OVERVIEW OF TATE'S THESIS

ALEX MINE

Abstract. This paper gives an overview of the main idea of John Tate's 1950 PhD thesis. I will explain the methods he used without going into too much technical detail. I will also briefly demonstrate how the results of the thesis can be applied in analytic number theory.

Contents

1. Preliminaries

1.1. **Locally Compact Groups and Haar Measures.** The main objects of study in this paper are **locally compact abelian groups**. This is an abelian topological group with the property that every point has a compact neighborhood, which is the general characterization of a locally compact topological space. One important property of locally compact groups is the existence of Haar measures.

Definition 1.1. A **left Haar Measure** is a measure μ on a locally compact topological group G with the property that for all Borel subsets $S \subset G$ and $g \in G$, $\mu(gS) = \mu(S)$.

As mentioned above, every locally compact group has a Haar measure, which is unique up to scaling. This fact, while an interesting theorem in its own right, will be assumed.

For example, the usual Lebesgue measure is a Haar measure of the group $(\mathbb{R}, +)$. The existence of Haar measures allows us to integrate on groups, which we can use to apply concepts from Fourier analysis.

1.2. **Completions and p-adic fields.** One important class of locally compact structures are *p-adic fields*, which are associated with a fixed prime p. These have multiple equivalent definitions. First, we can define the *p-adic* norm as follows: for

$a, b \in \mathbb{Z}$, let n_1, n_2 be such that $p^{n_1}|a$ but $p^{n_1+1} \nmid a$, and $p^{n_2}|b$ but $p^{n_2+1} \nmid b$. Then define

$$\left|\frac{a}{b}\right|_p = p^{n_2-n_1}.$$

One can check that this is a norm. We can then complete $\mathbb{Q}$ with respect to the metric induced by the norm. The result is a field which we call $\mathbb{Q}_p$. The p-norm naturally extends to this completion.

Note that the p-adic norm is very different from the usual norm. For instance, for all $x \in \mathbb{Z}$, $|x|_p \leq 1$. This indicates that $\mathbb{Q}_p$ looks very different from $\mathbb{R}$.

For the other construction, first consider the fact that any integer can be written in base p, i.e. as a finite polynomial in p with coefficients in $\{0, 1, \ldots, p-1\}$. We can thus extend this set to the set of formal power series in p with coefficients coming from the same set. We call this set $\mathbb{Z}_p$. It is easy to see how $\mathbb{Z}^+$ embeds into $\mathbb{Z}_p$, simply by writing the number in its base-p expansion. We can also embed negative integers into $\mathbb{Z}_p$–for instance, we can send -1 to $(p-1)+(p-1)p+(p-1)p^2+\cdots$. To see how to embed any integer into $\mathbb{Z}_p$, it is useful to view $\mathbb{Z}_p$ as the projective limit $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$. I won't go into detail about this construction, but one interesting property is that the projective limit of compact Hausdorff spaces is compact. We can endow $n\mathbb{Z}/p^n\mathbb{Z}$ with the discrete topology (which is Hausdorff), and we will actually end up with the same topology on $\mathbb{Z}_p$ as the projective limit as we get by using the p-norm. This can be used to show that, surprisingly, $\mathbb{Z}_p$ is compact, unlike the integers.

One can show that the units of the ring $\mathbb{Z}_p$ are precisely the elements with a nonzero p^0 coefficient. Thus, if we consider the fraction field of $\mathbb{Z}_p$–denoted $\mathbb{Q}_p$–we can identify the inverse of an element $x = a_n p^n + a_{n+1}p^{n+1} + \cdots$ (with $a_n \neq 0$) with $p^{-n}y$ where y is the inverse of the element $a_n + a_{n+1}p + \cdots$. Therefore, we can say

$$\mathbb{Q}_p = \bigcup_{n \in \mathbb{Z}} p^n \mathbb{Z}_p.$$

Because $\mathbb{Z}$ embeds into $\mathbb{Z}_p$, $\mathbb{Q}$ embeds into $\mathrm{Frac}(\mathbb{Z}_p) = \mathbb{Q}_p$. The field $\mathbb{Q}_p$ has a natural extension of the p-norm on $\mathbb{Q}$: considering elements of $\mathbb{Q}_p$ as formal power series, if we have an element $x = \sum_{k=j}^{\infty} a_k p^k$ with $j \in \mathbb{Z}$ and $a_j \neq 0$, we can let $|x|_p = p^{-j}$. We can show that this agrees with the definition on $\mathbb{Q}$.

