HOW DO ULTRAFILTERS ACT ON THEORIES? THE CUT SPECTRUM AND TREETOPS

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Abstract. We expand on and further explain the work by Malliaris and Shelah on the cofinality spectrum by doing a more thorough introduction to the problem from the perspective of ultrafilters. We present the main motivation for the study as well as the primary objects of study, the cut spectrum and the concept of treetops. Additionally, we present a more thorough proof of the first noteworthy result of their work.

Contents

1. Introduction

Ultrafilters play a significant role in model theory as they allow us to construct models of first-order theories that preserve the first order expressible structure and smooths the differences between models. Hence, it is only natural to want to study ultrafilters on their own as this may provide some insight on how we may use ultraproducts and ultrapowers to give answers to open questions in various areas of mathematics. Unfortunately, since the construction of non-principal ultrafilters requires the axiom of choice (via Zorn's lemma), we cannot study ultrafilters by looking directly at them and pondering about their structure as a set or another relevant mathematical object. Nonetheless, we know that ultrafilters act naturally on first order theories via ultraproducts, so we may study ultrafilters by looking at how different first order theories change or don't change when we take their ultraproducts.

We are interested in studying how the structure of both a model M and an ultrafilter $\mathcal{D}$ relate to the properties of the ultrapower $M^\lambda/\mathcal{D}$. One important property of models is saturation. For our purposes, saturation is a model theoretic concept that can be thought of as a measure of how 'complete' or 'full' a model is. In this paper we will work with two kinds of structures, linear orders and trees. For linear orders, the existence of elements realizing pre-cuts in the order is a notion of saturation; while the existence of upper bounds of increasing sequences plays a similar role in trees.

While it is not evident at first glance why these fullness conditions should be related in any form or way, it turns out that under suitable conditions, generally related to the regularity of the ultrafilter and the cardinals at play, many of these conditions are equivalent. The saturation properties of ultrapowers have been studied beginning in the 1960s; however, we will focus on the work by Malliaris and Shelah, who in [1] showed that a variety of fullness properties for regular ultrafilters are all equivalent and used this fact to prove a variety of open problems in set and model theory.

In this paper, we consider two kinds of interesting first-order theories, linear orders and trees, define fullness conditions for their ultrapowers under a particular ultrafilter $\mathcal{D}$ on λ and show the first steps towards proving that these conditions are equivalent. The work on this paper expands on the main relevant definitions and the proof of Theorem 2.2 in [1]. Hence, the reader might find this paper most useful when trying to understand the main work by Malliaris and Shelah, although it might be of interest on its own. We also assume that the reader has some understanding of the theory of ultrafilters and so we omit the most basic definitions and theorems in the main presentation of the work. However, the appendix contains a very concise introduction to the main definitions. A complete introduction to model theory, in general, can be found in [2] or if the reader wishes an introduction only to the relevant theory of ultrafilters [3] might be of use. Finally we would like to point out that throughout this paper we assume the regularity of ultrafilters and cardinals; while these assumptions are not necessary for this paper, they are necessary for the study of the cofinality spectrum, which is the central topic in [1].

2. The Cut Spectrum

If we want to look at how ultrafilters affect theories when we take the ultrapower, it makes sense first to look at well-known and straightforward theories. Linear orders are an excellent candidate for our study; we know what linear orders look like, and we have a good number of well-understood examples to start our study. Furthermore, linear orders have an intuitive notion of fullness from looking at cuts of the linear order (we give the relevant definitions below).

Definition 2.1. A *linear order* is a tuple $(X, <)$ where X is a set and $<$ is a total (also known as linear) order on X.

It should be easy to see that $(\mathbb{N}, <)$, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are examples of linear orders. Next, we define the notions of pre-cut and cut, which are intuitive ways of characterizing how "full" a linear order is.

Definition 2.2. Given a linear order $(X, <)$ such that $|X| = \lambda$ is regular, a pair of sequences $\langle a_\alpha \in X \mid \alpha < \kappa_1\rangle$ and $\langle b_\beta \in X \mid \beta < \kappa_2\rangle$ is a (κ_1, κ_2)*-pre-cut on* X if $a_\alpha < a_\gamma < b_\beta < b_\tau$ for all $\alpha < \gamma < \kappa_1$ and $\beta < \tau < \kappa_2$. Furthermore, we say that this is a (κ_1, κ_2)*-cut* if there does not exist a $c \in X$ satisfying $a_\alpha < c < b_\beta$ for all $\alpha < \kappa_1$ and $\beta < \kappa_2$.

