DIFFERENTIAL TOPOLOGY: MORSE THEORY AND THE EULER CHARACTERISTIC

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Abstract. This paper uses differential topology to define the Euler characteristic as a self-intersection number. We then use the basics of Morse theory and the Poincaré-Hopf Theorem to prove that the Euler characteristic equals the sum of the alternating Betti numbers.

Contents

1. Introduction

Differential topology, the subject of this paper, is the study of intrinsic topological properties of manifolds endowed with a smooth structure. Morse theory is the subfield of differential topology that accomplishes this by studying the analytic properties of the functions that can be defined on such manifolds.

As a first simple example, consider the height function on $S^1 \subseteq \mathbb{R}^2$; that is, the function that assigns to each point of S^1 its vertical coordinate. It is easy to see that it achieves a local maximum at exactly one point, and a local minimum at exactly one point. Now consider the 'smooth deformation' of S^1 depicted below, obtained by 'pushing in' from the top of the circle:

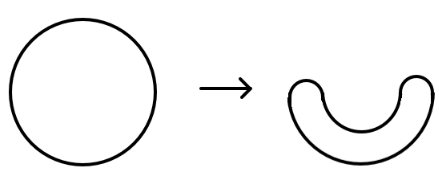

Figure 1. Smoothly deforming S^1.

The height function on the manifold shown on the right has four critical points, two local maximum and two local minimum points. Morse theory will allow us to prove that regardless of how we deform the circle, as long as we do it smoothly, the number of maximum points of the height functional will always be equal to the number of minimum points of that functional. This fact, which generalizes to a much more impressive result, is related to an important topological invariant of the manifold.

The topological invariant aforementioned is the **Euler characteristic**, which is usually defined as the alternating sum of the number of k-cells in a CW-complex homotopic to the manifold. For surfaces (closed, compact, 2-manifolds) that can be triangulated, that is, that can be cut into (not necessarily planar) triangles, the Euler characteristic is the familiar number $V - E + F$, where V is the number of vertices of the triangulation, E the number of edges, and F the number of faces of the triangulation.

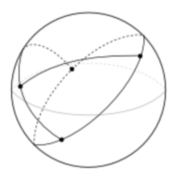

FIGURE 2. A triangulation of S^2.

In our approach, we will define the Euler characteristic using the differential structure of a manifold. Impressively, we will show that this new definition agrees with the purely topological definition mentioned in the previous paragraph. This will be achieved in two main steps: we will first use the Poincaré-Hopf Theorem to show that the 'differential' definition of the Euler characteristic is in fact a topological invariant, and then Morse theory will show us that this topological invariant is — as one would expect — the same as the Euler characteristic defined using CW-complexes.

Although prerequisites were reduced whenever possible, we assume the reader is familiar with the notion of homotopy, and with the definitions of a smooth manifold, of a tangent space and of the derivative of a function on a manifold. There is no formal prerequisite in algebraic topology and homological algebra, but since the focus of the paper is really on what puts the differential in 'differential topology', the exposition on cell complexes has been shortened as much as possible.

2. Basics of Smooth Manifolds

In this section, we develop preliminary results regarding regularity of maps between manifolds.

Definition 2.1. Let $f : X \to Y$ be a smooth map of manifolds. Then $y \in Y$ is a *critical value* of f if for some $x \in f^{-1}(y)$ the derivative $df_x : T_x(X) \to T_y(Y)$ is not of full rank.

Remark 2.2. In the above definition, a $n \times m$ rectangular matrix is said to be **full rank** if it is of full column rank and $n > m$ or if it is of full row rank and $m > n$.

The points $x \in X$ at which df_x fails to have full rank are called **critical points** of f. If $x \in X$ is not a critical point, it will be called a **regular point**; if $y \in Y$ is not a critical value, it will be called a **regular value**.

The following theorem from multivariable calculus tells us that regularity completely characterizes the behavior of a function, at least locally:

Theorem 2.3. (Inverse Function Theorem) *If $f : X \to Y$ is a smooth map between manifolds of the same dimension, and $x \in X$ is a regular point of f, then f is a local diffeomorphism at x.*

An important corollary is that if $x \in X$ has a parametrizable neighborhood, then so does $f(x) \in Y$, and vice versa. This can be extended to the case where $\dim X \neq \dim Y$.

For example, suppose that $k = \dim X < \dim Y$; if df_x is full rank at a point, we can still apply the Inverse Function Theorem to show that locally f looks like $f(x_1, ..., x_k) = (x_1, ..., x_k, 0, ..., 0)$ for some local coordinates of X and Y around x and $f(x)$ such that both x and $f(x)$ correspond to $(0, 0, .., 0)$.

However, this does not guarantee that $f(X)$ will be a submanifold of Y in general. Observe that f does not need to map the parametrizable neighborhood of x onto an *open* parametrizable set containing $f(x)$, in the relative topologies, so f does not necessarily map into a submanifold of Y.

The correct notion of a mapping of submanifolds of the domain into submanifolds of the codomain is that of an embedding. A function $f : X \to Y$, with $\dim X < \dim Y$ which has no critical points is called an **immersion**. It is an **embedding** if it is also injective and **proper** (a map is called proper the preimage under f of compact sets is compact).

Theorem 2.4. *If $f : X \to Y$ is an embedding, then the image of X in Y is a submanifold of Y.*

Proof. By the above considerations, we need only to prove that f maps an open neighborhood U of a point $x \in X$ diffeomorphically onto an open set V of $f(x) \in Y$.

Suppose V is not open. Then there is a sequence $\{y_i\}$ in $Y \setminus V$ such that $y_i \to y$ and $y \in V$. Now define $x_i = f^{-1}(y_i)$ (since f is injective) and consider the sequence $\{x_i\}$. The set $(\bigcup_i y_i) \cup \{y\}$ is certainly compact, so its preimage must be as well. Therefore, by replacing x_i by an appropriate subsequence we must have $x_i \to x$ for some x. Injectivity of f again implies that $f(x) = y$, so $x \in U$. Since U is open some of the x_i must also be in U. Then $f(x_i) = y_i \in V$, a contradiction. □

Similarly, when $\dim X \geq \dim Y$, we call a map f a **submersion** at $x \in X$ if x is a regular point of f. In this case if $\dim X = m$ and $\dim Y = n$ then the Inverse Function Theorem allows us to write $f(x_1, ..., x_m) = (x_1, ..., x_n)$ for appropriate local coordinates such that $x = (0, 0, ..., 0)$ and $f(x) = y = (0, 0, ..., 0)$. The preimage of $f(x) = 0$ is then the set of points $\{(0, ..., 0, x_{n+1}, ..., x_m)\}$ and it is open in the relative topology of the preimage of the neighborhood in which the coordinate system is defined. Therefore $f^{-1}(y)$ is a submanifold of X. Clearly, the dimension of $f^{-1}(y)$ is $m - n$.

The next theorem guarantees the existence, and in fact the abundance, of regular values in the image of a smooth function.

Theorem 2.5. (Sard's Theorem) *Let $f : X \to Y$ be a smooth map between the manifolds X and Y. Then the set of critical values of f has Lebesgue measure zero.*

The reader not acquainted with the definition of a measure can instead think of Sard's Theorem as stating that the set of regular values of f is dense in Y. This form of the theorem will suffice for all results in the paper. Although Sard's Theorem will be of great importance throughout the paper, we refer the reader to [2] for a proof since it consists mostly of techniques from multivariable calculus, such as Fubini's Theorem and Taylor's Theorem.

3. Transversality and Oriented Intersection Theory

Here we develop the main concepts of differential topology that will be used in future sections. The central definition of the section is:

Definition 3.1. Let X and Y be submanifolds of Z. Then X intersects Y **transversally** at $x \in X \cap Y$ if

$$T_x(X) + T_x(Y) = T_x(Z).$$

We say X and Y are **transverse** if they are transverse everywhere, and we write $X \pitchfork Y$. Note that two non-intersecting manifolds are trivially transverse.

