WHAT DOES A LIE ALGEBRA KNOW ABOUT A LIE GROUP?

HOLLY MANDEL

ABSTRACT. We define Lie groups and Lie algebras and show how invariant vector fields on a Lie group form a Lie algebra. We prove that this correspondence respects natural maps and discuss conditions under which it is a bijection. Finally, we introduce the exponential map and use it to write the Lie group operation as a function on its Lie algebra.

CONTENTS

1. INTRODUCTION

Lie groups provide a mathematical description of many naturally-occuring symmetries. Though they take a variety of shapes, Lie groups are closely linked to linear objects called Lie algebras. In fact, there is a direct correspondence between these two concepts: simply-connected Lie groups are isomorphic exactly when their Lie algebras are isomorphic, and every finite-dimensional real or complex Lie algebra occurs as the Lie algebra of a simply-connected Lie group. To put it another way, a simply-connected Lie group is completely characterized by the small collection of scalars that determine its Lie bracket, no more than $\frac{n^2(n-1)}{2}$ numbers for an n-dimensional Lie group.

In this paper, we introduce the basic Lie group-Lie algebra correspondence. We first define the concepts of a Lie group and a Lie algebra and demonstrate how a certain set of functions on a Lie group has a natural Lie algebra structure. We then construct the exponential map, a local diffeomorphism between a Lie group and its Lie algebra, by following certain flows on the Lie group. Finally, we explain a sense in which the exponential map and the Lie algebra together encode the group operation.

Date: August 2016.

2. Lie Groups and Lie Algebras

Intuitively, Lie groups are smoothly-varying collections of symmetries. For instance, while the transformations of the plane that map a regular polygon to itself are discrete, the rotations of a circle can be smoothly parameterized by angle.

Definition 2.1. A **Lie group** is an algebraic group $(G, \star)$ that is also a smooth manifold, such that:

(1) the inverse map $g \mapsto g^{-1}$ is a smooth map $G \to G$.
(2) the group operation $(g, h) \mapsto g \star h$ is a smooth map $G \times G \to G$.

Definition 2.2. A **Lie group homomorphism** is a smooth map between Lie groups that is a homomorphism of groups. A **Lie group isomorphism** is a Lie group homomorphism that is a diffeomorphism.

Perhaps the most familiar Lie groups are the **general linear group** $\mathrm{GL_n}(\mathbb{R})$ of invertible real $n \times n$ matrices, and its close relative $\mathrm{GL}(V)$, the set of automorphisms of a vector space V. Many interesting Lie groups arise as their subgroups.

Proposition 2.3. *$GL_n(\mathbb{R})$ is a Lie group.*

Proof. The collection $\mathrm{M_n}(\mathbb{R})$ of $n \times n$ real matrices is a vector space isomorphic to $\mathbb{R}^{n^2}$, so any linear isomorphism $f : \mathrm{M_n}(\mathbb{R}) \to \mathbb{R}^{n^2}$ gives a smooth structure on $\mathrm{M_n}(\mathbb{R})$ with the global chart $(\mathrm{M_n}(\mathbb{R}), f)$. We consider $\mathrm{GL_n}(\mathbb{R})$ as a subset of $\mathrm{M_n}(\mathbb{R})$.

The determinant function $\det \colon \mathrm{M_n}(\mathbb{R}) \to \mathbb{R}$ is a polynomial in the entries of a matrix. Under a certain choice of f, these entries correspond to the standard basis vectors of $\mathbb{R}^{n^2}$, so the coordinate representation of det is a polynomial on $\mathbb{R}^{n^2}$. Therefore, det is continuous. Since $\mathrm{GL_n}(\mathbb{R}) = \det^{-1}(\mathbb{R}\backslash\{0\})$, it follows that $\mathrm{GL_n}(\mathbb{R})$ is an open subset of $\mathrm{M_n}(\mathbb{R})$, so $\mathrm{GL_n}(\mathbb{R})$ is an embedded submanifold, inheriting the smooth structure generated by restricting f to $\mathrm{GL_n}(\mathbb{R})$.

$\mathrm{GL_n}(\mathbb{R})$ is an algebraic group under the operation of matrix multiplication. Cramer's formula in the given coordinates shows that matrix inversion is a smooth map $\mathrm{GL_n}(\mathbb{R}) \to \mathrm{GL_n}(\mathbb{R})$. The coordinates on $\mathrm{GL_n}(\mathbb{R})$ also induce natural coordinates on $\mathrm{GL_n}(\mathbb{R}) \times \mathrm{GL_n}(\mathbb{R})$, with respect to which matrix multiplication is a polynomial function $\mathbb{R}^{2n^2} \to \mathbb{R}^{n^2}$. □

The abstract definition of a Lie algebra, a vector space with an additional multiplication-like operation, seems unrelated to the concept of a Lie group. For instance, we can speak of Lie algebras over fields of finite characteristic, over which there can be no smooth structure.