The topology endowed on $\mathbb{Q}_p$ makes it locally compact. This gives $\mathbb{Q}_p$ many properties that will be illustrative in the following analysis.

2. Local Theory

We start with an algebraic number field–that is, a field that is a finite extension of $\mathbb{Q}$. On such a field we can define an absolute value, or in other words a norm on the field as a vector space over itself. In fact, we can do this in many ways. For instance, on the rationals, we can define the usual absolute value, or we can use the p-adic norms as described above. We want to consider the set of all such absolute values for a particular field, but we want to be careful not to double count ones that are equivalent, which means that they are multiples or powers of each other. So instead we consider equivalence classes of absolute values, which we call *places*. The places of $\mathbb{Q}$ containing the normal absolute value and the p-adic norms in fact exhaust the places of $\mathbb{Q}$.

With an algebraic number field k and a place v, we consider k_v, the completion of k with respect to v. This is well-defined despite v being an equivalence class of norms, because we can see that a sequence that converges with respect to one norm also converges with respect to any equivalent norm. We distinguish between *archimedean* and *nonarchimedean* places. The former are places that have the archimedean property, that for all $x, y \in k_v$, there is some positive integer n such that $|nx|_v > |y|_v$. The latter are places that lack this property. For example, the reals are archimedean. On the other hand, if we have $x \in \mathbb{Q}_p$ that is divisible by p, i.e. its power series looks like $a_1 p + a_2 p^2 + \cdots$, then for any positive integer n, nx will still be divisible by p, so its power series expansion doesn't have a p^0 term, so $|nx|_p \leq \frac{1}{p}$. Thus x and 1 are a counterexample to the archimedean property, so $\mathbb{Q}_p$ is nonarchimedean. It is no coincidence that the nonarchimedean places vastly outnumber the archimedean ones in the case of $\mathbb{Q}$; in general there are only finitely many nonarchimedean places and infinitely many archimedean places. Also, note that the nonarchimedean places of $\mathbb{Q}$ are associated with prime numbers, which correspond to the prime ideals of $\mathbb{Z}$. In general, the nonarchimedean places of k correspond to the prime ideals of the *ring of integers* of k, which is a special subring ring generalizing the role of the integers inside the rationals (specifically, it is the set of all elements which are roots of monic polynomials with integer coefficients, which in $\mathbb{Q}$ can easily be seen to be $\mathbb{Z}$). We denote this ring as $\mathcal{O}_k$.

Because k_v is more often nonarchimedean, we will focus on that case. (The archimedean case is also simpler; one can use basic complex analysis to get very similar results to the ones that follow when k_v is archimedean, because then k_v is either $\mathbb{R}$ or $\mathbb{C}$.)

Let $F = k_v$. We want to be able to use Fourier analysis to look at F. Let $S(F)$ be the space of Schwartz-Bruhat functions on F. When $F = \mathbb{R}$, this is the usual Schwartz space. When F is nonarchimedean, this is simply the space of locally constant functions with compact support. Because of the topology of nonarchimedean completions, such functions are actually continuous. Let $S'(F)$ be the space of continuous linear functionals on $S(F)$.

We can define an action of $F^\times$ on $S(F)$ as follows: for $a \in F^\times$ and $f \in S(F)$, define

$$(a \cdot f)(x) = f(ax).$$

This gives rise to an action of $F^\times$ on $S'(F)$ by

$$\langle a \cdot \lambda, f \rangle = \langle \lambda, a^{-1} \cdot f \rangle.$$

Now if we have a homomorphism $\omega : F^\times \to \mathbb{C}^\times$, called a *quasicharacter* of $F^\times$, we define the space of $\omega -$ *eigendistributions* as

$$S'(\omega) = \{\lambda \in S'(F) : a \cdot \lambda = \omega(a)\lambda \text{ for all } a \in F^\times\}.$$

Strangely, this space is 1-dimensional for all completions $F = k_v$. The proof of this will not be given. This fact plays a central role in the following analysis.

