A COMBINATORIAL APPROACH TO THE BROUWER FIXED POINT THEOREM

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Abstract. The Brouwer Fixed Point Theorem states that any continuous mapping from a closed ball in Euclidean space to itself has at least one fixed point. This theorem has a wide variety of applications in areas such as differential equations, economics, and game theory. Although this is fundamentally an analytic and topological statement, there exists an elegant combinatorial proof using Sperner's Lemma. We take the combinatorial approach of [3] for the one- and two-dimensional cases, then synthesize the methods of [1] and [2] for the general case.

Contents

1. Introduction

Definition 1.1. A **fixed point** of a function $f : X \to X$ is a point $x_0 \in X$ such that

$$f(x_0) = x_0.$$

Fixed point theory is the field of mathematics concerned with the existence of such points. For a closed interval, the existence of such a point is obvious by the Intermediate Value Theorem.

Theorem 1.2. *(Brouwer's Theorem for a Closed Interval.) Suppose $f : [a, b] \to [a, b]$ is a continuous function. Then there exists a point $x_0 \in [a, b]$ such that $f(x_0) = x_0$.*

Proof: If $f(a) = a$ or $f(b) = b$, we are done. Otherwise, $f(a) > a$ and $f(b) < b$. Consider the function $g(x) = f(x) - x$. Then $g(a) > 0$ while $g(b) < 0$. By the Intermediate Value Theorem, since g is also continuous, there exists $x_0 \in [a, b]$ such that $g(x_0) = 0$, or $f(x_0) = x_0$. ■

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Remark 1.3. Note that Theorem (1.2) does not hold for open intervals; the function $f(x) = x^2$ has no fixed point on the open interval $(0, 1)$.

Generalizations of this result to n-dimensional Euclidean space also hold, but are not as easy to prove. Rather than using analysis and topology, we turn instead to Sperner's Lemma, which provides an elegant and unexpected combinatorial approach to the general case of Brouwer's Theorem, which states that any continuous function $f : X \to X$ has a fixed point, where X is a closed n-ball. In fact, the claim holds for any convex compact subset X of $\mathbb{R}^n$.

2. Sperner's Lemma in One and Two Dimensions

In this section, we prove two combinatorial lemmas that are special cases of Sperner's Lemma.

Lemma 2.1. *(Sperner's Lemma in One Dimension.) Consider a closed interval containing a finite number of points partitioning it into subintervals. We color the points of the interval with two colors by the following rule: the leftmost point must be colored by 1, the rightmost point by 2, and all points in between by 1 or 2. Then there is at least one subinterval whose two endpoints are colored by two different numbers (we call such a subinterval a rainbow interval). In fact, there must exist an odd number of rainbow intervals.*

Proof: First we show existence. Consider all interior points of our partitioned closed interval, i.e., all points other than the leftmost and rightmost points. Either they are all colored by 1, or at least one is colored by 2. In the first case, the rightmost subinterval must be colored by 1 on the left and 2 on the right, and hence is a rainbow interval. In the second case, consider the leftmost point colored by 2; then all points left of that must be colored by 1, including the point immediately left of it. The subinterval bordered by these two points is a rainbow interval.

Now we show that there are an odd number of rainbow intervals. Counting along the whole interval from left to right, the first rainbow interval has its left endpoint colored by 1 and its right endpoint colored by 2. The second rainbow interval has its right endpoint colored by 2 and its right endpoint colored by 1. Each new rainbow interval switches the rightmost color as we count along the entire line. But the right endpoint of the last rainbow interval must be colored by 2, so we have an odd number of rainbow intervals. ■

To prove Sperner's Lemma in two dimensions, we provide a useful analogy. Imagine a house consisting of a finite number of rooms (figure 1). Each room may have either 0, 1, or 2 doors. A room with exactly 1 door is called a *dead end*, while a room with exactly 2 doors is called a *communicating room.* A door may either lead to another room (in which case we call it an *inside door*), or it may lead outside (an *outside door*); note that these two options are mutually exclusive. Furthermore, we do not allow any room to have two outside doors, and no two rooms may be connected by more than one door.

Proposition 2.2. *The number of dead ends and the number of outside doors have the same parity, i.e., either both are odd or both are even.*

Proof: We consider all possible walks through the rooms of our house. A walk is an undirected path (despite the arrows of figure 1) through the rooms and doors of a

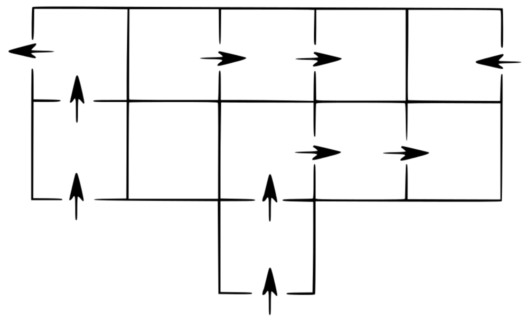

FIGURE 1. A house and all possible walks through it [3]. Despite the arrows, the walks are undirected.

house according to the following rules: each door can be passed through only once, and a walk must begin at an outside door or at a dead end. The walk continues through communicating rooms and terminates either at an outside door or at a dead end. Our conditions guarantee unique paths from every starting point; as a walk passes through a door, it either ends outside or at a dead end (which ends the walk), or it enters a communicating room, in which case there is only one door through which the walk may continue. With these conditions, three types of walks are possible:

(1) from an outside door to a dead end (or the other way around, as we disregard the direction of a walk);
(2) from an outside door to another outside door;
(3) from a dead end to a dead end.