Definition 2.4. A **Lie algebra** is a vector space L over a field F with a bilinear operation $L \times L \to L$, denoted by brackets: $(x, y) \mapsto [x, y]$. The bracket operation must satisfy:

(1) $[x, x] = 0$ for all $x \in L$.
(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Definition 2.5. A **Lie algebra homomorphism (isomorphism)** is a linear homomorphism (isomorphism) between Lie algebras $f : L_1 \to L_2$ such that $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L_1$.

When char $F \neq 2$, bilinearity can be used to show that the condition (1) in Definition 2.4 is equivalent to **antisymmetry**, the condition that $[x, y] = -[y, x]$.

Condition (2) is called the **Jacobi identity**, and it replaces associativity as a constraint on triple products.

Example 2.6. The real vector space $\mathrm{M_n}(\mathbb{R})$ with the bracket operation given by the commutator $[A, B] = AB - BA$ is a Lie algebra, denoted $\mathfrak{gl}_n(\mathbb{R})$. We will prove in Section 3 that this is the Lie algebra of $\mathrm{GL_n}(\mathbb{R})$.

Example 2.7. [**2**, Section 1.3] Let $\mathfrak{U}$ be a vector space with a binary operation $(u, v) \mapsto u * v$ that is bilinear but not necessarily associative. Let $D(\mathfrak{U})$ be the set of **linear derivations** on $\mathfrak{U}$, linear operations $\pi : \mathfrak{U} \to \mathfrak{U}$ satisfying the product rule $\pi(a * b) = a * \pi(b) + \pi(a) * b$. Then $D(\mathfrak{U})$ is a Lie algebra under the commutator $[\pi, \psi] = \pi \circ \psi - \psi \circ \pi$, where $\circ$ denotes function composition. The verification amounts to checking that the bracket of two derivations is a derivation. By contrast, the composition of two derivations is not necessarily a derivation.

Example 2.8. If V is a Lie algebra and $f : V \to W$ is a vector space isomorphism, then W is a Lie algebra under the bracket operation $[w_1, w_2] = f([f^{-1}(w_1), f^{-1}(w_2)])$. In this paper, we call the resulting operation on W the **induced bracket** of f.

3. Invariant Vector Fields

The cooperation of Lie algebras with linear derivations suggests a possible relationship to the tangent bundle. To define a Lie algebra on a Lie group, we look at a special class of its vector fields. The proofs in this section follow John Lee [**5**, Chapter 8].

In what follows, we take "vector field" to mean "smooth vector field." For a vector field V on a smooth manifold M, we use subscripts for the point of evaluation: loosely, $V_p \in T_pM$, the tangent space to M at p. When V acts on $f \in C^\infty(M)$, we write the resulting smooth function simply as Vf, so $Vf(g) = [V_g](f)$.

For a smooth map $F : M \to N$ and $p \in M$, let $D(F)_p : T_pM \to T_{F(p)}N$ be the differential of F at p. Given a vector field X on M, if there exists a vector field Y on N such that $Y_{F(p)} = D(F)_pX_p$ for all $p \in M$, we write $Y = (F)_*X$. Such a Y exists for every X if F is a diffeomorphism [**5**, Proposition 8.19]. The crucial vector fields on a Lie group are those for which this operation gives a symmetry.

Definition 3.1. Let $(G, \star)$ be a Lie group and let $L_g : G \to G$ be the left multiplication diffeomorphism $h \mapsto g \star h$. A vector field X is called **left-invariant** if $(L_g)_*X = X$ for all $g \in G$.

This condition is very restrictive, so much so that a left-invariant vector field is uniquely determined by its value at any point in G. Let **Lie(*G*)** be the set of all left-invariant vector fields on a Lie group G and define π to be the evaluation map $V \mapsto V_e$, where e is the identity of G.

Proposition 3.2. *The evaluation map π: Lie(G) $\to T_eG$ is a linear isomorphism.*

Proof. [**5**, Proposition 8.37] By definition vector fields add and scale pointwise, so evaluation at e is linear.