One should notice that transversality is defined relative to the ambient manifold in which the intersecting submanifolds are embedded: two non-parallel lines in the xy-plane intersect transversally when regarded as submanifolds of $\mathbb{R}^2$, but not when considered as embedded in $\mathbb{R}^3$.

In a sense, transversality restricts the notion of an intersection. It guarantees that the two manifolds are not only 'touching', but actually 'crossing' at their intersection. This requirement turns out to be much more topologically relevant than simple intersection. For instance, the next theorem shows that transverse intersections are also smooth manifolds:

Theorem 3.2. *Let X and Y be non-vacuously intersecting submanifolds of Z such that $X \pitchfork Y$. Then $X \cap Y$ is also a submanifold of Z and*

$$\text{codim } (X \cap Y) = \text{codim } X + \text{codim Y}.$$

Proof. Let dim $X = k$, dim $Y = m$ and dim $Z = n$. We need to show that every point $p \in X \cap Y$ has a parametrizable neighborhood U. By the considerations in Section 2, the inclusion map $i : X \hookrightarrow Z$ can be written as $i(x_1, ..., x_k) = (x_1, ..., x_k, 0, ..., 0)$ for some local coordinates of U. Consider the map $f : U \to \mathbb{R}^{n-k}$ given by $f(x_1, ..., x_m) = (x_{m-k+1}, ..., x_m)$ on the same local coordinates; then we have $f^{-1}(0) = U \cap X$. Similarly, one can construct a map $g : U \to \mathbb{R}^{n-m}$ such that $g^{-1}(0) = U \cap Y$. By construction, the derivatives df_z and dg_z are surjective everywhere on U, so zero is a regular value of both f and g.

Now we have a natural map to parametrize $X \cap Y$. By hypothesis, p is a regular point for the map $(f, g) : U \to \mathbb{R}^{2n-m-k}$, and so (f, g) has no critical points in a neighborhood $\tilde{U}$ of p. Then by considerations on Section 2, $(f, g)|_{\tilde{U}}^{-1}(0, 0) = X \cap Y \cap \tilde{U}$ is a submanifold of Z. □

Next we would like to secure the existence of transverse intersections. The next few theorems will show that, even if two manifolds do not intersect transversally, it is possible to deform one of them very slightly and obtain a transverse intersection.

Definition 3.3. A **deformation** of a submanifold X of Y is a smooth function $i : X \times S \to Y$, where S is an open ball around 0 in $\mathbb{R}^n$, such that $i_s(x) := i(x, s)$ is an embedding for all $s \in S$ and i_0 is the inclusion map $X \hookrightarrow Y$.

Before moving on to the proof that deformations 'almost always' generate transverse intersections, we show that deformations themselves are in fact very easy to construct:

Lemma 3.4. *Let X be compact, and let $i : X \times S \to Y$ be a smooth function such that $i_0(x) := i(x, 0)$ is the embedding inclusion map $X \hookrightarrow Y$. Then if $\epsilon > 0$ is small enough, i is a deformation of X when restricted to $X \times S^\epsilon$ (S^ϵ is the open ball around zero with radius ϵ).*

Proof. Since all $i_s(x) = i(x, s)$ are automatically proper by the compactness of X, we need to show that they are immersions and one-to-one, for all small enough s.

For each point $x \in X$, associate an open set $U_x \times S^{\epsilon_x} \subseteq X \times S$ such that $d(i_s)_{x'}$ is full rank for all $s \in S^{\epsilon_x}$ and all $x' \in U_x$. This must exist because $d(i_0)_x$ is full rank at all points, and the determinant is a continuous function; so if $d(i_0)_{x'}$ has a square submatrix with nonvanishing determinant so does $d(i_s)_{x'}$ for small enough s, since $i(x, s)$ is smooth in s and x. Since X is compact, we can cover X with finitely many of these neighborhoods, and take the minimum of the ϵ_x to find an ϵ such that if $s \in S^\epsilon$, the map i_s is actually an immersion.

Now assume that for all $\epsilon > 0$ there exists $s \in S^\epsilon$ such that i_s is not injective. Define a map $F : X \times S \to Y \times S$ by $F(x, s) = (i_s(x), s)$. Consider two pointwise distinct sequences of points in X, $\{x_i\}$ and $\{y_i\}$ such that $F(x_i, s_i) = F(y_i, s_i)$, where $\{s_i\}$ is any sequence $s_i \to 0$. Then by passing to a subsequence, compactness of X guarantees that $x_i \to x$ and $y_i \to y$, that is, the sequences converge. We have $x = y$ since i_0 maps them to the same value and i_0 is, by hypothesis, injective. At $(x, 0)$, $dF_{(x,0)}$ must be injective, since i_0 is injective; so by the Inverse Function Theorem, F is actually injective on a neighborhood of $(x, 0)$. This contradicts the fact that $x_i \neq y_i$ for all i. □

The next theorem, proved in detail in [1], expresses a geometric fact regarding neighborhoods of manifolds that will be useful in proofs that consist of perturbing manifolds on a small open set:

Theorem 3.5. (ϵ-Neighborhood Theorem) *Let X be a compact boundaryless manifold embedded in $\mathbb{R}^m$. Let:*

$$X^\epsilon := \{z \in \mathbb{R}^m : |z - x| < \epsilon, \text{ for some } x \in X\}.$$

Then there exists a smooth map $\pi : X^\epsilon \to X$ which sends $z \in X^\epsilon$ to the unique closest point to z in X. Moreover, π is a submersion; that is, it has no critical points.

Finally we can prove:

Theorem 3.6. *Let X be a boundaryless compact submanifold of Y, a manifold embedded in $\mathbb{R}^n$ with an ϵ-neighborhood Y^ϵ, and a map $\pi : Y^\epsilon \to Y$. Define a deformation $i : X \times B^n \to Y$ (B^n denotes the unit ball in $\mathbb{R}^n$) of X:*

$$i_s(x) := i(x, s) = \pi(x + s\epsilon).$$

Let Z be any submanifold of Y. Then for almost every $s \in S$ the manifold X_s defined by the embedding $i_s(X)$ satisfies $X_s \pitchfork Z$.

Proof. First we note that i is a submersion. This follows from the fact that π is a submersion, by the ϵ-neighborhood Theorem, and by observing that even for fixed x the map $(x, s) \mapsto x + \epsilon s$ spans all directions of Y^ϵ so it is also a submersion; as a composition of submersions, i is itself a submersion. Therefore every point in Y is a regular value of i, and thus $i^{-1}(Z)$ is a submanifold of $X \times B^n$.

Now consider the projection map $\rho : X \times B^n \to B^n$ given by $(x, s) \mapsto s$. We claim that when $s \in B^n$ is a regular value of the map $\rho|_{h^{-1}(Z)}$, we have $X_s \pitchfork Z$; then since $h^{-1}(Z)$ is indeed a manifold, Sard's Theorem finishes our proof.

Let us now prove our claim. For the sake of simplicity, let $W := h^{-1}(Z)$. Our hypothesis of regularity at $s \in B^n$ implies that for every $(x, s) \in W$, the map $d\rho_{(x,s)}|_W$ is surjective. Therefore if we add the kernel of $d\rho_{(x,s)}|_W$, which sits inside $T_x(X) \times 0$, to the tangent space of the space of W, we get the full tangent space of $X \times B^n$:

$$(T_x(X) \times 0) + T_{(x,a)}W = T_x(X) \times \mathbb{R}^n. \tag{1}$$

Notice that $T_{(x,s)}W = dh^{-1}_{(x,s)}[T_{\pi(x+s\epsilon)}(Z)]$. To see this, let $j : W \to X$ be the natural inclusion map. Then $h \circ j$ is a submersion, so

$$d(h \circ j) : T_{(x,s)}W \to T_{\pi(x+s\epsilon)}(Z)$$

is a surjective map of the tangent spaces, and our assertion follows using the chain rule and noting that $dj_{(x,s)} = \text{id}$. Applying this to (1), we get:

$$\begin{aligned}(T_x(X) \times 0) + dh^{-1}_{(x,s)}[T_{\pi(x+s\epsilon)}(Z)] &= T_x(X) \times \mathbb{R}^n \\ \Rightarrow dh_{(x,s)}(T_x(X) \times 0) + T_{\pi(x+s\epsilon)}(Z) &= dh_{(x,s)}[T_x(X) \times \mathbb{R}^n] \\ \Rightarrow T_{\pi(x+s\epsilon)}(X_s) + T_{\pi(x+s\epsilon)}(Z) &= T_{\pi(x+s\epsilon)}(Y)\end{aligned}$$

where the last equality follows from the fact that the map $dh_{(x,s)} : T_x(X) \times \mathbb{R}^n \to T_{\pi(x+s\epsilon)}(Y)$ is a surjection. The last equality is the transversality condition, and our proof is complete.