Now we may ask if ultrapowers of linear orders have cuts and in that case which kinds of cuts. To simplify this problem let us restrict ourselves to the case where $\mathcal{D}$ is a regular ultrafilter and $(\mathbb{N}, <)$ is our linear order. Then we might define the first main object of our study: the cut spectrum.

Definition 2.3. Given a regular ultrafilter $\mathcal{D}$ on λ we define the *cut spectrum* of $\mathcal{D}$ as

$$C(\mathcal{D}) = \{(\kappa_1, \kappa_2) \mid \kappa_1, \kappa_2 \text{ are regular, } \kappa_1 + \kappa_2 < \lambda \text{ and } (\mathbb{N}, <)^\lambda/\mathcal{D} \text{ has a } (\kappa_1, \kappa_2) \text{ cut}\}$$

Then the first central question of this paper raises itself: when is the cut spectrum of some ultrafilter empty? Moreover, what are the implications of having an empty cut spectrum? We will return to these questions later after we introduce another way of looking at ultrafilters. Nonetheless, it is worth noting at this point that if we assume that $\mathcal{D}$ is regular, then we might replace $\mathbb{N}$ for $\mathbb{Z}$ or $\mathbb{Q}$ and the cut spectrum will remain the same.

3. Ultrapower of Trees

If we want to study ultrafilters by looking at how different structures change when we take a particular ultrapower, then it is important to look at various kinds of structures. Trees, as we will explain below, are another useful object of study since they can also exhibit a form of saturation under ultrapowers that turns out to be of particular importance.

Definition 3.1. A tuple $(\mathcal{T}, \trianglelefteq)$, where $\mathcal{T}$ is a set, is a *Tree* if $\trianglelefteq$ defines a partial order on $\mathcal{T}$ such that for any $t \in \mathcal{T}$, the set of predecessors of t is well-ordered under $\trianglelefteq$.

An example of trees are finite sequences of natural numbers with $\trianglelefteq$ defined by the initial segment. More generally, given a cardinal κ and a set A, the set of all κ-indexed sequences of elements in A is a tree when ordered by their initial segment.

The fullness condition for ultrapowers of trees that we will use in this paper is determined by the existence, or lack thereof, of upper bounds in this ordering. Consider then a tree $(\mathcal{T}, \trianglelefteq)$ and take its ultrapower with respect to some ultrafilter $\mathcal{D}$ on some infinite cardinal λ. Note that $\mathcal{M} = (\mathcal{T}, \trianglelefteq)^\lambda/\mathcal{D}$ is no longer a tree, since the idea of well-ordering cannot be expressed in first-order logic. However, $\trianglelefteq$ still partially orders $\mathcal{M}$ and for any $t \in (\mathcal{T}, \trianglelefteq)^\lambda/\mathcal{D}$ the set of predecessors is still linearly ordered. Next, suppose you have an increasing sequence $(c_\alpha)_{\alpha<\kappa}$ of elements of $\mathcal{M}$ with $\kappa < \lambda$. One might ask whether or not this sequence has an upper bound on $\mathcal{M}$. If it always does, then we have a notion of the ultrapower being full or saturated in some sense.

Definition 3.2. Let $\mathcal{D}$ be a regular ultrafilter defined on λ. We say that $\mathcal{D}$ has λ *treetops* if for any tree $(\mathcal{T}, \trianglelefteq)$ and any infinite regular cardinal $\kappa < \lambda$, any strictly increasing κ-indexed sequence in $(\mathcal{T}, \trianglelefteq)^\lambda/\mathcal{D}$ has an upper bound.

The previous definition means that $\mathcal{D}$ induces the realization of upper bounds, which we call treetops in this context, for all trees and all κ-indexed sequences with $\kappa < \lambda$. The existence of treetops is relevant to our study as it is independent of the tree we choose, and so we might consider as a property of $\mathcal{D}$ and not of trees per se. Therefore, we might begin studying ultrafilters by looking at which filters have λ treetops and what are the consequences of exhibiting this behavior.

Finally, it is important to notice that this is not the only possible fullness condition. For example, we might look at the set of predecessor of a point in the tree, which happens to be a linear order, and look at the cut spectrum. Then, we might say that a tree is full or saturated if for every point in the ultrapower of the tree, the linear order defined by its predecessors has an empty cut spectrum. We will not use this notion in the rest of the paper, but this shows that fullness can be thought of in different ways.

4. Trees and the Cut Spectrum

We have introduced two different ways in which an ultrafilter might “fill up” the first-order theory via ultrafilters. Then, we may ask whether these notions are related in some way and what the necessary and sufficient conditions are for this relation to exist. The simplest relationship between the cut spectrum and the idea of treetops is presented in Lemma 2.2 of [1]. In this section, we give a more thorough proof of this lemma.