To show that π is surjective, take $v \in T_eG$ and define the map V that sends g to $D(L_g)_ev$. We claim that V is a smooth vector field. This is equivalent to the statement that Vf is smooth for all $f \in C^\infty(G)$ [**5**, Proposition 8.14]. Let γ be a curve such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then

$$v(f) = \gamma'(0)f = \left(D(\gamma)_0 \frac{d}{dt}\bigg|_{t=0} \right) f = \frac{d}{dt}\bigg|_{t=0} f \circ \gamma(t).$$

For all $f \in C^\infty(G)$ and $g \in G$,

$$Vf(g) = V_g(f) = D(L_g)_e v(f) = \frac{d}{dt}\bigg|_{t=0} f \circ L_g \circ \gamma(t).$$

Define m to be the multiplication map $m(g,h) = g \star h$ and take n to be the smooth map $G \times (-\epsilon, \epsilon) \to G \times G$ given by $(g,t) \mapsto (g, \gamma(t))$. Then $f \circ L_g \circ \gamma(t) = f \circ m \circ n(g,t)$. This is a composition of smooth maps, so its derivative in t is a smooth function of g. Therefore, Vf is a smooth function on G. To see that V is left-invariant, we compute for any $g, h \in G$:

$$((L_h)_* V)_{hg} = D(L_h)_g V_g = D(L_h)_g \circ D(L_g)_e v = D(L_{hg})_e v = V_{hg}.$$

This shows that $(L_h)_* V$ and V are equal at hg. But hg ranges over G, so these vector fields are equal. Therefore V is an element of Lie(G) and $\pi(V) = v$.

To prove that π is injective, we show that V as defined above is the only possible left-invariant vector field evaluating to v at the identity. For if W is any such vector field, then left invariance implies that

$$W_g = ((L_g)_* W)_g = D(L_g)_e v = V_g$$

for all $g \in G$. Therefore, $W = V$. □

Given $v \in T_e G$, we write $\pi^{-1}(v)$ as L^v. That is, L^v is the left-invariant vector field such that $L^v_e = v$. Because every tangent space to an n-dimensional manifold is an n-dimensional real vector space, Proposition 3.2 determines the vector space structure of Lie(G).

Corollary 3.3. *Let G be a Lie group that is an n-dimensional real manifold. Then Lie(G) is an n-dimensional real vector space.*

It remains to find an appropriate binary operation. We cannot simply compose vector fields, since the resulting object is not a derivation. However, the commutator of two vector fields is a vector field, and this operation preserves left-invariance.

Proposition 3.4. *Lie(G) is a Lie algebra under the bracket operation $[X, Y] = X \circ Y - Y \circ X$.*

Proof. The vector fields on a manifold M are exactly the derivations of $C^\infty(M)$ [**5**, Proposition 8.15]. Therefore, Example 2.7 shows that the space of vector fields on G form a Lie algebra with the given bracket.

To see that the subspace Lie(G) is itself a Lie algebra, it suffices to prove that it is closed under the bracket operation, meaning that $[X, Y]$ is left-invariant when X and Y are. We use a more general property of the bracket.

Lemma 3.5. *If $X' = (F)_* X$ and $Y' = (F)_* Y$, then $[X', Y'] = (F)_* [X, Y]$.*

Proof. [**5**, Propositions 8.30, 8.33] Applying X' and $(F)_* X$ to an arbitrary smooth function f, we see that $X' = (F)_* X$ exactly when $(X'(f)) \circ F = X(f \circ F)$ for every $f \in C^\infty(M)$. Therefore the hypotheses on X' and Y' imply that

$$X \circ Y(f \circ F) = X(Y'(f) \circ F) = (X' \circ Y')(f) \circ F$$

and $Y \circ X(f \circ F) = (Y' \circ X')(f) \circ F$ by the same argument. Then

$$\begin{aligned} [X, Y](f \circ F) &= X \circ Y(f \circ F) - Y \circ X(f \circ F) \\ &= (X' \circ Y')(f) \circ F - (Y' \circ X')(f) \circ F = [X', Y'](f) \circ F \end{aligned}$$

for all $f \in C^\infty(M)$. This proves that $[X', Y'] = (F)_* [X, Y]$. □

The proposition follows because $X = (L_g)_* X$ and $Y = (L_g)_* Y$ for all $g \in G$. □

Using Example 2.8 and the identification of Proposition 3.2, we can induce a Lie algebra structure on T_eG consistent with the bracket on vector fields.

Corollary 3.6. *Take π as in Proposition 3.2. T_eG is a Lie algebra under the induced bracket of π, and π is a Lie algebra isomorphism $Lie(G) \to T_eG$.*

Definition 3.7. Let G be a Lie group. The **Lie algebra of** G, denoted $\mathfrak{g}$, is T_eG under the induced bracket operation.