□

We now turn our attention to the theory of intersection in oriented manifolds. On a real n-dimensional vector space, one can choose bases $a = \{a_1, ..., a_n\}$ and $b = \{b_1, ..., b_n\}$. Then the change of bases map from a to b is an $n \times n$ matrix. If the determinant of this matrix is positive, we say a has the same orientation as b and we denote $[a] = [b]$. Similarly, if the determinant is negative, we say a and b have opposite orientations and we write $[a] = -[b]$.

Clearly, having the same orientation defines an equivalence class for bases of a vector space; we can then choose arbitrarily one of the classes and call it an **orientation** of the vector space. An **orientation on a manifold** X is a smooth choice of orientations for all the tangent spaces $T_x(X)$. Here smoothness is to be understood as the condition that for every $x_0 \in X$ there exists a parametrization $\phi : U \to X$ of a neighborhood of x_0 such that the map $d\phi_x : \mathbb{R}^k \to T_x(X)$ preserves orientation for all $x \in U$.

If V and W are oriented vector spaces, there is a natural orientation induced on the product vector space $V \times W$. Let $[v_1, ..., v_n]$ be an orientation V and $[w_1, ..., w_m]$ an orientation for W. Then

$$[(v_1, 0), ..., (v_n, 0), (0, w_1), ..., (0, w_m)]$$

is the naturally induced orientation for $V \times W$. We call this the **product orientation** of $V \times W$. Note that the product orientation also induces an orientation for products of manifolds. Moreover, direct sums also induce product orientations;

that is, if $[v_1, ..., v_n]$ is an orientation for V and $[w_1, ..., w_m]$ an orientation for W, the combined basis $[v_1, ..., v_n, w_1, ..., w_m]$ gives an orientation for $V \oplus W$.

For the rest of this section, X and Z denote compact boundaryless submanifolds of Y. Moreover, we will assume dim $X +$ dim $Z =$ dim Y, unless otherwise noted. The condition on the dimensions implies that if X intersects Z at x transversally their intersection is a 0-manifold (a discrete set of points) and also that:

$$T_x(X) \oplus T_x(Z) = T_x(Y).$$

If $T_x(Y)$ is given an orientation that agrees with the direct sum orientation of $T_x(X)$ and $T_x(Z)$, we define the **orientation number** of $x \in X \cap Z$ as $+1$, and if the orientations do not match, we define it as -1.

Note that the orientation number depends on the order in which we write the direct sum; if instead we consider the direct sum orientation given by $T_x(Z) \oplus T_x(X) = T_x(Y)$, all orientation numbers change sign, and we denote this intersection by $Z \cap X$ or $-(X \cap Z)$. The global way of counting orientation numbers is:

Definition 3.7. Let $X \pitchfork Z$. The **intersection number** of X and Z, $I(X, Z)$ is the sum of the orientation numbers of the points in $X \cap Z$.

We can extend this definition to manifolds that do not intersect transversally if we show that the intersection number is a homotopy invariant. Our next lemma will establish this invariance.

But before proving the lemma, we need a remark. If $X \pitchfork Z$ and dim $X+$dim $Z =$ dim $Y + 1$, then $X \cap Z$ is an oriented 1-manifold. By noting that all 1-manifolds are diffeomorphic to either circles or segments (see [1]), it can be proven that the intersection numbers at the boundary of the 1-manifold is zero.

Lemma 3.8. *Let $i : X \times [0, 1] \to Y$, which we also write as $i_s(x)$, be a function smooth in X and continuous in $[0, 1]$. Then if $X_0 = i_0(X)$ and $X_1 = i_1(X)$, and both $X_0 \pitchfork Z$ and $X_1 \pitchfork Z$, we have:*

$$I(X_0, Y) = I(X_1, Y).$$

Proof sketch. First, note that $W := i(X, [0, 1])$, i.e., the image of all $X \times [0, 1]$ under i, is itself a manifold satisfying dim $W +$ dim $Y =$ dim $Z + 1$. By Theorem 3.6, we can take a deformation W' of W satisfying $W' \pitchfork Y$. Moreover, since $X_0 \pitchfork Z$ and $X_1 \pitchfork Z$, this deformation can be made such that $W' = W$ outside of some $X \times [\epsilon, 1 - \epsilon]$, for some $\epsilon > 0$, by multiplying the deformation of Theorem 3.6 by a bump function that vanishes outside of $[\epsilon, 1 - \epsilon]$.

Since $W' \pitchfork Y$, the intersection $W' \pitchfork Y$ is a 1-manifold, and its boundary is given by $i(X \times \{1\}) - i(X \times \{0\}) = X_1 - X_0$. But by our remark preceding the lemma, the sum of the intersection numbers in the boundary of a 1-manifold is always 0, so $I(X_1, Y) - I(X_0, Y) = 0$, completing the proof. □

Therefore, if X is not transverse to Z, we can still, using Theorem 3.6, deform Z into some homotopic Z' such that $X \pitchfork Z'$. Then we define $I(X, Z) = I(X, Z')$. We conclude this section with definitions that will be used in the proof of the Poincaré-Hopf Theorem.

Definition 3.9. Let $f : X \to Y$ be a smooth map of manifolds. If $y \in Y$ is a regular value of f, the **degree** of f at y is given by:

$$\deg_y f = \sum_{x \in f^{-1}(y)} \text{sign} \det df_x.$$

Lemma 3.10. *For all $y_0, y_1 \in Y$ that are regular values of f*

$$\deg_{y_0} f = \deg_{y_1} f$$

Therefore, we can define a global degree of f, deg f. Moreover, this degree is a homotopy invariant; that is, if f is homotopic to g then deg f = deg g

Proof. The idea of the proof is very similar to that of Lemma 3.8, as the degree of a function is, in a sense, an intersection number. See [2], pp. 28. □

Finally, we define the Euler characteristic as a self-intersection number, as promised:

Definition 3.11. *(Euler Characteristic)* We define the **Euler characteristic** $\chi(X)$ of a manifold X as:

$$\chi(X) := I(\Delta, \Delta)$$

where $\Delta = \{(x, x) : x \in X\}$ is regarded as a submanifold of $X \times X$.

By the remark preceding Definition 3.9, we see that $I(\Delta, \Delta)$ is well defined, even though Δ is not transverse to itself.

4. The Poincaré-Hopf Theorem

Now we are ready to prove the celebrated Poincaré-Hopf Theorem, which concerns **vector fields** on a manifold X; that is, smooth maps that assign a vector in $T_x(X)$ to each point $x \in X$. The theorem reveals a surprising connection between the topology of a manifold and the vector fields that can be constructed on it. For instance, the reader can verify that it is much easier to construct a non-vanishing vector field on the torus than on the sphere (which is actually an impossible task).

Let us now set up definitions necessary to state the theorem. If x is a zero of a vector field on a subset of $\mathbb{R}^n$, that is, if $\vec{v}(x) = 0$, then we define the **index** of $\vec{v}$ at x, $\text{ind}_x(\vec{v})$, as the degree of the map $x \mapsto \vec{v}(x)/|\vec{v}(x)|$ from a small sphere (small enough to contain only one zero) around x to S^{k-1}. Intuitively, the index of a zero of a vector field counts how many times the vector field wraps around a sphere on a small neighborhood of the zero.