Theorem 4.1 (Lemma 2.2 in [1])**.** *Suppose $\mathcal{D}$ is a regular ultrafilter on λ with λ^+-treetops, and let $\kappa < \lambda^+$ be a regular cardinal. Then $C(\mathcal{D})$ has no (κ, κ)-cuts, i.e. $(\kappa, \kappa) \notin C(\mathcal{D})$.*

Proof. We will mostly follow the notation on [1]; however, we will diverge from it in the interest of clarity whenever it seems appropriate. Let $M = (N, <)$ and define M_1 to be the ultraproduct of M with respect to $\mathcal{D}$ (i.e. $M_1 = M^\lambda/\mathcal{D}$). Now, assume for the sake of contradiction that the theorem fails, which implies that we have two sequences $\mathbf{a} = \langle a_\alpha \in X \mid \alpha < \kappa\rangle$ and $\mathbf{b} = \langle b_\alpha \in X \mid \alpha < \kappa\rangle$ such that $M_1 \models a_\alpha < a_\beta < b_\beta < b_\alpha$ for all $\alpha < \beta < \kappa$ and there does not exist $c \in M_1$ satisfying $a_\alpha < c < b_\alpha$ for all $\alpha < \kappa$.

Since our main assumption refers to trees and not to linear orders, we want to construct a tree structure that is somehow related to this cut. In a suitable construction, we should be able to use the treetops hypothesis to build an element of M_1 that satisfies the cut, and thus provide us with a contradiction. Define $\mathcal{T}^M$ to be the set consisting of finite sequences of pairs of natural numbers, i.e.

$$\mathcal{T}^M = \{\langle (a_i, b_i) \in \mathbb{N} \times \mathbb{N} \mid i < n\rangle \mid n \in \mathbb{N}\}.$$

We can order $\mathcal{T}^M$ in a natural way by looking at their initial segment since our sequences are indexed by an initial segment of $\mathbb{N}$ as explain in Section 3. This implies that $(\mathcal{T}^M, \trianglelefteq)$ is indeed a tree.

Since the proof can get very involved at some stages, we give a brief overview here. Our plan is to construct a tree that models the cut we believe to exists and then use the treetops hypothesis to prove that the cut actually has a point in it, and therefore it is not a cut. To do this, we'll need to be able to talk about sequences in a formal sense, meaning that we need some function and relation symbols that operate on the kinds of sequences we want to talk about. So we will expand the language and add function and relation symbols (along with their interpretation in the new language); doing this in $\mathbb{N}$ is straightforward once you know what relevant information you want to be able to express in first-order logic and it is the subject of the next couple of paragraphs. Once we have an expansion on the base model M, we will use the fact that ultrapowers commute with reducts to see that the expansion will naturally transfer to the ultrapower, along with any first order properties of the expanded symbols (we will give the details of this below). Once we have the expansion on the ultrapower, we will use transfinite induction to construct a sequence that models our cut on a tree that is models cuts in M_1. Finally, we will use the treetops hypothesis to find an upper bound to this sequence and show that this upper bound fills the cut.

Notice that our current language only allows us to talk about ordering and not about the kind of finite sequences we find in M. Hence, let us expand the language by adding the necessary tools to be able to reason about these sequences.[1] Some work using cardinal arithmetic allows us to show that there are only countably many functions from some initial segment of $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ (proof of this fact can be found in the appendix), so we may associate each function of this type with a number $d \in \mathbb{N}$. Given a $d \in \mathbb{N}$, let us define the *length* of f_d to be the smallest natural number for which f_d is not defined. Since f_d is always a function from some initial segment of the naturals, this number always exists and we denote it by $\mathbf{lg}(d)$. It is important to note that technically $\mathbf{lg} : \mathbb{N} \to \mathbb{N}$ but since each natural number is associated with a unique function of this type we will abuse notation and use either $\mathbf{lg}(d)$ or $\mathbf{lg}(f_d)$ interchangeably. One important property of $\mathbf{lg}$ is that for any $d, n, m \in \mathbb{N}$ such that $n < \mathbf{lg}(f_d) \leq m$, we know that $f_d(n)$ is defined but $f_d(m)$ is not.

[1] At this point we diverge from the proof presented in [1] as the expansion presented and the one we will construct are different. Again, this is done for the sake of clarity; however, the main idea of the proof remains the same.