We saw in Proposition 2.3 that $\mathrm{GL_n}(\mathbb{R})$ is an open subset of $\mathrm{M_n}(\mathbb{R})$, so $T_e\mathrm{GL_n}(\mathbb{R}) \simeq T_e\mathrm{M_n}(\mathbb{R})$. By the standard identification of the tangent space to a vector space with itself (see [**5**, Proposition 3.13]), we therefore have $T_e\mathrm{GL_n}(\mathbb{R}) \simeq \mathrm{M_n}(\mathbb{R})$. This isomorphism is easily visualized: if $\gamma(t) = (\gamma_{ij}(t))$ is a path in $\mathrm{GL_n}(\mathbb{R})$, we identify $\gamma'(t)$ with $(\gamma'_{ij}(t)) \in \mathrm{M_n}(\mathbb{R})$. Then, the bracket on $\mathrm{GL_n}(\mathbb{R})$ takes a simple form.

Proposition 3.8. *Let $G = GL_n(\mathbb{R})$. The induced Lie bracket on T_eG is the commutator under matrix multiplication: $[A, B] = AB - BA$.*

Proof. [**4**, Proposition 5.62] Let F be the space of linear functionals $\mathrm{M_n}(\mathbb{R}) \to \mathbb{R}$ and define for each $X \in \mathrm{M_n}(\mathbb{R})$ the linear map $P_X : F \to F$ such that $[P_X(f)](A) = f(AX)$ for all $f \in F$ and all $A \in \mathrm{M_n}(\mathbb{R})$. The restriction of any $f \in F$ to the open subset $G \subset \mathrm{M_n}(\mathbb{R})$ is a smooth map on G, so we can apply a vector field $L^X \in \mathrm{Lie}(G)$ as a derivation of f. Now since G is open, $\gamma(t) = I + tX$ defines a curve in G for small t with $\gamma(0) = I$ and $\gamma'(0) = X$. We evaluate

$$L^X f(A) = D(L_A)_e X(f) = D(L_A)_e \gamma'(0) f.$$

Computing as we did in the proof of Proposition 3.2, we find

$$D(L_A)_e \gamma'(0) f = \left.\frac{d}{dt}\right|_{t=0} f \circ L_A \circ \gamma(t) = \left.\frac{d}{dt}\right|_{t=0} f(A + tAX) = f(AX)$$

where the last equality uses the linearity of f. Therefore

$$L^X f = P_X(f) \qquad \forall f \in F,\ \forall X \in \mathrm{M_n}(\mathbb{R}). \tag{3.9}$$

Using Equation 3.9 and keeping track of which bracket operation is in effect, we compute

$$P_{[X,Y]}(f) = L^{[X,Y]} f = [L^X, L^Y] f = L^X(L^Y f) - L^Y(L^X f). \tag{3.10}$$

Two applications of Equation 3.9 show that

$$L^X(L^Y f) - L^Y(L^X f) = P_X(P_Y(f)) - P_Y(P_X(F)) = P_{XY-YX}(f), \tag{3.11}$$

where the last equality follows from the definition of P. But $P_X(f)(I) = f(X)$, so combining Equations 3.10 and 3.11 gives $f([X, Y]) = f(XY - YX)$ for all $f \in F$. Since two elements of a vector space are equal exactly when their values under all linear functionals are equal, this implies $[X, Y] = XY - YX$. □

4. Induced Homomorphisms

We now have a way to assign a Lie algebra to a Lie group. To help justify this choice, we observe that natural maps between Lie groups induce natural maps between Lie algebras.

Theorem 4.1. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and let $F : G \to H$ be a Lie group homomorphism. Then the differential $D(F)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. [**5**, Theorem 8.44] $D(F)_e$ is by definition a linear homomorphism $\mathfrak{g} \to \mathfrak{h}$. We need to show that $D(F)_e[v, w] = [D(F)_e\, v, D(F)_e\, w]$ for all $v, w \in \mathfrak{g}$.

By Definition 3.7, $D(F)_e[v, w] = D(F)_e[L^v, L^w]_e$. If F were assumed to be a diffeomorphism, we could write these expressions in terms of $(F)_*[L^v, L^w]$. In fact, a similar situation holds for Lie group homomorphisms.