To extend this definition to zeros of a vector field of a manifold, one chooses a local parametrization $\phi : U \to M$ around x, the zero of $\vec{v}$, and then defines $\text{ind}_x\ (\vec{v}) := \text{ind}_{\phi^{-1}(x)}\ (\phi^*\vec{v})$, where $\phi^*\vec{v}$ is the pullback of v. We define the pullback of v through ϕ by:

$$\phi^*\vec{v} := d\phi_u^{-1}\vec{v}(\phi(u))$$

Now we can state Poincaré-Hopf:

Theorem 4.1. (Poincaré-Hopf) *Let X be a compact oriented manifold and $\vec{v}$ a vector field on X with finitely many zeros, $\{x_1, ..., x_n\}$. Then,*

$$\chi(X) = \sum_{i=1}^{n} \text{ind}_{x_i}(\vec{v}).$$

For our proof, we will need:

Definition 4.2. The **tangent bundle** of an n-dimensional smooth manifold is the manifold.

$$T(X) := \{(x, v) \in X \times \mathbb{R}^n : v \in T_x(X)\}.$$

That the tangent bundle is actually a manifold is not obvious, but the proof consists only of finding an appropriate parametrization of the open sets of $T(X)$ based on parametrizations of X. Intuitively, the tangent bundle attaches a copy of the tangent space to each point of X.

A vector field $\vec{v}$ on X naturally defines a map $f_{\vec{v}} : X \to T(X)$ by $f_{\vec{v}}(x) = (x, \vec{v}(x))$. Since X is assumed to be compact, $f_{\vec{v}}$ is proper; and since it is clearly injective (its first component is the identity), $f_{\vec{v}}$ embedds X as $X_{\vec{v}}$, the image of X under $f_{\vec{v}}$, in $T(X)$. Therefore $X_{\vec{v}}$ is a manifold diffeomorphic to X.

It is clear that the zeros of $\vec{v}$ correspond to the intersection points of $X_{\vec{v}}$ with $X_0 = \{(x, 0)\}$. We call such zeros **nondegenerate** if $d\vec{v} : T_x(X) \to T_x(X)$ is actually a bijection (the fact that $d\vec{v}$ actually maps $T_x(X)$ into itself is not at all obvious; see [2]).

Lemma 4.3. *If x is a zero of $\vec{v}$, then it is nondegenerate if and only if $X_{\vec{v}} \pitchfork X_0$ at $(x, 0)$. In this case, $ind_x(\vec{v})$ is the orientation number of the point $(x, 0)$ in $X_0 \cap X_{\vec{v}}$.*

Proof. We first want to show that $\vec{v}$ is nondegenerate at x if and only if:

$$T_{(x,0)}(X_{\vec{v}}) + T_{(x,0)}(X_0) = T_{(x,0)}(T(X)) = T_x(X) \times T_x(X).$$

The tangent space of $X_{\vec{v}}$ at $(x, 0)$ is seen to be the graph of the linear map $d\vec{v}$, that is, $\{(w, d\vec{v}_x(w)) : w \in T_x(X)\}$, whereas the tangent space of X_0 is simply $\{(w, 0) : w \in T_x(X)\}$ which, as expected, looks like $T_x(X)$. So we see that the transversality condition is satisfied if and only if $d\vec{v}$ is bijective.

For the second part, note that the orientation number of the point $(x, 0)$ equals to $+1$ if $d\vec{v}_x$ preserves orientation and -1 if it reverses. To see this, let

$$[(\alpha_1, d\vec{v}_x(\alpha_1), ..., (\alpha_n, d\vec{v}_x(\alpha_n))]$$

be a positively oriented basis for the tangent space of $X_{\vec{v}}$ and consider the induced basis for $T_{(x,0)}(X_0) + T_{(x,0)}(X_{\vec{v}})$ on $X_0 \cap X_{\vec{v}}$ given by:

$$\begin{aligned} &[(\alpha_1, 0), ..., (\alpha_n, 0), (\alpha_1, d\vec{v}_x(\alpha_1)), ..., (\alpha_1, d\vec{v}_x)] \\ &= [(\alpha_1, 0), ..., (\alpha_n, 0), (0, d\vec{v}_{(x,0)}(\alpha_1)), ..., (0, d\vec{v}_x(\alpha_n))] = \text{sign } \alpha \cdot \text{ sign } d\vec{v}_x(\alpha). \end{aligned}$$

So the orientation number of the intersection point depends only on whether $d\vec{v}_x$ preserves or reverses orientation.

Around a zero x of $\vec{v}$ we can write $\vec{v}(x+w) = d\vec{v}_x(w)+\epsilon(w)$, where $\epsilon(w)/|w| \to 0$ as $w \to 0$. Consider the homotopy:

$$F_t(w) = \frac{d\vec{v}_x(w) + t\epsilon(w)}{|d\vec{v}_x(w) + t\epsilon(w)|}.$$

Here F_t are smooth maps $F_t : S_\epsilon \to S^k$. At $t = 1$, the degree of this map is $\text{ind}_x(\vec{v})$. At $t = 0$, the map is simply $d\vec{v}_x(w)/|d\vec{v}_x(w)|$. Since $d\vec{v}_x$ is a linear isomorphism of a vector space isomorphic to $\mathbb{R}^k$, it is either homotopic to the identity or to the reflection map, so at $t = 0$ the degree of the map $F_0(w) = d\vec{v}_x(w)/|d\vec{v}_x(w)|$ is ± 1 according to whether $d\vec{v}_x$ reverses or preserves orientation. Since the degree of a map is a homotopy invariant, this completes the proof. □

Now we show that we can assume that the zeros of $\vec{v}$ are nondegenerate:

Lemma 4.4. *Suppose that x is a zero of $\vec{v}$ and U is a neighborhood of x in X containing no other zero. Then there exists $\vec{v}_1$ agreeing with $\vec{v}$ outside some compact subset of U and such that $\vec{v}_1$ has only nondegenerate zeros inside U.*

Proof. The idea is to use Sard to choose $a \in \mathbb{R}^k$ such that $-a$ is a regular value of $\vec{v}$. Then $\vec{v}_1(z) = \vec{v}(z) + a$ only has nondegenerate zeros, since if $\vec{v}_1(x) = 0$ then $\vec{v}(x) = -a$, so $d\vec{v}(x)$ is full rank, and so is $d\vec{v}_1(z)$ since it differs from $d\vec{v}(z)$ by a constant. Now choose a smooth function $\rho(z)$ compactly supported in U that assumes the value 1 on some neighborhood of x. The new modified function

$$\vec{v}_1(z) = \vec{v}(z) + \rho(z)a$$

completes the proof. Note that by Sard and by the assumption that x is an isolated zero, we can choose a small enough that $\vec{v}_1(x) = 0$ only on the $\rho(z) = 1$ region, so our previous considerations are unaltered. □

By construction, note that $\vec{v}_1$ is homotopic to $\vec{v}$ with $\vec{v}_t = \vec{v}(z) + t\rho(z)a$, so that we can define the intersection number of $\vec{v}$ at x to be the sum of the intersection numbers of $\vec{v}_1$ in U.

By Lemma 4.4 we can find some $\vec{v}$ such that $X_{\vec{v}}$ only intersects X_0 transversally. But any $X_{\vec{v}}$ can be smoothly deformed into X_0 by the homotopy that mutiplies $\vec{v}$ by a number smoothly varying from 0 to 1. Therefore $I(X_0, X_0) = I(X_0, X_{\vec{v}})$, which, also by Lemma 3.3, corresponds to $\sum_{i=1}^n \mathrm{ind}_{x_i}(\vec{v})$. The next theorem will show that $I(X_0, X_0) = I(\Delta, \Delta)$, which will complete the proof of Poincaré-Hopf, since:

$$\chi(X) = I(\Delta, \Delta) = I(X_0, X_0) = I(X_0, X_{\vec{v}}) = \sum_{i=1}^n \mathrm{ind}_{x_i}(\vec{v})$$

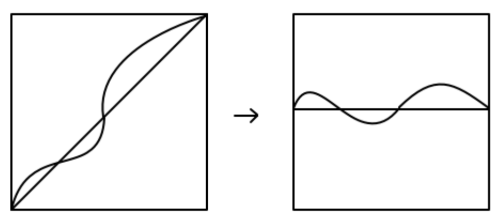

Figure 3. Visualizing Poincaré-Hopf

The equalities above are depicted in Figure 3. The horizontal and vertical axes of the first square represent X; the diagonal of the square, naturally, represents Δ. To calculate $I(\Delta, \Delta)$, we find an $f : X \to X$ homotopic to the identity such that graph(f), shown as the wavy curve in the first square, satisfies graph$(f) \pitchfork \Delta$. Then Lemma 3.8 shows that $I(\Delta, \Delta) = I(\Delta, \mathrm{graph}(f))$

Similarly, the second square depicts X_0 as a horizontal line embedded in $T(X)$ (the full square), with the vertical coordinates at each point $x \in X_0$ corresponding to $T_x(X)$. The wavy curve here represents $X_{\vec{v}}$, so we want to show that the vector field determining $X_{\vec{v}}$ is associated with the function f of the first square, giving $I(\Delta, \mathrm{graph}(f)) = I(X_0, X_{\vec{v}})$. We will do this in what follows.