With this in mind, we may also define the **maxdom** (an abbreviation of the maximum of the domain) of this kind of functions by setting $\mathbf{maxdom}(d) = \mathbf{maxdom}(f_d) = \max(dom(f_d)) = \mathbf{lg}(f_d) - 1$. Hence, $\mathbf{maxdom}(f_d)$ is the largest natural number for which f_d is defined.

Next, define a quaternary relation symbol $R(a, b, c, d)$ that holds whenever $a \leq \mathbf{lg}(f_d)$, meaning that $f_d(a)$ is defined, and $f_d(a) = (a, b)$. Notice that this predicate is subject to a few axioms that come as a consequence of the fact that we are considering all possible functions from initial segments to $\mathbb{N} \times \mathbb{N}$. We enumerate the most relevant ones:

(1) For all $a, d \in \mathbb{N}$ with $a \leq \mathbf{maxdom}(f_d)$, there exists $b, c \in \mathbb{N}$ satisfying $R(a, b, c, d)$.
(2) If $R(a, b, c, d)$ for some $a, b, c, d \in \mathbb{N}$, then for all $a' \leq a$ there exists $b', c' \in \mathbb{N}$ such that $R(a', b', c', d)$.
(3) For any $k \in \mathbb{N}$ and for any collection of numbers $a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_k$, there exists some $d \in \mathbb{N}$ such that $R(a_i, b_i, c_i, d)$ for $i = 1, \ldots, k$.
(4) Given any $d \in \mathbb{N}$ and any two $b, c \in \mathbb{N}$, there exists some $d' \in \mathbb{N}$ such that $f_d(n) = f_{d'}(n)$ for $n \leq \mathbf{maxdom}(f_d)$, $f_{d'}(\mathbf{lg}(f_d)) = (b, c)$ and $\mathbf{lg}(f_{d'}) = \mathbf{lg}(f_d) + 1$. This means that we can always extend our sequence by one pair of elements (and hence by finitely many).

Finally, we may also define an evaluation function $E : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that $E(d, a) = (b, c)$ iff $R(a, b, c, d)$. This also gives rise to a projection function $P(d, a, n) : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is defined by $P(d, a, 0) = b$ and $P(d, a, 1) = c$ whenever $R(a, b, c, d)$ and it is not defined otherwise.

Hence, we may expand to a language $\mathscr{L} = (\leq, \mathbf{lg}, \mathbf{maxdom}, R, E, P)$ and consider $\mathbb{N}$ in this new language with the interpretations given above. We call this set M^+ and observe that in this extension all of the following are true:

(1) $\mathbb{N}$ is still the domain set of M^+ as we did not add new elements and it is a definable set.
(2) $\mathcal{T}^{M^+}$ is a definable set with a definable ordering relation $\trianglelefteq$.
(3) The elements of $\mathcal{T}^{M^+}$ are functions from an initial segment of $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, which happen to be all the sequences considered by the predicate defined above.
(4) After expanding the model, we may define the following functions, uniformly, for all $x \in \mathcal{T}^{M^+}$:
   (i) A function $\mathbf{lg} : \mathcal{T}^{M^+} \to \mathbb{N}$ that gives the length of the sequence (i.e. the smallest $n \in \mathbb{N}$ for which the pair (a_n, b_n) is not defined).
   (ii) The function $\mathbf{maxdom}(x)$, which gives us the largest number for which the sequence is defined; i.e. $\mathbf{lg}(x) - 1$.
   (iii) For any $n \leq \mathbf{maxdom}(x)$, an evaluation function $x(n) \in \mathbb{N}^2$.
   (iv) For any $n \leq \mathbf{maxdom}(x)$, two projection functions denoted $x(n, 0)$ and $x(n, 1)$, which evaluate to the first and second coordinates of $x(n)$ respectively.

   Here the reader should be able to see that this is true since any $x \in \mathcal{T}^{M^+}$ corresponds to a unique f_d and all of the above are defined for f_d in the extended language. The addition of these symbols, along with their interpretation, will allow us to control the behavior of the sequences f_d even in the ultrapower, where the existence of hyperintegers can make their behavior more unpredictable.

Let us now consider a way to induce the same expansion on M_1 (i.e. M_1^+). Once we have an extension on M, we have to wonder how to transfer this expansion to the ultraproduct, and wether there is a natural (or correct) way of doing this. It turns out that since ultrapowers commute with reducts (see A.8 for the formal statement), there is a natural way in which our extension of the base model transfers to the ultrapower. Intuitively, ultrapowers commute with reducts states that expanding the language and then taking the ultrapower is the same as taking the ultrapower and the expanding the language, as far as first order logic is concerned. Therefore, once we know how the expansion looks and how the ultrapower of that expansion looks (and we do because of Łoś theorem); we also know how the expansion of the ultrapower looks like. It happens to be the same as the ultrapower of the expansion.