Lemma 4.2. *Let $F : G \to H$ be a Lie group homomorphism. Then for all $X \in$ Lie(G), there exists $Y \in$ Lie(G) such that $Y = (F)_*X$.*

Proof. Because F is a group homomorphism, we can compute

$$F(L_g(h)) = F(g) \star F(h) = L_{F(g)}(F(h))$$

for all h, so $F \circ L_g = L_{F(g)} \circ F$. Therefore we have the following equality for all h, g:

$$D(F)_{L_g(h)} \circ D(L_g)_h = D(L_{F(g)})_{F(h)} \circ D(F)_h.$$

Given $X = L^v$ for some $v \in T_eG$, let $Y = L^{D(F)_e v}$. We then have

$$Y_{F(g)} = D(L_{F(g)})_e D(F)_e v = D(F)_g D(L_g)_e v = D(F)_g X_g$$

for all g, so $Y = (F)_*X$. □

This allows us to write $D(F)_e[L^v, L^w]_e$ as $((F)_*[L^v, L^w])_e$, which by Lemma 3.5 is equal to $[(F)_*L^v, (F)_*L^w]_e$. Since $(F)_*L^v$ is a left-invariant vector field taking $D(F)_e v$ at the identity, $(F)_*L^v = L^{D(F)_e v}$, and similarly for w. Thus

$$D(F)_e[L^v, L^w]_e = [(F)_*L^v, (F)_*L^w]_e = [L^{D(F)_e v}, L^{D(F)_w}]_e$$

which by Definition 3.7 implies the result. □

Corollary 4.3. *If $F : G \to H$ is a Lie group isomorphism, then $D(F)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism.*

One consequence of Theorem 4.1 is that a representation of a Lie group induces a corresponding representation of its Lie algebra. Recall that End(V) is the space of endomorphisms of a vector space V.

Definition 4.4. Let G be a Lie group and V a finite-dimensional real vector space. A **Lie group representation** is a Lie group homomorphism $G \to \mathrm{GL}(V)$.

Definition 4.5. Let $\mathfrak{g}$ be a Lie algebra with base field F and let V be an F-vector space. A **Lie algebra representation** is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the Lie algebra End(V) under the commutator operation.

By analogy with Proposition 3.8, $\mathfrak{gl}(V)$ is the Lie algebra of $\mathrm{GL}(V)$. Therefore, Theorem 4.1 shows that the differential of a Lie group representation $F : G \to \mathrm{GL}(V)$ gives a Lie algebra representation $D(F)_e : \mathfrak{g} \to \mathfrak{gl}(V)$.

We introduce a useful representation of a Lie group G in the general linear group of its Lie algebra, which gives a representation of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$. This representation is obtained by repeatedly differentiating the action of conjugation in G. After several steps, we recover the Lie bracket in the differential of this representation. Our discussion follows Jeffrey Lee [**4**, Section 5.5].

First, define for all $g \in G$ the conjugation map $C_g : G \to G$ that sends x to gxg^{-1}. Then C_g is a Lie group isomorphism: it is a classic group automorphism, and it is diffeomorphic by the smoothness of the group operation. We define Ad_g to be $D(C_g)_e$. By Corollary 4.3, Ad_g is a Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}$.

Since Lie algebra isomorphisms are a special type of linear isomorphisms, Ad_g is an element of $\mathrm{GL}(\mathfrak{g})$. Therefore the map $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$ maps G into $\mathrm{GL}(\mathfrak{g})$. To show that it is smooth, we cite a simple geometric result [**4**, Lemma 5.85].

Lemma 4.6. *Let $f : M \times N \to N$ be smooth and say there exists $y_0 \in N$ such that $f(x, y_0) = y_0$ for all $x \in M$. Define $f_x(y) = f(x, y)$. Then $x \mapsto D(f_x)_{y_0}$ is a smooth function $M \to GL(T_{y_0}N)$.*

Proposition 4.7. *$Ad : G \to GL(\mathfrak{g})$ is a Lie group homomorphism.*

Proof. Ad is an algebraic homomorphism:

$$\mathrm{Ad}(gh) = D(C_{gh})_e = D(C_g \circ C_h)_e = D(C_g)_e \circ D(C_h)_e = \mathrm{Ad}(g) \circ \mathrm{Ad}(h).$$

Let $C(g, x) = gxg^{-1}$. Then C is a smooth map $G \times G \to G$ and $C(g, e) = e$ for all g. Therefore Lemma 4.6 implies that $\mathrm{Ad} : g \mapsto D(C_g)e$ is smooth. □

We call Ad the **adjoint representation**. By Theorem 4.1 and Proposition 4.7, $D(\mathrm{Ad})_e$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Let **ad** be this differential. Then $\mathrm{ad}(x)$ is an element of $\mathfrak{gl}(\mathfrak{g})$, the Lie algebra of linear transformations of $\mathfrak{g}$. In fact, it is a linear transformation we have already seen.