The idea of the following definitions and arguments is to create diffeomorphic neighborhoods of X_0 and of Δ, in their respective ambient manifolds, so that we can perturb them inside this neighborhood to obtain a transverse intersection.

Definition 4.5. Let Z be a submanifold of Y, a manifold embedded in $\mathbb{R}^n$. The **normal bundle to** Z **in** Y is the set

$$N(Z; Y) = \{(z, v) : z \in Z, v \in T_x(Y) \text{ and } v \perp T_z(Z)\}$$

which is actually also a manifold, as the reader can verify.

Theorem 4.6. (Tubular Neighborhood Theorem) *There exists a diffeomorphism from an open neighborhood of Z in $N(Z;Y)$ onto an open neighborhood of Z in Y.*

Proof. Let $Y^\epsilon \xrightarrow{\pi} Y$, where π is the projection map from the ϵ-Neighborhood Theorem. Consider the map $h : N(Z;Y) \to \mathbb{R}^m$ given by $h(z,v) = z + v$. Then $W = h^{-1}(Y^\varepsilon)$ is an open neighborhood of Z in $N(Z,Y)$. Moreover the sequence of functions

$$W \xrightarrow{h} Y^\epsilon \xrightarrow{\pi} Y$$

is the identity on Z, so by the Inverse Function Theorem (see [1], pp. 56) h is a diffeomorphism from an open neighborhood of Z in $N(Z;Y)$ onto an open neighborhood of Z in Y. □

In particular, the orthogonal complement to $T_{(x,x)}(\Delta)$ in $T_{(x,x)}(X,X)$ is preceisely the collection of vectors $\{(-v,v) : v \in T_x(X)\}$; this is easily seen taking inner products. The map sending $T(X) \to N(\Delta, X \times X)$ defined by $(x,v) \mapsto ((x,x),(v,-v))$ is a diffeomorphism, for it is clearly smooth with smooth inverse. Therefore the Tubular Neighborhood Theorem proves that there is a diffeomorphism of a neighborhood of X_0 in $T(X)$ with a neighborhood of Δ in $X \times X$, extending the usual diffeomorphism $(x,0) \mapsto (x,x)$.

To finish the proof, using Theorem 3.6 we can deform X_0 inside its neighborhood in $T(X)$ into a X' embedded in $T(X)$ such that X_0 and X' are homotopic and $X_0 \pitchfork X'$. Then the set of points $\Delta' = \{(x,x') : x \in X_0, x' \in X'\}$, which can be seen as the graph of a function $x \mapsto x'$, intersects Δ when X_0 intersects X', and with the same orientation, since the neighborhoods of Δ and X_0 are diffeomorphic. Thus:

$$I(X_0, X_0) = I(X_0, X_0') = I(\Delta, \Delta') = I(\Delta, \Delta)$$

This completes the proof of the Poincaré-Hopf Theorem. To show a typical application of the theorem, we sketch a proof of:

Corollary 4.7. (Hairy Ball Theorem) *Every smooth vector field on S^2 vanishes at some point.*

Proof. Consider the vector field that travels along the latitudinal lines of S^2 from the North to the South pole. There are only two zeros of this vector field, both with index 1 (as the reader can verify); one at the North pole and one at the South pole. Therefore, by Poincaré-Hopf, $\chi(S^2) = 2$. So every vector field on S^2 must have at least one zero, for otherwise the sum of its indices is zero, contradicting the Poincaré-Hopf Theorem. □

5. Morse Functions and Gradient Flows

In this section we introduce Morse theory, which will be used to prove the homotopic equivalence of manifolds with cell complexes.

We start with definitions. A critical point x of a smooth function $f : M \to \mathbb{R}$ is called **nondegenerate** if the Hessian of f with respect to some local coordinates is invertible at x. It is not hard to check that this notion is well-defined: the Jacobian matrix of the transition map to a different coordinate system is always invertible, so our definition does not depend on coordinate choice.

Definition 5.1. A smooth function $f : M \to \mathbb{R}$ is called a **Morse function** if all its critical points are nondegenerate. The **index** λ of a critical point x of f is the number of negative entries of the Hessian $H_f(x)$, after diagonalization.

Remark 5.2. The definition of index of a Morse function should not be confused with the index of a zero of a vector field.

For surfaces in $\mathbb{R}^3$, the prime example of a Morse function is the height function — that is, the function projecting each point onto one of its coordinates. As an example, consider a copy of $\mathbb{T}^2$ standing vertically (that is, with its symmetry axis lying parallel to the horizontal plane) in $\mathbb{R}^3$. The height function on $\mathbb{T}^2$ is a Morse function with four critical points: the topmost and the bottom-most points of the torus have indices 2 and 0, respectively, and the top most and the bottom-most points of the hole inside the torus are saddle points, so both have index 1.

The existence of such Morse functions in every manifold is guaranteed by:

Theorem 5.3. *Let $f : U \to \mathbb{R}$ be a smooth function on an open set $U \subseteq \mathbb{R}^k$. Then for almost all $a = (a_1, ..., a_k)$ in $\mathbb{R}^k$, the function*

$$f_a = f + a_1x_1 + ... + a_kx_k$$

is a Morse function on U.

The idea of the proof for some f defined on $U \subset \mathbb{R}^k$ is very similar to that of Lemma 4.4; see [1] for a proof that extends the result to any manifold.

The following theorem specifies the local behavior of Morse functions around one of its critical points, according to its index.

Lemma 5.4. (Morse) *In a neighborhood of a critical point p with index λ of a Morse function f, we can write f as:*

$$f = f(p) - y_1^2 - ... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2$$

where $(y_1, ...y_n)$ is some coordinate system of p.

Although Lemma 5.4 is central to Morse theory, its proof is rather technical and lengthy and will be omitted. In short, the idea is to represent f in quadratic form, and then eliminate the non-y_i^2 terms using the nondegeneracy of the Hessian. See [4] for a complete proof.

Corollary 5.5. *Critical points of a Morse function f are isolated. In a compact manifold, the number of critical points of f is finite.*

Proof. For the first part, note that the partial derivatives must have only finitely many zeros in a neighborhood of p because of the form of f in the neighborhood where the coordinate system is defined.

If the manifold is also compact, then the number of zeros must be finite since otherwise we can choose a sequence of critical points $\{p_n\}$ that converges to some p: if p is a critical point, we contradict the first part of the corollary; if p is not a critical point, we contradict continuity of the partial derivatives of f. $\square$

A function $f : M \to \mathbb{R}$ generates a vector field called the **gradient** of f; it is defined as the unique vector field such that at all points $x \in X$ and all $w \in T_x(X)$ we have:

$$df_x(w) = \nabla f \cdot w$$

In particular, for Morse functions on a compact manifold we see that the zeros of this gradient vector field are isolated and finite. This fact, along with the Poincaré-Hopf Theorem, gives us a simple way of calculating the Euler characteristic of a manifold based on any Morse function defined on it:

Theorem 5.6. *Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold, and denote by k_λ the number of critical points of f with index λ. Then the Euler characteristic of M is given by:*

$$\chi(M) = \sum_{\lambda=0}^{n} (-1)^\lambda k_\lambda$$

Proof. If $\phi : U \to X$ is a parametrization of some neighborhood of a critical point, the induced pullback of the gradient vector field determined by f is given by:

$$\phi^* \nabla f = \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{\partial (f \circ \phi)}{\partial x_i} g_{ij} e_j$$

where $g_{ij}(u) = d\phi_u(e_i) d\phi_u(e_j)$. As the functions g_{ij} never vanish, we notice that x is a critical point of f if and only if

$$\frac{\partial (f \circ \phi)(0)}{\partial x_i} = 0$$

and this is if and only if $\phi^* \nabla f(0) = 0$, assuming $\phi(0) = x$.