Applying the previous observation to this concrete case, we can consider a related model in which we do the operations in reverse: the model $(M^+)^\lambda/\mathcal{D}$ has an unambiguous definition since we know what the expansion looks like on M and we know what the ultraproduct of any model looks like as long as we only consider sentences in first order logic, and in this case we do. Furthermore, in this model we can speak about the same kind of sequences (now defined on the non-standard naturals) and all of the properties we defined above are still true. Hence, we have one canonical choice: define the expansion on M_1 to be that which makes it so that $(M^+)^\lambda/\mathcal{D} = (M^\lambda/\mathcal{D})^+ = M_1^+$. This choice makes sense since the ultrapower did not add new symbols; it just created new interpretations for them in the ultrapower, so our expansion on M_1 will add the same symbols and give the same interpretation that we obtained via Łoś theorem by taking the ultrapower of M^+ (this should provide the reader with a reason as to why ultrapowers commute with reducts might be true). Therefore, we can now talk about the tree $\mathcal{T}^{M_1^+}$ as the set of all functions from an initial segment of the nonstandard integers to pairs of nonstandard integers such that (1)-(4) above still hold. Notice that all of the functions above might produce a non-standard result as output now.

Let $\varphi(x)$ be a sentence in first order logic stating that for $x \in \mathcal{T}^{M_1^+}$ and $n < m \leq \mathbf{maxdom}(x)$ implies $x(n,0) < x(m,0) < x(m,1) < x(n,1)$. Notice that φ states that the first coordinate is an ascending sequence, the second one is a descending sequence and that every element in the first coordinate is less than every element of the second coordinate; i.e. x is describing a pre-cut on M_1^+ with the first coordinate expressing the ascending sequence and the second coordinate expressing the descending one. If we consider the subset of $\mathcal{T}^{M_1^+}$ that satisfies φ we obtain a subtree of $\mathcal{T}^{M_1^+}$, which we will henceforth refer to as $\mathcal{T}_*$. It is easy to see that $\mathcal{T}_*$ is a subtree since if $\mathcal{T}^{M_1^+} \models \varphi(c)$ for some $c \in \mathcal{T}^{M_1^+}$ then $\mathcal{T}^{M_1^+} \models \varphi(c\restriction_n)$ for any $n \leq \mathbf{maxdom}(c)$.

Next, we want to model our cut as a sequence of elements of $\mathcal{T}_*$. Hence, we want to construct a sequence of elements $c_\alpha \in \mathcal{T}_*$ and a sequence of $n_\alpha \in M_1$ for $\alpha < \kappa$ such that for $\alpha < \beta < \kappa$ we can deduce $c_\alpha \trianglelefteq c_\beta$ in $\mathcal{T}_*$, $n_\alpha = \mathbf{maxdom}(c_\alpha)$, $c_\alpha(n_\alpha, 0) = a_\alpha$ and $c_\alpha(n_\alpha, 1) = b_\alpha$ for $\alpha < \kappa$. Notice that this implies that c_α has the first α elements of $\mathbf{a}$ in its first coordinate and the first α elements of $\mathbf{b}$ in the second coordinate. So the c_α serve as a representation of the cut on $\mathcal{T}_*$. To construct this sequence we proceed by induction:

Base Case: Let $c_0 = \langle (a_0, b_0) \rangle$ and $n_0 = 0$.

Inductive Step when $\alpha = \beta + 1$: If α is a successor ordinal, we have an element c_β satisfying the conditions above. Then, we can just add (a_α, b_α) to the end of the sequence c_β. More formally, $c_\alpha = c_\beta \frown \langle (a_\alpha, b_\alpha) \rangle$ and $n_\alpha = n_\beta + 1$. Notice that up to β we satisfied φ by the definition of $\mathbf{a}$ and $\mathbf{b}$. Next, notice that the pair (a_α, b_α) also satisfies φ by the definition of the sequences. Hence, c_α satisfies φ and we get $c_\alpha \in \mathcal{T}_*$.