Proposition 4.8. *$ad(x)y = [x, y]$.*

Proof. [**4**, Proposition 5.89] First, let R_g be the right multiplication map in G: $h \mapsto h \star g$. For $w \in T_eG$, we define the vector field $R^w = D(R_g)_e w$. The proof that $w \mapsto R^w$ is linear and that R^w is a smooth right-invariant vector field is identical to the proof for left-invariant vector fields. Now

$$L_g = R_g \circ (R_{g^{-1}} \circ L_g) = R_g \circ C_g$$

which implies that for any $v \in T_eG$ and $g \in G$,

$$L^v_g = D(L_g)_e v = D(R_g)_e \circ D(C_g)_e v = (R^{\mathrm{Ad}_g(v)})_g. \tag{4.9}$$

We first use this to evaluate the right-hand side. Let $b_1, ..., b_n$ be a basis for $\mathfrak{g}$ and let ϵ_{jk} be a basis for $\mathrm{End}(\mathfrak{g})$, so $\epsilon_{jk}(y) = \sum_{i=1}^n c^i_{jk} b_i$ for constants c^i_{jk}. Proposition 4.7 implies that Ad has a coordinate representation $\mathrm{Ad}(g) = \sum_{j,k} \alpha_{jk}(g)\epsilon_{jk}$ with α_{jk} smooth functions $G \to \mathbb{R}$. Then

$$\mathrm{Ad}_g(y) = \sum_{j,k} \alpha_{jk}(g) e_{jk}(y) = \sum_{i=1}^n a_i(g) b_i \tag{4.10}$$

where each $a_i(g) = \sum_{j,k} \alpha_{jk}(g) c^i_{jk}$ is smooth. Using Equation 4.9 to subsitute $R^{\mathrm{Ad}_g(y)}$ for L^y and expanding the expression for $\mathrm{Ad}_g y$ gives

$$[L^x, L^y](g) = [L^x, R^{\sum_{i=1}^n a_i(g) b_i}](g) = \sum_{i=1}^n [L^x, a_i(g) R^{b_i}](g).$$

The product rule allows us to expand the last term:

$$\sum_{i=1}^n [L^x, a_i(g) R^{b_i}] = \sum_{i=1}^n a_i(g)[L^x, R^{b_i}] + L^x(a_i) R^{b_i}(g).$$

We state without proof that the bracket of a left-invariant and right-invariant vector field is zero (this follows from [**5**, Theorem 9.42 and Proposition 20.8(h)]). Therefore the first term on the right vanishes. Evaluating the resulting expression at e gives

$$[x, y] = [L^x, L^y](e) = \sum_{i=1}^{n} (L^x(a_i)R^{b_i})(e) = \sum_{i=1}^{n} x(a_i)b_i.$$

To evaluate the left-hand side, take $\gamma(t)$ such that $\gamma(0) = 0$ and $\gamma'(0) = x$. Then, recalling the identification $\mathrm{M_n}(\mathbb{R}) \simeq T_e\mathrm{GL_n}(\mathbb{R})$ discussed before Proposition 3.8,

$$\mathrm{ad}(x)y = \left(\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\gamma(t)) \right) y = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\gamma(t))y.$$

Finally, by Equation 4.10,

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\gamma(t))y = \sum_{i=1}^{n} \frac{d}{dt}\bigg|_{t=0} a_i(\gamma(t))b_i = \sum_{i=1}^{n} x(a_i)b_i.$$

□

5. A General Exponential Map

In the previous section, we demonstrated that the Lie algebra of a Lie group is determined by its algebraic structure. Now, we work in the opposite direction to show that the group is locally determined by its Lie algebra. For this purpose, we introduce notation to describe the flows of left-invariant vector fields.

Definition 5.1. Let $(G, \star)$ be a Lie group. A **one-parameter subgroup** is a Lie group homomorphism $(\mathbb{R}, +) \to G$.

Proposition 5.2. *The one-parameter subgroups of a Lie group are exactly the maximal integral curves of its left-invariant vector fields, initialized at e.*

Proof. [**5**, Theorem 20.1] Let $\gamma(t)$ be a one-parameter subgroup. Then a calculation shows that for all $t \in \mathbb{R}$,

$$D(L_{\gamma(t)})_e \gamma'(0) = (L_{\gamma(t)} \circ \gamma)'(0) = \gamma'(t).$$

Therefore, $\gamma'(t) = L^{\gamma'(0)}_{\gamma(t)}$ for all t, so by definition $\gamma(t)$ is an integral curve of $L^{\gamma'(0)}$.