Given a Haar measure dx of F, one can see that $d(ax) = |a|dx$ for $a \in F^\times$, i.e. the measure of a scaled set is the measure of the original set scaled by the absolute value of the scalar. This is obvious in the reals, but requires a bit more work in other cases. From this one can show that a Haar measure $d^\times x$ on $F^\times$ is a multiple of $|x|^{-1}dx$, say by μ. This makes sense because then

$$d^\times(ax) = \mu|ax|^{-1}d(ax) = \mu|a|^{-1}|x|^{-1}|a|dx = \mu|x|^{-1}dx = d^\times x$$

which is exactly the translation invariance property of a Haar measure.

Let $\omega_s(x) = |x|^s$, so that ω_s is a quasicharater of $F^\times$. We define the *zeta integrals* as follows: for ω a character of $F^\times$ and $f \in S(F)$, define

$$z(s,\omega;f) = \int_{F^\times} f(x)\omega\omega_s(x)d^\times x$$

where by $\omega\omega_s$ we simply mean the quasicharacter acquired by multiplying ω and ω_s. This integral is absolutely convergent when $\mathrm{Re}(s) > 0$. Thus, this defines a distribution $z(s,\omega)$ on $S(F)$. In fact,

$$\begin{aligned}\langle a \cdot z(s,\omega), f\rangle &= \int_{F^\times} f(a^{-1}x)\omega\omega_s(x)d^\times x \\ &= \int_{F^\times} f(x)\omega\omega_s(ax)d^\times ax \\ &= \omega\omega_s(a)\int_{F^\times} f(x)\omega\omega_s(x)d^\times x = \omega\omega_s(a)\langle z(s,\omega), f\rangle\end{aligned}$$

so $z(s,\omega) \in S'(\omega\omega_s)$.

We now want to specify quasicharacters with a certain property. A quasicharacter ω is *unramified* if it is trivial on $\mathcal{O}_F$. If this is the case, we can write $\omega(s) = q^s$ for some $q \in \mathbb{C}^\times$. This case is easier to study than the ramified case, and later I will mention why it makes sense to focus on this case. We will focus on the unramified nonarchimedean case, i.e. v a nonarchimedean place and ω an unramified quasicharacter.

One can show that $\mathcal{O}_F$ is a discrete valuation ring, i.e. a principal ideal domain with a unique maximal ideal. Let $\mathcal{P}$ be the maximal ideal, and let π be a generator of $\mathcal{P}$. Then $\mathcal{P}$ is a neighborhood of 0, and raising $\mathcal{P}$ to higher powers yields smaller neighborhoods. Let $\omega(\pi) = t$. If f has compact support in $F^\times$, then the integral defining $z(s,\omega;f)$ is entire. Thus if we can replace a general f with something that has compact support in $F^\times$, we get an entire function. Let $f^*(x) = f(x) - f(x\pi^{-1})$. Because $f \in S(F)$, it is constant in a neighborhood of 0. Let $\mathcal{P}^r$ be a neighborhood small enough that f is constant in it. Then for $x \in \mathcal{P}^{r+1}$, $f^*(x) = f(x) - f(x\pi^{-1}) = 0$. So the support of f^* is bounded away from 0, and thus is compact in $F^\times$, as it is already compact in F.

Now define the distribution $z_0(s,w)$ by $\langle z_0(s,\omega), f\rangle = \langle z(s,\omega), f^*\rangle$. By the above, $\langle z_0(s,\omega), f\rangle$ is entire. In addition, on the half-plane $\mathrm{Re}(s) > 0$, we have

$$\begin{aligned}\langle z_0(s,\omega), f\rangle &= \int_{F^\times} (f(x) - f(x\omega^{-1}))\omega\omega_s(x)d^\times x \\ &= \int_{F^\times} f(x)\omega\omega_s(x)d^\times x - \int_{F^\times} f(x\omega^{-1})\omega\omega_s(x)d^\times x \\ &= (1 - tq^{-s})\langle z(s,\omega), f\rangle.\end{aligned}$$

This makes sense, because $z(s,\omega)$ and $z_0(s,\omega)$ are in $S'(\omega\omega_s)$, a 1-dimensional space. We know there is some constant of proportionality between them, and the above specifies it. We write $L(s,\omega) = (1-tq^{-s})^{-1}$, so that $z(s,\omega) = L(s,w)z_0(s,\omega)$. Because $z_0(s,\omega)$ is entire, this gives a meromorphic continuation of $z(s,\omega)$ to the whole plane, which was originally only defined on the right half-plane. Similar results hold in the ramified nonarchimedean, and archimedean cases.