Inductive Step when α is a limit ordinal: Notice that the sequence $\langle c_\beta \mid \beta < \alpha < \kappa < \lambda \rangle$ is an ascending sequence on $\mathcal{T}_*$. Then, by our hypothesis of λ^+-treetops, we can find an element $c_* \in \mathcal{T}$ (and not $\mathcal{T}_*$, but we will deal with this later) that is an upper bound to our sequence (i.e. $c_\beta \trianglelefteq c_*$ for $\beta < \alpha$). Let $n_* = \mathbf{maxdom}(c_*)$ and by the definition of $\trianglelefteq$ we deduce that

$$a_\beta = c_\beta(n_\beta, 0) = c_*(n_\beta, 0) < c_*(n_*, 0) < c_*(n_*, 1) < c_*(n_\beta, 1) = c_\beta(n_\beta, 1) = b_\beta; \quad \text{for } \beta < \alpha.$$

However, since we obtained c_* from our hypothesis, we are not guaranteed that at every step in the sequence c_* models the cut; that is, we might have introduced some additional points that are not in the cut when we obtained the upper bound. Recall that c_* is only guaranteed to be an upper bound not a least upper bound or an optimal upper bound; it might be an upper bound with uncountably many points past α. Moreover, it is possible that $c_*(n_*, 0) > a_\alpha$ or $c_*(n_*, 1) < b_\alpha$. This is problematic since we might have missed the cut altogether. To fix this, observe that the set

$$N = \{n \leq n_* \mid c_*(n, 0) < a_\alpha \wedge b_\alpha < c_*(n, 1)\}$$

contains all $n \leq n_*$ for which c_* still models the cut. Moreover, this set is definable in M_1^+ and is bounded above by n_*. Thus, $N \subset M_1^+$ is a nonempty bounded set, but remember that in $\mathbb{N}$ every nonempty bounded set has a maximum; hence, by Łoś theorem, and more precisely the transfer principle (see A.7 for the formal statement), it follows that this set has an upper bound m_*. We may now consider only the restriction $c_* \restriction_{m_*}$. Notice that since, by hypothesis, we modeled the cut for each $\beta < \alpha$, so that $c_\beta \trianglelefteq c_* \restriction_{m_*}$ for each $\beta < \alpha$. This implies that up until α we modeled the cut perfectly, and starting at α we might have an unknown number of points. However, they all respect the cut in the sense that we may still add (a_α, b_α) at the end of the sequence and obtain a sequence that is increasing in the first coordinate and decreasing on the second one. xTherefore, we can define $c_\alpha = c_* \restriction_{m_*} {}^\frown \langle (a_\alpha, b_\alpha) \rangle$ and $n_\alpha = m_*$. Finally, we have to wonder if $c_\alpha \in \mathcal{T}_*$, but notice that the restriction we did above, along with the definition of $\mathbf{a}$ and $\mathbf{b}$, guarantees that c_α satisfies φ and so $c_\alpha \in \mathcal{T}_*$.

Therefore, we have a sequence $\langle c_\alpha \mid \alpha < \kappa \rangle$ in $\mathcal{T}_*$ that represents the cut. Again, we may use the treetops hypothesis to obtain an upper bound $c_\kappa \in \mathcal{T}_*$ for which $n_\kappa = \mathbf{maxdom}(c_\kappa)$ and $c_\alpha \trianglelefteq c_\kappa$ for all $\alpha < \kappa$. Now, the fact that $c_\kappa \in \mathcal{T}_*$ implies that c_κ satisfies φ. Therefore, for all $\alpha < \kappa$ it is true that

$$a_\beta = c_\beta(n_\beta, 0) = c_\kappa(n_\beta, 0) < c_\kappa(n_\kappa, 0) < c_\kappa(n_\kappa, 1) < c_\kappa(n_\beta, 1) = c_\beta(n_\beta, 1) = b_\beta.$$

As a result, both $c_\kappa(n_\kappa, 0)$ and $c_\kappa(n_\kappa, 1)$ are elements in M_1^+ that realize the cut. But since M_1^+ is just an extension of M_1, it is also true that $c_\kappa(n_\kappa, 0)$ and $c_\kappa(n_\kappa, 1)$ are elements of M_1 that realize the cut. Hence, we have obtained a contradiction. □

It is important at this point to underscore the most vital parts of this proof. First, from the theory of ultraproducts we used Łoś 's theorem to carry desirable properties to the ultraproduct, like every nonempty bounded set of nonstandard natural numbers has a greatest element. We also expanded the language to be able to talk about sequences of pairs of natural numbers in such a way that there exists a natural correspondence between natural numbers, the kind of sequences we want to talk about and elements in the tree, which then we applied Łoś to carry to the ultraproduct. These techniques are valuable beyond this proof and come up both in the rest of the work in [1] and, more generally, in other works in model theory.