Now we compute an element of the matrix representing the derivative of $\phi^* \nabla f$:

$$\frac{\partial [\phi^* \nabla f]_j}{\partial x_m} = \frac{\partial}{\partial x_m} \sum_{i=1}^{k} \frac{\partial (f \circ \phi)}{\partial x_i} g_{ij} = \sum_{i=1}^{k} \frac{\partial^2 (f \circ \phi)}{\partial x_i \partial x_m} g_{ij}$$

This means that the derivative of $\phi^* \nabla f$ is given by the matrix product of the Hessian of $f \circ \phi$ and the matrix with elements (g_{ij}). Since the determinant of the matrix (g_{ij}) does not vanish, we conclude that u is a nondegenerate zero of $\phi^* \nabla f$ and only if if it is a nondegenerate zero of f.

Now if ∇f has a non-degenerate zero at x, then $\text{ind}_x(\nabla f) = \text{sign}(\det d(\nabla f))$. This is a direct consequence of Lemma 4.3. Since $d(\phi^* \nabla f)$ is just the product of the Hessian of f with the matrix (g_{ij}), the Poincaré-Hopf Index Theorem concludes: for a Morse function, all critical points are non-degenerate, and they correspond precisely to the points where ∇f vanishes and is non-degenerate.

The critical points with odd index, where the determinant of the Hessian is negative, subtract 1 from $\chi(M)$, and similarly even index points add one. Since $\text{ind}_x(\nabla f)$ corresponds to the sign of the determinant of the Hessian of f at that point, summing over all x that are critical points of f we get $\chi(M)$.

□

Example 5.7. The considerations following Remark 5.2 immediately show that the Euler characteristic of the 2-torus is 0.

We now turn our discussion to that of gradient flows; these will be our main tools in the following proofs. A **curve** $c : [a, b] \to M$ on the manifold M is called an integral curve of the smooth vector field $\vec{v}$ on M if for all $t \in [a, b]$ we have:

$$c'(t) = \vec{v}(c(t))$$

A well-known theorem in the theory of ordinary differential equations (pp. 443, [5]) guarantees the existence of an integral curve $c : \mathbb{R} \to M$ of the vector field $\vec{v}$ passing through a specified point x_0 at time zero, that is, c satisfies $c(0) = x_0$.

Theorem 5.8. *Let $f : M \to \mathbb{R}$ be a Morse function with no critical values in $[a, b]$. Define*

$$M_{[a,b]} := \{x : f^{-1}(x) \in [a, b]\}$$

Then $M_{[a,b]}$ is diffeomorphic to $f^{-1}(a) \times [a, b]$.

Proof. For each $x \in f^{-1}(a)$, we can find an integral curve c_x of the smooth vector field $\nabla f/||\nabla f||^2$ such that $c_x(0) = x$. For all t such that c stays in $M_{[a,b]}$, we have

$$\frac{d}{dt}[f(c_x(t))] = df_{c_x(t)}(c_x'(t)) = \nabla f_{c_x(t)} \cdot c_x'(t) = 1$$

The fundamental theorem of calculus then gives $c_x(b - a) \in f^{-1}(b)$. Now define a diffeomorphism $\phi : f^{-1}(a) \times [a, b] \to M_{[a,b]}$ by $\phi(x, t) = c_x(t - a)$.

Injectivity follows from the non-intersection of integral curves, and surjectivity from the existence of integral curves. Smoothness in the t parameter follows from the definition of $c_x(t)$, whereas smoothness in x and smoothness of the inverse follow from the quoted theorem in the theory of ordinary differential equations. □

Corollary 5.9. *The level sets $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.*

Remark 5.10. In full formality, to define $|| \cdot ||$ one would have to consider a Riemannian metric on M, but we do not go into the details of that here; see [4].

The idea of the above proof was to let $f^{-1}(a)$ 'flow' into $f^{-1}(b)$ using the gradient vector field. The change in diffeomorphism type when there is a critical value in $[a, b]$ is not as simple, but will also be treated with gradient flows. From now on, we adopt the notation from Theorem 5.8 for $M_{[a,b]}$ and moreover we define $M_a = M_{(-\infty,a]}$. Let us also define:

Definition 5.11. *(Handle attachment)* Let M be a smooth m-manifold with boundary ∂M, and let $\varphi : \partial D^\lambda \times D^{m-\lambda} \to \partial M$ be an embedding where D^k is the k-dimensional disk. The quotient space

$$M' = \left(M \amalg D^\lambda \times D^{m-\lambda}\right) / \sim$$

where $\sim$ identifies $\varphi(x) \sim x \in \partial D^\lambda \times D^{n-\lambda}$ is called M **with a λ-handle attached** and sometimes we denote it simply by

$$M' = M \cup D^\lambda \times D^{m-\lambda}$$

The next theorem describes how the topology of a manifold changes as we cross a critical level of one of its Morse functions:

Theorem 5.12. (Crossing critical levels) *Let $f : M \to \mathbb{R}$ be a Morse function and c be a critical value of f. Assume that $f^{-1}(c) = \{p\}$, and choose $\varepsilon > 0$ such that p is the only critical point in $M_{[c-\varepsilon,c+\varepsilon]}$. If the index of p is λ, then*

$$M_{c+\varepsilon} \cong M_{c-\varepsilon} \cup D^\lambda \times D^{m-\lambda}$$

where $\cong$ denotes manifolds of the same diffeomorphism type.

Proof Sketch. Let us assume $0 < \lambda < m$. By Lemma 4.3., we can write f on a neighborhood U of p with respect to some local coordinates as:

$$f = c - y_1^2 - ... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2$$

Now, the set $M_{c-\varepsilon} \cap U$ corresponds to the points $(y_1, ..., y_n)$ that satisfy the inequality:

$$y_1^2 + ... + y_\lambda^2 - y_{\lambda+1}^2 - ... - y_m^2 \geq \varepsilon$$

and analogously for $M_{c+\varepsilon} \cap U$, the inequality is:

$$-y_1^2 - ... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n \leq \varepsilon$$

In the illustration below, taken from [4], the darkest region corresponds to the first inequality, and the dotted region to the second. The horizontal axis maps the values $y_1^2 + ... + y_\lambda^2$ and the vertical axis the values $y_{\lambda+1}^2 + ... + y_m^2$.

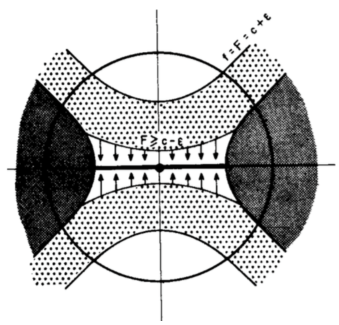

FIGURE 4. Attaching of a λ-handle

The dark region then corresponds to $M_{c-\varepsilon} \cap U$ and the dotted region to $M_{c+\varepsilon} \cap U$. Now consider the region given by the simultaneous inequalities:

$$\begin{cases} y_1^2 + ... + y_\lambda^2 - y_{\lambda+1}^2 - ... - y_m^2 \leq \varepsilon \\ y_\lambda^2 + y_{\lambda+1}^2 + ... + y_m^2 \leq \delta \end{cases}$$

for some $\delta > 0$ much smaller than ε. The region can be shown to be diffeomorphic to $D^\lambda \times D^{m-\lambda}$, which corresponds to a (very thin) rectangle on Figure 4, with the boundary $\partial D^\lambda \times D^{m-\lambda}$ attached to $M_{c-\varepsilon} \cap U$. In the illustration, the handle corresponds to the white horizontal 'bar' connecting the two sides of $M_{c-\varepsilon} \cap U$. The corners of this attachment are not smooth at first, but by making the corners smooth, as shown in Figure 4, one can make:

$$M_{c+\varepsilon} \cong M_{c-\varepsilon} \cup D^\lambda \times D^{m-\lambda}$$

using an argument similar to that of Theorem 5.8, that is, by using the gradient flow through $M_{[c-\varepsilon, c+\varepsilon]}$.