2.1. **Fourier Analysis.** Fix a nontrivial continuous character ψ of F^+. Then one can show that every continuous character of F^+ can be written as $x \mapsto \psi(xy)$ for some $y \in F$ (the trivial character is given by $y = 0$). For $f \in S(F)$, define the *Fourier transform* of f as:

$$\hat{f}(x) = \int_F f(y)\psi(xy)dy.$$

One can show that $\hat{f} \in S(F)$. In fact, the map $f \mapsto \hat{f}$ is an isomorphism $S(F) \to S(F)$. In addition, there is a unique choice of the Haar measure dx such that we have $\hat{\hat{f}}(x) = f(-x)$. We take dx to be this measure. We can define Fourier transforms on distributions as

$$\langle \hat{\lambda}, f \rangle = \langle \lambda, \hat{f} \rangle.$$

Lemma 2.1. *For $\lambda \in S'(\omega)$ we have $\hat{\lambda} \in S'(\omega^{-1}\omega_1)$.*

Proof. We have $\lambda \in S'(\omega)$, i.e. for all $a \in F^\times$, $a \cdot \lambda = \omega(a)\lambda$. Now

$$\langle a \cdot \hat{\lambda}, f \rangle = \langle \hat{\lambda}, a^{-1} \cdot f \rangle = \langle \lambda, \widehat{a^{-1} \cdot f} \rangle.$$

This gives

$$\begin{aligned} \widehat{a^{-1}f}(x) &= \int_F f(a^{-1}y)\psi(xy)dy \\ &= \int_F f(y)\psi(xay)d(ay) \\ &= |a| \int_F f(y)\psi(xay)dy \\ &= |a|\hat{f}(ax) = |a|(a \cdot \hat{f})(x) \end{aligned}$$

Thus we have $\langle a \cdot \hat{\lambda}, f \rangle = \langle \lambda, |a|(a \cdot \hat{f}) \rangle = \langle |a|\lambda, a \cdot \hat{f} \rangle = \langle |a|(a^{-1} \cdot \lambda), \hat{f} \rangle = \langle |a|\omega(a^{-1})\hat{\lambda}, f \rangle$. Therefore $a \cdot \hat{\lambda} = \omega^{-1}\omega_1(a)\hat{\lambda}$, as desired. □

We know that $z_0(s, \omega) \in S'(\omega\omega_s)$, so the above lemma tells us that $\widehat{z_0(s,\omega)} \in S'(\omega^{-1}\omega_{-s}\omega_1) = S'(\omega^{-1}\omega_{1-s})$. But $z_0(1-s, \omega^{-1}) \in S'(\omega^{-1}\omega_{1-s})$. In addition, this space is 1-dimensional, so the two differ by a constant multiple. Rearranging, this give the main result of this section, the local functional equation:

$$z_0(\widehat{1-s,\omega^{-1}}) = \epsilon(s,\omega,\psi)z_0(s,\omega)$$

where we view $\epsilon(s,\omega,\psi)$ as a constant depending on s, ω, and ψ.

3. Global (Adelic) Theory

Now we want to apply similar methods in a more global way, that sees all the information from all the places simultaneously. To do this, we construct an object called the *adèle ring*, denoted by $\mathbb{A}_k$, or simply $\mathbb{A}$ when the field we are working with has already been specified and is clear. This is defined as follows:

$$\mathbb{A} = \prod_v{}' k_v$$

where the product is indexed over all places of k and the prime symbol indicates that the product is *restricted*: for each element of $\mathbb{A}$, for all but finitely many places v, the corresponding entry lies in the ring of integers of k_v. By restricting the product in this way, we can define a topology on A that retains local compactness by using Tychonoff's theorem. This allows us to define a Haar measure and apply similar methods to those used in the local case. We can also define the *idèle* ring as

$$\mathbb{A}^\times = \prod_v{}' k_v^\times$$

where we now restrict to $\mathcal{O}_{k_v}^\times$ instead of $\mathcal{O}_{k_v}$.