5. Conclusion

The primary objective of this paper is to explain in detail one of the first, but not the most evident or trivial, results in [1]. This theorem serves as the opening to a much greater and richer and complex theory concerning regular ultrafilters and their properties. In fact, one of the main discoveries by Malliaris and Shelah generalizes the work on this paper as follows:

Theorem (Theorem 10.25 in [1])**.** *Let $\mathcal{D}$ be a regular ultrafilter on $\mathcal{I}$, with $|\mathcal{I}| = \lambda$. Then the following are all equivalent:*

(1) $\mathcal{D}$ has λ^+ treetops.
(2) $C(\mathcal{D}) = \emptyset$
(3) If $\kappa < \lambda$ is a regular cardinal, then $(\kappa, \kappa) \notin C(\mathcal{D})$
(4) $\mathcal{D}$ is λ^+-good according to Kiesler's definition of good ultrafilters

Of note are two important consequences of this theorem. First, the fact that considering symmetric cuts is sufficient to determine whether or not the cut spectrum is empty is a non-trivial fact and underscores the importance of the theorem we proved above. Furthermore, knowing that regular ultrafilters with an empty cut spectrum (or equivalently λ^+ treetops) are also λ^+-good is a powerful tool when trying to understand Kiesler's order and the structure of ultrafilters themselves. Also, as good ultrafilters exhibit many interesting and useful properties that can be used when solving more convoluted problems in model theory, the previous theorem provides us with a powerful tool to find good ultrafilters. If the reader wishes to continue exploring this theory, the 1991 paper by Malliaris and Shelah further explores this topic and also solves some then considered open problems in model theory and set theory.

Appendix

Definition A.1. An *Ultrafilter* $\mathcal{D}$ over a set $\mathcal{I}$ with $|\mathcal{I}| = \lambda$ is a set $\mathcal{D} \subset \mathscr{P}(\mathcal{I})$ satisfying all of the following properties:

(1) $\emptyset \notin \mathcal{D}$ and $\mathcal{I} \in \mathcal{D}$.
(2) If $X \in \mathcal{D}$ and $X \subset Y$, then $Y \in \mathcal{D}$.
(3) If $X, Y \in \mathcal{D}$, then $X \cap Y \in \mathcal{D}$.
(4) For all $X \in \mathcal{I}$ either $X \in \mathcal{D}$ or $X^c \in \mathcal{D}$ but not both.

Definition A.2. An ultrafilter $\mathcal{D}$ over $\mathcal{I}$ is said to be *Non-Principal* if it is not of the form $\{X \subset \mathcal{I} \mid x \in X\}$ for some $x \in \mathcal{I}$. Conversly, an ultrafilter is said to be principal if it is of the form specified above. Non-Principal ultrafilters are also known as *free* ultrafilters.

The existence of non-principal ultrafilters over infinite sets can be proven using Zorn's lemma.

Definition A.3. Two elements $f, g \in \prod_{i \in I} A_i$ are said to be *modulo $\mathcal{D}$ equivalent* if $\{i \in I \mid f(i) = g(i)\} \in \mathcal{D}$ and we write $f =_{\mathcal{D}} g$ to indicate this relationship. Furthermore, we define $[f]_{\mathcal{D}}$ as the equivalence class of all functions $g \in \prod_{i \in I} A_i$ such that $g =_{\mathcal{D}} f$ for some function $f \in \prod_{i \in I} A_i$.

Definition A.4. The *ultraproduct of $\{A_i\}_{i \in I}$ modulo $\mathcal{D}$* is

$$\prod_{i \in I} A_i \,/\mathcal{D} := \{[f]_{\mathcal{D}} \mid f \in \prod_{i \in I} A_i\}.$$

Furthermore, If we let $A_i = A$ for some set A, then the *ultrapower of A modulo $\mathcal{D}$* is

$$A^I/\mathcal{D} = \prod_{i \in I} A \,/\mathcal{D} = \{[f]_{\mathcal{D}} \mid f \in \prod_{i \in I} A\}.$$

Definition A.5. Let $\mathcal{D}$ be an ultrafilter over some set I and let $M_i = (A_i, I_i)$ be an $\mathcal{L}$-structure of some language $\mathcal{L}$ for all $i \in I$. Then the ultraproduct $M = \left(\prod_{i \in I} A_i/\mathcal{D}, I\right)$ is also an $\mathcal{L}$-structure with an interpretation function I and defined as follows:

- If c is a constant in $\mathcal{L}$, then $I(c) = [\langle I_i(c) \mid i \in \mathcal{I}\rangle]_{\mathcal{D}}$.