Finally, we deal with the cases where $\lambda = 0$ and $\lambda = m$. If $\lambda = 0$, then on a neighborhood of p we can choose local coordinates such that $f = c + y_1^2 + ... + y_n^2$. The region $M_{c+\varepsilon} \cap U$ is then given by the equation $y_1^2 + ... + y_n^2 \leq \varepsilon$, which is just $D^n \cong D^0 \times D^n$, whereas $M_{c-\varepsilon} \cap U = \varnothing$. So in this case $M_{c+\varepsilon}$ is the disjoint union (since the atttaching map of $\partial D^0 \times D^n$ is 'empty') of $M_{c-\varepsilon}$ and D^n. The analysis of $\lambda = m$ is very similar, and corresponds to capping the complement of a disk

$$y_1^2 + ... + y_m^2 \geq \varepsilon$$

which corresponds to $U \cap M_{c-\varepsilon}$ with an m-handle, that is, a disk 'facing down'. □

Remark 5.13. The process of making the corners smooth is not quite so simple, but only makes use of standard smooth 'bump' function arguments; see [4] for details.

We conclude this section with a result, given without proof (see [3]), that will be useful in later sections.

Theorem 5.14. (*Arrangement of critical points*) *Let M be an m-dimensional closed manifold and f a Morse function on it. Then we can perturb f so as to make the increasing order of critical values agree with the increasing order of indices. In other words, if p_i, p_j are critical points of f with $f(p_i) < f(p_j)$ then the index of p_i is less or equal to the index of p_j.*

We will call a function f as above an ordered Morse function.

6. Cell Homology of Manifolds

In the previous section we saw that we can think of a manifold as the attachment of various 'handles' of the form $D^\lambda \times D^{m-\lambda}$. Moreover, we can choose our Morse function such that we attach these handles in increasing order of index. It turns out that there is a certain type of topological space that is, in a sense, a special case of this handle decomposition of manifolds: these are called **finite CW-complexes**; which can be generalized to CW-complexes in the infinite case.

In what follows, we denote the interior of the i-dimensional disk D^i by e^i, and call it an i-cell; $\bar{e}^i$ is the closure of the cell. By convention, e^0 is a singleton.

Definition 6.1. We define a finite CW-complex inductively:

(i) A 0-dimensional cell complex is a space of the form $\bar{e}^0_1 \sqcup ... \sqcup \bar{e}^0_n$.

(ii) An i-dimensional cell complex X is defined by attaching a disjoint union of i-cells $\bar{e}^i_1 \sqcup ... \sqcup \overline{e}^i_n$ to an $(i-1)$-dimensional cell complex Y:

$$X = Y \cup_{h_i} (\bar{e}^i_1 \sqcup ... \sqcup \bar{e}^i_n) := (Y \amalg (\bar{e}^i_1 \sqcup ... \sqcup \bar{e}^i_n))/\sim$$

where h_i is a continuous map $h_i : \partial\bar{e}^i_1 \sqcup ... \sqcup \partial\bar{e}^i_n \to Y$, and the equivalence relation $\sim$ is generated by identifying $x \in \partial\bar{e}^i_1 \sqcup ... \sqcup \partial\bar{e}^i_n$ with $h_i(x) \in Y$.

Remark 6.2. One can orient an i-cell the same way a manifold can be oriented. We denote the oriented i-cell by $\langle\bar{e}^i\rangle$.

The cell complex generated after n steps in Definition 6.1 is called an **n-skeleton** of the cell complex X, denoted by X^n. Let k_q be the number of q-cells in X. Then the formal sum:

$$c = a_1\langle e^q_1\rangle + ... + a_{k_q}\langle e^q_{k_q}\rangle$$

is called a q-chain of X. The set of all formal sums is the **q-dimensional chain group of** X, denoted by $C_q(X)$.

Definition 6.3. *(Boundary homomorphism)* Let X be a cell complex with h_i as in Definition 6.1. Define a homomorphism $\partial_q : C_q(X) \to C_{q-1}(X)$, by the formula:

$$\partial_q(\langle e^q_k\rangle) = a_{k1}\langle e^{q-1}_1\rangle + ... + a_{kk_{q-1}}\langle e^{q-1}_{k_{q-1}}\rangle$$

where a_{kl} is defined as follows.

Since $\partial\bar{e}_k^q$ is diffeomorphic to S^{q-1}, we can regard the attaching map h_k attaching $e_k^q \to X^{q-1}$ as a map $h_k : S^{q-1} \to X^{q-1}$. Pick some open $U \subset e_l^{q-1}$ and perturb h_k continuously so that h_k restricted to $h_k^{-1}(U)$ becomes a C^∞ map. Then we define a_{kl} as the degree of the C^∞ map h_k restricted to $h_k^{-1}(U)$.

One can check that with this definition:

Lemma 6.4. *$\partial_{q-1} \circ \partial q = 0$, for all q.*

So the boundary operator defines a *chain complex* of X:

$$\dots \xrightarrow{\partial_{q+2}} C_{q+1}(X) \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} \dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \{0\}$$

From which we define the group of **q-dimensional cycles** on X:

$$Z_q(X) = \text{Ker } \partial_q$$

and the group of **q-dimensional boundaries** of X:

$$B_q(X) = \text{Im } \partial_{q+1}$$

Finally, we call the quotient group $H_q(X) = Z_q(X)/B_q(X)$ the **q-dimensional homology group of** X. The elements of $H_q(X)$ are called **homology classes**, and they are equivalence classes defined by identifying elements of $Z_q(X)$ whose difference lies in $B_q(X)$.

To make all these definitions less abstract we apply all the new concepts to an example:

Example 6.5. (S^n as a cell complex) The n-sphere can be regardaded as a cell complex with two cells, namely, the n-disk $\bar{e}^n$ with its boundary attached to a 0-cell, a point $\bar{e}^0$. Note that:

$$C_q(X) = \begin{cases} \mathbb{Z} \text{ , if } q = n \text{ or } q = 0 \\ 0 \text{ , otherwise} \end{cases}$$

The chain complex of S^n can be represented by the diagram:

$$\xrightarrow{\partial_{q+2}} \{0\} \xrightarrow{\partial_{q+1}} \mathbb{Z} \xrightarrow{\partial_q} \{0\} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_2} \{0\} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} \{0\}$$

It is obvious that $Z_q(X) = C_q(X)$, except when $n = 1$ since $\partial_q = 0$ for all q. In the case $n = 1$, we have $\partial_1\langle\bar{e}^1\rangle = \langle\bar{e}^0\rangle - \langle\bar{e}^0\rangle = 0$, since the boundary of $\bar{e}^1$ consists of two points attached with opposite orientations to the 0-cell, so $Z_q(X) = C_q(X)$ in this case as well. But $\partial_q = 0$ also implies that $B_q(X)$ is trivial for all q and n. Therefore:

$$H_q(X) = \begin{cases} \mathbb{Z} \text{ , if } q = n \text{ or } q = 0 \\ 0 \text{ , otherwise} \end{cases}$$

Returning to our general discussion, the Fundamental Theorem of Finitely Generated Abelian Groups tells us that $H_q(X)$ has the form:

$$H_q(X) \cong \mathbb{Z}^n \oplus T$$

where T is a finite abelian group, called the torsion part of $H_q(X)$. We denote the number n the previous equation by $b_q(X)$, and call it the *q-dimensional Betti number of* X. Note that if $H_q(X)$ is torsion-free, that is, $H_q(X) \cong \mathbb{Z}^n$, then $H_q(X)$ has a well-defined notion of dimension, with dim $H_q(X) = b_q(X) = n$.

