb{F})$, and therefore showed that there is a scalar multiplication and an addition operation contained in the subspace. However, a typical algebra has a multiplication operation as well, and it turns out matrix multiplication in G will define a certain operation in $\mathfrak{g}$. We call this operation a commutator, and we will formally define it before going into some of it's properties. First, we define the conjugation map and the adjoint.

Definition 4.16. Let G be a matrix group. For all g and $a \in G$, the conjugation map $C_g : G \to G$ is defined as,

$$C_g(a) = gag^{-1}$$

Note that since G is a group, then this implies that $C_g(a) \in G$ for all $g,$ and $a \in G$. The conjugation map is differentiable and the adjoint is defined as the derivative of the conjugation map evaluated at the identity. It is typically denoted Ad_g. To attain a simpler formula for Ad_g, recall that any $X \in \mathfrak{g}$ can be written as $X = x'(0)$ for some differentiable path in G with $x(0) = I$. Then

$$Ad_g(X) = dC_g(X)\Big|_{t=0} = \frac{d}{dt}\Big|_{t=0} gx(t)g^{-1} = gXg^{-1}$$

You'll notice that if the elements of G commute with X, then the adjoint map is the identity map on $\mathfrak{g}$. So generally, one can view the adjoint map as a measure of how much g commutes with elements in G near the identity in the direction of X.

Proposition 4.17. *For any $g \in G$ and for any $X \in \mathfrak{g}$, then $Ad_g(X) \in \mathfrak{g}$.*

Proof. This is clear from the fact that $e^{Ad_g(X)} = e^{gXg^{-1}} = ge^X g^{-1}$ by proposition 4.8, and since $ge^X g^{-1} \in G$ then $Ad_g(X) \in \mathfrak{g}$. □

Definition 4.18. The Lie Bracket for two vectors X and $Y \in \mathfrak{g}$ is given by,

$$[X, Y] = \frac{d}{dt}\Big|_{t=0} Ad_{x(t)}(Y)$$

where $x(t)$ is a differentiable path in G such that $x(0) = I$ and $x'(0) = X$.

It is important to notice that $[X, Y] \in \mathfrak{g}$ which is seen by the prior proposition. Since $x(t)$ is a path in G and $Y \in \mathfrak{g}$ then $Ad_{x(t)}(Y) \in \mathfrak{g}$.

Proposition 4.19. *For all $X, Y \in \mathfrak{g}$,*

$$[X, Y] = XY - YX$$

Before we prove this, we must first prove a simple lemma.

Lemma 4.20. *Let $x(t)$ be a differentiable path such that $x(0) = I$. Then*

$$\frac{d}{dt}\Big|_{t=0} x^{-1}(t) = -x'(0)$$

Proof. First note that $x(t) \cdot x^{-1}(t) = I$. Then by the product rule,

$$x'(t)x^{-1}(t) + x(t)(x^{-1})'(t) = 0$$

or similarly,

$$x(t)(x^{-1})'(t) = -x'(t)x^{-1}(t)$$

Evaluating at $t = 0$ gives us the equality,

$$\left.\frac{d}{dt}\right|_{t=0} x^{-1}(t) = -x'(0)$$

□

Proof (Proposition 4.19). Recall that $x(t)$ is a differentiable path through G such that $x(0) = I$ and $x'(0) = X$. Then,

$$\begin{aligned}[X, Y] &= \left.\frac{d}{dt}\right|_{t=0} Ad_{x(t)}(Y) \\ &= \left.\frac{d}{dt}\right|_{t=0} x(t)Yx^{-1}(t)\end{aligned}$$

by the product rule and the lemma,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} x(t)Yx^{-1}(t) &= x'(0)Yx^{-1}(0) + x(0)Y(x^{-1})'(0) \\ &= XY - YX\end{aligned}$$

□

From this, it is easy to note that $[X, Y] = 0$ only when X, Y commute. This commutativity in the Lie algebra corresponds to how elements in G commute in the X and Y direction. Using the above fact, we can discuss its properties.

Definition 4.21. Let G be a matrix Lie group, and $\mathfrak{g}$ be its corresponding Lie algebra. The **Lie bracket** of two elements, $a, b \in \mathfrak{g}$ is defined to be their commutator in A. The Lie bracket is a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and satisfies the following properties:

(1) For all scalars $a, b \in \mathbb{F}$ and all $x, y, z \in \mathfrak{g}$,
$$[ax + by, z] = a[x, z] + b[y, z], \; and \; [z, ax + by] = a[z, x] + b[z, y]$$

(2) For all $x \in \mathfrak{g}$
$$[x, x] = 0$$

(3) For all $x, y \in \mathfrak{g}$
$$[x, y] = -[y, x]$$

(4) For all $x, y, z \in \mathfrak{g}$ the Lie bracket satisfies the Jacobi identity,
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

Acknowledgements. I would like to think my mentor Holly Mandel for helping me throughout the entire process of writing this paper. It would not have been possible without her patience and knowledge in helping me better understand difficult concepts, and for her incredibly helpful feedback while I wrote my paper. For that, I am extremely grateful. I would also like to thank Peter May for organizing the REU and giving me an opportunity to further explore mathematics this summer.

References

[1] Hall, Brian C. Lie Groups, Lie Algebras, and Representations: An Elementary Introduction. New York: Springer, 2015.

[2] Tapp, Kristopher. Matrix Groups for Undergraduates. Providence, RI: American Mathematical Society, 2005. Print.