- If f is a function symbol of arity n and $g_1, \ldots, g_n \in \prod_{i \in \mathcal{I}} A_i$, then

$$I(f)([g_1]_{\mathcal{D}}, \ldots, [g_n]_{\mathcal{D}}) = [\langle I_i(f)(g_1(i), \ldots, g_n(i)) \mid i \in \mathcal{I} \rangle]_{\mathcal{D}}$$

- If R is a relation symbol of arity n and $g_1, \ldots, g_n \in \prod_{i \in \mathcal{I}} A_i$. then

$$([g_1]_{\mathcal{D}}, \ldots, [g_n]_{\mathcal{D}}) \in I(R) \quad \textit{iff} \quad \{i \in I \mid (g_1(i), \ldots, g_n(i)) \in I_i(R)\} \in \mathcal{D}.$$

From here on out we assume all formulas φ can be expressed in first-order logic.

Theorem A.6 (Łoś 's Theorem)**.** *Let $\mathcal{L}$ be a language, $\mathcal{I}$ be a set with some ultrafilter $\mathcal{D}$ on $\mathcal{I}$ and A_i be an $\mathcal{L}$-structure for all $i \in \mathcal{I}$. Then for all $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and each $f_1, \ldots, f_n \in \prod_{i \in \mathcal{I}} A_i$ we have that*

$$M = \left(\prod_{i \in I} A_i / \mathcal{D}, I \right) \models \varphi([f_1]_{\mathcal{D}}, \ldots, [f_n]_{\mathcal{D}}) \quad \textit{iff} \quad \{i \in I \mid M_i \models \varphi(f_1(i), \ldots, f_n(i))\} \in \mathcal{D}.$$

Corollary A.7 (Transfer Principle)**.** *Let $\mathcal{L}$ be a language, I be a set with some ultrafilter $\mathcal{D}$ on I and M be an $\mathcal{L}$-structure. Then for all sentence φ of $\mathcal{L}$ (meaning no free variable) we have that $M^I/\mathcal{D} \models \varphi$ iff $M \models \varphi$. In other words, $M^I/\mathcal{D}$ and M satisfy exactly the same theory.*

Corollary A.8 (Ultrapowers commute with reducts)**.** *Let $\mathcal{L}, \mathcal{L}'$ be languages in first-order logic such that $\mathcal{L} \subset \mathcal{L}'$. Furthermore, let M be a $\mathcal{L}'$-structure and $\mathcal{D}$ an ultrafilter on $\lambda > \aleph_0$. Then*

$$(M^\lambda/\mathcal{D}) \mid_{\mathcal{L}} = (M \mid_{\mathcal{L}})^\lambda/\mathcal{D}.$$

Definition A.9. An ultrafilter $\mathcal{D}$ over some infinite set $\mathcal{I}$ is said to be *regular* if there exists a collection of sets $\langle X_\kappa \mid \kappa < \lambda \rangle$, also known as a regularizing family, such that each $X_\kappa \in \mathcal{D}$ and such that for each $i \in \mathcal{I}$, i belongs to finitely many X_κ.

Definition A.10. A cardinal λ is said to be *regular* if

$$\lambda = \inf\{\kappa \mid \exists \langle a_\gamma \mid \gamma < \kappa \rangle \text{ such that } \lim_{\gamma \to \kappa} a_\gamma = \lambda\}.$$

By taking the sequence of all cardinals less than λ, it is clear that the infimum of this set exists and it is at most λ. However, if the infimum is less than λ, we say that λ is a *singular* cardinal.

Theorem A.11. *There are only countably many functions from an initial segment of* $\mathbb{N}$ *to* $\mathbb{N} \times \mathbb{N}$

Proof. We rely heavily on cardinal arithmetic for this proof and assume that $\mathbb{N}$ includes zero. Notice that for any $n \in \mathbb{N}$ the set $\{g : \{0, \ldots, n\} \to \mathbb{N} \times \mathbb{N}\}$ has cardinality $\aleph_0^n$ as $\mathbb{N} \times \mathbb{N}$ is countable. Hence, the set of all functions from an initial segment of $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ must have cardinality

$$\sum_{n \in \mathbb{N}} \aleph_0^n$$

But notice that the fundamental theorem of cardinal arithmetic implies that $\aleph_0^n = \aleph_0$ (or equivalently the reader should know that the cartesian product of finitely many countable sets is countable). This implies that $\sum_{n \in \mathbb{N}} \aleph_0^n = \sum_{n \in \mathbb{N}} \aleph_0 = \aleph_0 \cdot |\mathbb{N}| = \aleph_0^2$ and this is just $\aleph_0$ by the fundamental theorem of cardinal arithmetic. This completes the proof. □

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