The following theorem is a special case of the Euler-Poincaré formula. It relates, very loosely, the number of cells in the complex to its Betti numbers.

Theorem 6.6. (Euler-Poincaré) *Let X be an m-dimensional cell complex, and $k_q = dim\ C_q(X)$ the number of q cells of X. Suppose $H_q(X)$ is torsion-free for all $q \leq m$. Then*

$$\sum_{q=0}^{m}(-1)^q k_q = \sum_{q=0}^{m}(-1)^q b_q(X)$$

Proof sketch. The rank-nullity theorem gives dim Ker ∂_q+dim Im ∂_q = dim $C_q(X) = k_q$ which translates to

$$k_q = \text{dim } Z_q(X) + \text{dim } B_{q-1}(X)$$

Moreover, by definition of $H_q(X)$ the rank-nullity theorem again gives dim $H_q(X) =$ dim $Z_q(X) -$ dim $B_q(X)$, which translates to

$$b_q(X) = \text{dim } Z_q(X) - \text{dim } B_q(X)$$

Therefore:

$$\sum_{q=0}^{\infty}(-1)^q[k_q - b_q(X)] = \sum_{q=0}^{\infty}(-1)^q[\text{dim } B_{q-1}(X) + \text{dim } B_q(X)]$$
$$= \text{dim } B_{-1}(X) = 0$$

by definition. We can sum over all q since the groups vanish after $q > m$. □

This result also holds for homology groups that have a torsion part; the proof in that case is essentially the same but with small modifications to fit the group theoretic terminology. To keep prerequisites to a minimum, we only present the proof as above.

Let's now take a step back from algebraic topology, and justify our introduction of cell complexes in the context of Morse theory by proving the following result:

Theorem 6.7. (Identification of manifolds with cell complexes) *Let M be an m-dimensional manifold. By choosing an ordered Morse function f with k_i critical points of index i, Theorem 4.10 tells us that M is diffeomorphic to*

$$N = (h^0_1 \sqcup h^0_{k_0}) \cup (h^1_1 \sqcup h^1_{k_1}) \cup ... \cup (h^l_1 \sqcup h^l_{k_l})$$

where $h^\lambda = D^\lambda \times D^{m-\lambda}$.

If, as in the equation above, the maximum index of the handles of N is l, then N is homotopy equivalent to a certain l-dimensional cell complex X. Moreover, there is a bijection between the i-handles of N and the i-cells of X.

Remark 6.8. Since we are mostly concerned with compact manifolds, Corollary 5.5 justifies our assertion from the beginning of the chapter that we only need to deal with finite cell complexes.

Before giving the proof of Theorem 6.7, we prove the following lemma:

Lemma 6.9. *Let K and M be topological spaces. The* **mapping cylinder** *of $h : K \to M$ is the space $M_h = M \cup K \times [0, 1]$ with $x \in K \times \{0\}$ identified with $h(x) \in M$; that is:*

$$M_h = (([0, 1] \times K) \amalg M)/\sim$$

where $\sim$ is the equivalence relation generated by $(x, 0) \sim h(x)$. Then M_h is homotopic equivalent to M.

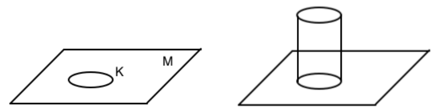

FIGURE 5. The mapping cylinder of h

Proof. We show that M is a deformation retract of M_h. Consider the map $F : M_h \times [0,1] \to M_h$ which is the identity on M and for $(x,s) \in K \times [0,1]$ let $F_t(x,s) = (x, ts)$. Clearly the map is continuous, F_1 is the identity, F_0 maps into M and by definition F_t is the identity on M. □

Now we prove 6.7:

Proof. We use induction on the maximum index of the handles, l. In the case $l = 0$, N is the disjoint union of disks, the boundary of which can be collapsed into a point so the disks are homotopy equivalent to the mapping cylinder of the collapsing map. The points to which the disks are collapsed form a 0-dimensional cell complex, and are homotopy equivalent to the mapping cylinders, so the $l = 0$ proof is complete.

Assume that Theorem 6.7 is proved for handlebodies of maximal index $l - 1$. Moreover, assume this is done by finding a continuous mapping from the boundary of these handlebodies to some cell complex such that the mapping cylinder of this map is homeomorphic to the handlebody itself.

Let N be a handlebody of maximal index l. Suppose it can be written as

$$N = H \cup_\psi D^l \times D^{m-l}$$

where H is a handlebody of maximal index $l - 1$. That is, for simplicity, assume N has only one l-handle. By induction hypothesis, there is a cell complex Y and a mapping $g : \partial H \to Y$ such that H is homeomorphic to the mapping cylinder of g.

The handle $D^l \times D^{m-l}$ itself can be identified with the mapping cylinder of $c : D^l \times \partial D^{m-l} \to D^l \times \{0\}$ given by $(x, y) \mapsto (x, 0)$. Now the boundary $D^l \times \partial D^{m-l}$ embedded by ψ is a submanifold of ∂H; and, so is the l-cell $D^l \times \{0\}$. Therefore the restriction of g to $D^l \times \{0\}$ attaches a l-cell to the cell complex Y.

From now on we denote

$$K = \partial H \setminus (\partial D^l \times \text{int } D^{m-l})$$

So that (see Figure 6):

$$\partial N = K \cup D^l \times \partial D^{m-l}$$

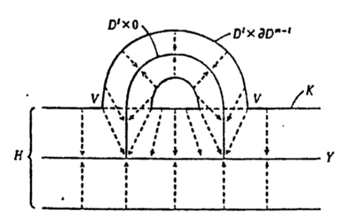

FIGURE 6. Finding the map h for l; illustration from [3]

If we can define a continuous map $h : \partial N \to X$ (represented in Figure 4 by the dotted arrows) such that N is homeomorphic to the mapping cylinder of h, then we are done. On the region $D^l \times \partial D^{m-l}$ we set $h = c$. Automatically, the portion $D^l \times D^{m-l}$ is then homeomorphic to the l-cell we want to attach to Y.

Now we define h on K. Using Theorem 5.8, there is a neighborhood V of ∂K such that $V \cong \partial K \times [0,1]$ (See Fig 6). Let V_c be the mapping cylinder of the restriction of c to ∂K; which by definition of K can be naturally identified with $\partial D^l \times D^{l-m}$. Then by definition of the mapping cylinder, V_c is homeomorphic to $\partial D^l \times D^{m-l} \cup V$, with V being attached 'around' $\partial D^l \times D^{m-l}$. Let $j : V_c \to \partial D^l \times D^{m-l} \cup V$ be this homeomorphism. Moreover, let $i : V \hookrightarrow V_c$ be the natural inclusion map.

Finally, define:

$$h(p) = \begin{cases} g \circ j \circ i(p), \text{ if } p \in K \times [0,1] \\ g(p), \text{ if } p \in \partial H \setminus (K \times [0,1)) \end{cases}$$

The function h is adjusted to glue appropriately the natural attaching map g on ∂H and the natural attaching map c on the attached handle. Note that it accomplishes this, and continuously since j and i are continuous, because $j \circ i(x,1) = c$ and $j \circ i(x,0) = \text{id}$. Now N is by construction homeomorphic to the mapping cylinder M_h, and the proof is complete.

□

It is a well known fact that homotopy equivalent cell complexes have the same homology groups. Therefore for our purposes we can define the homology groups of a manifold M to be those of the cell complex obtained from it using Theorem 6.7.

The following theorem finally uses the results of this section to connect the Euler characteristic computed from Theorem 5.6 to the alternating sum of the Betti numbers.

Theorem 6.10. *The Euler characteristic of a manifold is given by the alternating sum of its Betti numbers:*

$$\chi(M) = \sum_{q=0} (-1)^q b_q(M)$$

Proof. Identify M with a cell complex using 6.7, and use Theorem 5.6 and Theorem 6.6 to conclude. □

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