Conversely, for any $v \in T_eG$, let $\gamma_v(t)$ be the maximal integral curve of L^v such that $\gamma_v(0) = e$. In general, the maximal integral curve of a vector field is only guaranteed to exist on some neighborhood of 0. However, using invariance to extend the domain, one can show that any integral curve of a left-invariant vector field is defined on all of $\mathbb{R}$ (see [**5**, Lemma 9.15, Theorem 9.18]). For any $s \in \mathbb{R}$, define $\eta(t) = L_{\gamma_v(s)} \circ \gamma_v(t)$. Then, since $\gamma_v'(t) = L^v_{\gamma(t)}$,

$$\eta'(t) = D(L_{\gamma_v(s)})_{\gamma_v(t)} \gamma_v'(t) = D(L_{\gamma_v(s)})_{\gamma_v(t)} L^v_{\gamma_v(t)} = L^v_{\eta(t)}$$

by the left-invariance of L^v. Therefore, both $\eta(t)$ and $\gamma_v(s+t)$ are integral curves for L^v initialized at $\gamma_v(s)$. By uniqueness, this implies that $\eta(t) = \gamma(s+t)$, so for all $t, s \in \mathbb{R}$,

$$\gamma(s+t) = L_{\gamma_v(s)} \circ \gamma_v(t) = \gamma_v(s) \star \gamma_v(t).$$

□

Since every vector field has a unique maximal integral curve and every $v \in T_eG$ corresponds to a unique left-invariant vector field, the following is well-defined.

Definition 5.3. Let $\gamma_v(t)$ be the unique one-parameter subgroup such that $\gamma_v'(0) = v$. The **exponential map** is the function $\exp : T_eG \to G$ that sends v to $\gamma_v(1)$.

Proposition 5.4. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. There exists a neighborhood $U \subset \mathfrak{g}$ contaning 0 and a neighborhood $V \subset G$ containing e such that $exp|_U : U \to V$ is a diffeomorphism.*

Proof. [**5**, Proposition 20.8] The smooth dependence of a differential equation on initial conditions can be used to prove that exp is smooth (see [**5**, Proposition 20.8(a)]). It remains to prove that exp is a local diffeomorphism at 0. By the inverse function theorem, it is sufficient to show that $D(\exp)_0 : T_0\mathfrak{g} \to \mathfrak{g}$ is a linear isomorphism. In fact, identifying $T_0\mathfrak{g}$ with $\mathfrak{g}$ as in Proposition 3.8, $D(\exp)_0$ is the identity map. For given $X \in \mathfrak{g}$, the curve $\eta(t) = tX$ is a map from some interval containing 0 into $\mathfrak{g}$ such that $\eta(0) = 0$ and $\eta'(0) = X$. We use η to compute

$$D(\exp)_0 X = D(\exp)_0 \frac{d}{dt}\bigg|_{t=0} \eta(t) = \frac{d}{dt}\bigg|_{t=0} \exp(tX).$$

Let $\gamma_X(t)$ be the one parameter subgroup for X. By the chain rule, $\gamma_X(\lambda t)$ is an integral curve for $L^{\lambda X}$. Therefore $\gamma_{\lambda X}(1) = \gamma_X(\lambda)$ by the uniqueness of integral curves, so

$$\frac{d}{dt}\bigg|_{t=0} \exp(tX) = \frac{d}{dt}\bigg|_{t=0} \gamma_{tX}(1) = \frac{d}{dt}\bigg|_{t=0} \gamma_X(t) = X.$$

□

Example 5.5. [**1**, Exercises 12.40-12.52] Let $G = \mathrm{GL_n}(\mathbb{R})$ with Lie algebra $T_eG \simeq \mathrm{M_n}(\mathbb{R})$. In this case, the exponential map has an explicit power series form using the multiplicative operation on $\mathrm{M_n}(\mathbb{R})$:

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \tag{5.6}$$

One can verify that this power series converges componentwise to a smooth function $\mathrm{M_n}(\mathbb{R}) \to \mathrm{GL_n}(\mathbb{R})$, that $e^{A(t+s)} = e^{At}e^{As}$, that $e^0 = I$, and that

$$\frac{d}{dt}\bigg|_{t=0} e^{At} = A. \tag{5.7}$$

Therefore $t \mapsto e^{At}$ is the unique integral curve for L^A, so $e^A = \exp(A)$.