Similarly to the local case, we can define $S(\mathbb{A})$ as the Schwarz-Bruhat functions on $\mathbb{A}$. This space is very difficult to describe fully, but we can describe some of its elements. The space $S(\mathbb{A})$ contains all functions of the form $\bigotimes_v f_v$, where each $f_v \in S(k_v)$ and all but finitely many f_v are the characteristic function of $\mathcal{O}_{k_v}$, which we refer to as f_v^0. These functions are well-defined because for $f_v \in S(k_v)$ and x in $\mathcal{O}_{k_v}$, $f_v(x) = 1$, so if $x \in \mathbb{A}$ and f is of the form above, we have

$$f(x) = \left(\bigotimes_v f_v\right) \cdot (x_v) = \prod_v f_v(x_v).$$

But because the entries of every element of $\mathbb{A}$ are in the corresponding ring of integers AND all but finitely many f_v are characteristic functions of that ring, all but finitely many terms of the product are 1, so the product is well-defined. Because $S(\mathbb{A})$ is a vector space, it contains all finite linear combinations of such factorizable functions. In fact, these are dense inside $S(\mathbb{A})$, which allows us to be able to focus on them when we want to talk about $S(\mathbb{A})$. We can then define $S'(\mathbb{A})$ as the space of continuous linear functionals on $S(\mathbb{A})$.

The following lemma (which won't be proved) does a lot of work for us in the global case:

Lemma 3.1. *If for every place v of k we have some distribution $\lambda_v \in S'(k_v)$, such that for all but finitely many v we have $\langle \lambda_v, f_v^0 \rangle = 1$, then there exists a unique $\lambda \in S'(\mathbb{A})$ such that for all* ***factorizable*** *$f = \bigotimes_v f_v \in S(\mathbb{A})$ we have*

$$\langle \lambda, f \rangle = \prod_v \langle \lambda_v, f_v \rangle$$

An important property of the adèle ring is that k embeds into it in a natural way. To illustrate this, think about $\mathbb{A}_\mathbb{Q}$. For any positive integer b, b is invertible in $\mathbb{Z}_p$ whenever the expansion of b has a nonzero p^0 coefficient, or in other words, when p does not divide b. If this is the case, for any integer a, $\frac{a}{b}$ is in $\mathbb{Z}_p$ because $a \in \mathbb{Z}_p$ and $b^{-1} \in \mathbb{Z}_p$. Thus any rational number is in $\mathbb{Z}_p$ for all but finitely many primes p, which are exactly those primes that divide its denominator. Because $\mathbb{Z}_p$ is the ring of integers of $\mathbb{Q}_p$, for $q \in \mathbb{Q}$, the element $(q, q, q, \ldots)$ is thus in $\mathbb{A}_\mathbb{Q}$ because it satisfies the restriction criterion. For a general field k, the idea is the same: we can embed k into $\mathbb{A}_k$ diagonally. Similarly, we can embed $k^\times$ into $\mathbb{A}_k^\times$.

We can also define quasicharacters on $\mathbb{A}^\times$. In particular, we want to look at homomorphisms $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ that are trivial on the image of $k^\times$ under the diagonal embedding. Such a function is called a *Hecke character*. Given a Hecke character ω, we can look at its restriction ω_v to the field k_v for a place v. This gives a quasicharacter on k_v. It turns out that for all but finitely many places, this character is unramified. This is the reason that it makes sense to focus on the unramified archimedean case.

We can define spaces of ω-eigendistributions in the same way as we did in the local theory. In addition, because we have $\omega = \bigotimes_v \omega_v$, we can see that $S'(\omega) = \bigotimes_v S'(\omega_v)$. This means that $S'(\omega)$ also has dimension 1 in the global case. Also, we have $z_0(s,\omega) = \bigotimes_v z_0(s\omega_v)$ spanning $S'(\omega\omega_s)$ (ω_s can be defined on $\mathbb{A}^\times$ as the product of ω_s at all the local places because that will all but finitely many times equal 1).

We can also define global zeta-integrals. We can construct a Haar measure on $\mathbb{A}^\times$ as a product (in a suitable sense) of local Haar measures. This allows us to define, for ω a character of $\mathbb{A}^\times$ trivial on $k^\times$ and $f \in S(\mathbb{A})$,

$$z(s,\omega;f) = \int_{\mathbb{A}^\times} f(x)\omega\omega_s(x)dx.$$

For factorizable f, this is equal to the product of the local zeta integrals of the factors of f. But it turns out that these integrals are trivial at all the unramified places, so the product is actually finite. This means that it converges when the product converges absolutely, which is when $\text{Re}(s) > 1$.