6. The Campbell-Baker-Hausdorff Formula

The exponential map gives a diffeomorphism between a neighborhood of the identity V of G and a neighborhood U containing 0 in $\mathfrak{g}$. By the continuity of the group operation, this implies that for sufficiently small $v, w \in \mathfrak{g}$, the product $\exp(v) \star \exp(w)$ should be an element of V, and therefore take the form $\exp(z)$ for some $z \in U$. Therefore we have the map

$$T(v, w) = \exp^{-1}(\exp(v) \star \exp(w))$$

defined for small v, w, such that $\exp(T(v, w)) = \exp(v) \star \exp(w)$. We say that the function T encodes the group law locally, since close to the identity, we can use T and exp to evaluate multiplication on G.

Incredibly, the function $T(v, w)$ is given by a **Lie polynomial**, a power series whose terms involve only iterated brackets of v and w. The following formulas are quoted from Jeffrey Lee [**4**, pp. 223-224].

Theorem 6.1. *(Campbell-Baker-Hausdorff) Let $T(v,w) = exp^{-1}(exp(v) \star exp(w))$. Then in a neighborhood of 0, $T(v,w)$ is given by the power series*

$$T(v,w) = v + w + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \int_0^1 \left(\sum_{k,l>0, k+l\geq 1} \frac{t^k}{k!l!} (ad\ v)^k (ad\ w)^l \right)^n v\, dt. \tag{6.2}$$

Computing the first few terms gives

$$T(v,w) = v + w + \frac{1}{2}[v,w] + \frac{1}{12}([[v,w],w] + [[w,v]v]) + ...$$

Therefore the Lie bracket appears as the first nonlinear term in the "logarithmic" expansion of the group law.

We can obtain global information from this theorem if G is connected.

Proposition 6.3. *Let G be a connected Lie group and let U be an open set containing e. Then U generates G.*

Proof. (See [**3**, Corollary 2.10]) Let H be the algebraic subgroup of G generated by U. Then for all $h \in H$, the preimage $L_{h^{-1}}^{-1}(U) = h \star U$ is an open set containing h and contained in H. Therefore, H is an open subset of G and an embedded submanifold. On the other hand, it is a theorem that an algebraic subgroup of a Lie group is an embedded submanifold exactly when it is closed ([**5**, Corollary 20.13]). Since G is connected and H is nonempty, $H = G$. □

It follows that the Lie algebra determines the group operation on a connected Lie group. As an application, we show that a Lie group is abelian exactly when its Lie algebra has a trivial Lie bracket (such a Lie algebra is also called **abelian**).

Corollary 6.4. *A connected Lie group G with Lie algebra $\mathfrak{g}$ is abelian if and only if $\mathfrak{g}$ is abelian.*

Proof. On the one hand, if G is abelian, then the conjugation map is the identity: $C_g(x) = x$ for all $g, x \in G$. Therefore $\mathrm{Ad}_g = D(C_g)e$ is the identity map on T_eG, for all g. From this we compute $\mathrm{Ad}(g) = \mathrm{Id}_e$ for all g, and so $\mathrm{ad} = D(\mathrm{Ad})_e = 0$. By Proposition 4.8, this implies that $[v,w] = 0$ for all $v, w \in \mathfrak{g}$.

On the other hand, say $\mathfrak{g}$ is abelian. Then all nonlinear terms in Equation 6.2 are zero, so $T(v,w) = v + w$. This implies that $\exp(v) \star \exp(w) = \exp(v+w) = \exp(w+v) = \exp(w) \star \exp(v)$ for all small w, v. Since G is connected, it is generated by any neighborhood of e. But an algebraic group with a commuting generating set is abelian. □

Acknowledgments. I am very grateful to my mentor Tori Akin for all her guidance in learning this material and writing this paper. Thanks also to the Chicago REU for opening its doors to students from other schools.

References

[1] Dummit, D.S. and Foote, R.M., "Abstract Algebra," Second Edition, Wiley, 1999.
[2] Humphreys, J.E., "Introduction to Lie Algebras and Representation Theory," Springer-Verlag, 1972.
[3] Kirillov, A., "Introduction to Lie Groups and Lie Algebras," Cambridge University Press, 2008.
[4] Lee, Jeffrey M., "Manifolds and Differential Geometry," American Mathematical Society, 2009.
[5] Lee, John M., "An Introduction to Smooth Manifolds," Springer-Verlag, 2003.