We can also define the global L-function $\Lambda(s,\omega) = \prod_v L_v(s,\omega)$, which allows us to write $z(s\omega) = \Lambda(s,\omega)z_0(s\omega)$ in the global sense, with convergence on $\text{Re}(s) > 1$.

3.1. **Adèlic Fourier Analysis.** Again, we pick ψ to be an additive character of $\mathbb{A}$ that's trivial on the image of k. We again can identify $\mathbb{A}$ with the group of characters, and we can define a Fourier transform that's compatible with the local ones, i.e. if $f \in S(\mathbb{A})$ is factorizable, then $\hat{f} = \bigotimes_v \hat{f}_v$. We define Fourier transforms on distributions as before as well.

The conclusion of this section relies on a fact that Tate proved in his thesis, which he called the main theorem. His result makes use of a generalization of the Poisson summation formula. This formula, which holds for functions f on the adèles that satisfy certain convergence criteria and $a \in \mathbb{A}^\times$, is

$$\frac{1}{|a|}\sum_{x\in k}\hat{f}(x/a) = \sum_{x \in k} f(ax)$$

which looks very similar to the standard Poisson summation formula in $\mathbb{R}^n$. Here the absolute value in $\mathbb{A}^\times$ is the product of the absolute values of each component of $a \in \mathbb{A}^\times$, i.e.

$$|a| = \prod_v |a|_v$$

where the product is over all places v. This is a well-defined product because it ends up being a finite product: at each nonarchimedean place, every unit of the ring of integers has absolute value 1, so all but finitely many terms of the product are 1.

Tate used the Poisson formula to prove the following theorem:

Theorem 3.2. *The global zeta integral has a meromorphic continuation to the whole complex plane and*

$$z(\widehat{1-s,\omega^{-1}}) = z(s,\omega).$$

We can define $\epsilon(s,\omega) = \prod_v \epsilon(s,\omega_v,\psi_v)$. This ends up not depending on ψ.

Now: we can take the products of the local functional equations to find that globally,

$$z_0(\widehat{1-s,\omega^{-1}}) = \epsilon(s,\omega)z_0(s,\omega).$$

In addition, we have

$$\Lambda(1-s,\omega^{-1})z_0(\widehat{1-s,\omega^{-1}}) = z(\widehat{1-s,\omega^{-1}}) \qquad = z(s,\omega) = \Lambda(s,\omega)z_0(s\omega)$$

by making the appropriate substitutions. Making one more, we see

$$\Lambda(1-s,\omega^{-1})\epsilon(s,\omega)z_0(s,\omega) = \Lambda(s,\omega)z_0(s\omega).$$

But $z_0(s,\omega)$ is never 0, so we end up with the **Global Functional Equation**:

Theorem 3.3. $\Lambda(1-s,\omega^{-1})\epsilon(s,\omega) = \Lambda(s,\omega)$.

This is what Tate set out to prove. He was not the first to do so; in fact Hecke, whose name is attached to some of the objects in this paper, first proved it, although in a much more laborious fashion. Tate was the first to utilize the nice properties of the adèles to form a much more elegant and natural proof.

4. Some Applications in Analytic Number Theory

The global functional equation looks daunting because of its connection with the strange structure of the adèles, but it can be used easily to find some very concrete results. For example, we let $k=\mathbb{Q}$ and ω be the trivial character. Then for each nonarchimedean place, the local L function is $(1-1/p^s)^{-1}$. The product of these over all primes is, due to Euler, the Reimann ζ function. One can show that the L function for the remaining place is exactly what one would want it to be in order for the global functional equation to yield the familiar functional equation for ζ:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\frac{1}{2}\pi s)\Gamma(s)\zeta(s).$$

The usual proofs of this fact rely on clever, tricky manipulations of power series, whereas this method gives it directly.

The global functional equation can also be used to prove Dirichlet's theorem: if a and b are relatively prime positive integers, there are infinitely many primes of the form $a+nb$. This can be reduced to showing that the (Dirichlet) L-function on a Dirichlet character does not vanish at $s=1$. First, one shows that the classical Dirichlet L-function coincides with the global L-function used above. Then one can compute the local L-factors with relative ease. The functional equation lets us evaluate the function at 1 by evaluating it at 0, which is much easier. In this way, the global functional equation helps us attack this problem as well.

Acknowledgments. I would like to thank my mentor Daniel Johnstone for guiding me through the process of learning this material.

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