Denote the number of these three possible paths by m, n, and p, respectively (without double-counting for direction). Counting the number of outside doors, there is 1 outside door for every path of type (1) and there are 2 for every path of type (2), so we have $m + 2n$ outside doors. Similarly, there is 1 dead end for every path of type (1) and there are 2 for every path of type (3), so we have $m + 2p$ dead ends. It follows that the number of doors and dead ends are of the same parity. ■

We can use this analogy to prove Sperner's Lemma in two dimensions. Consider an arbitrary triangle subdivided into smaller subtriangles under the following rule: any pair of subtriangles either share a common vertex, share a common edge, or share no points at all. We call this subdivision a *triangulation*, we call subtriangles *faces*, and we call the edges of the subtriangles *edges* of the triangulation (figure 2).

Lemma 2.3. *(Sperner's Lemma in Two Dimensions.) Consider a triangulation of a triangle T. We color the vertices of the triangulation with three colors by the following rules:*

(1) the three vertices of T are colored by 1, 2, and 3;
(2) if a vertex lies on an edge of T, it must be colored by one of the two numbers coloring the endpoints of the edge.

Then there is at least one face of the triangulation whose three vertices are colored by three different numbers (we call such a face a rainbow face). In fact, there must exist an odd number of rainbow faces (figure 3).

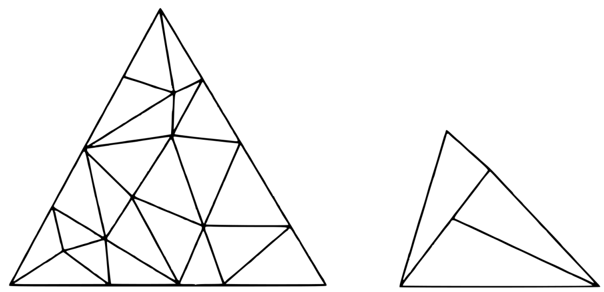

Figure 2. A proper (left) and improper (right) triangulation [3].

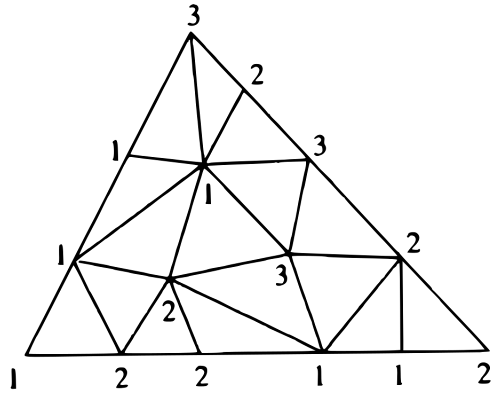

Figure 3. A proper coloring of a triangulation [3]. Note a rainbow face in the middle, and a total of five rainbow faces.

Proof: We will prove by reduction to Proposition (2.2). We may consider a triangle T as a house and its faces as rooms. We call an edge of the triangulation a door if its endpoints are colored by (1,2). A door is an outside door if it is on an edge of the triangle T itself (not its triangulation), and an inside door otherwise. Only a rainbow face (with vertices (1,2,3)) has exactly one edge colored by 1 and 2, so rainbow faces are analogous to dead ends. Similarly, faces with vertices (1,2,2) and (1,1,2) are communicating rooms. Any other faces are analogous to rooms without doors. We have thus reduced the triangulation of T to the house of Proposition (2.2).

By (2.2), the number of rainbow faces (dead ends) and the number of outside doors have the same parity. It remains to show that there is at least one outside door, and that the number of outside doors is odd. Note that outside doors can only exist on the edge of T colored by (1,2). But by Lemma (2.1), there is at least one edge of the triangulation colored by (1,2), and in fact an odd number of such edges. ■

Though it is possible to prove Brouwer's Theorem for a triangle from this result, we believe that a proof for a square is more easily visualized, so we now prove a version of Sperner's lemma for a square.

Lemma 2.4. *Consider a square S subdivided into smaller squares by lines parallel to its sides. We color the vertices of the subdivison with four colors by the following rules:*

(1) the four vertices of S are colored by 1, 2, 3, and 4;
(2) if a vertex lies on an edge of S, it must be colored by one of the two numbers coloring the endpoints of the edge.

Then there is at least one face colored by three different numbers (we call such a face a rainbow face).

Proof: We subdivide each square face by a diagonal line to make two triangles, which results in a triangulation of S. We call an edge with endpoints 1 and 2 a door. A door is an outside door if it is on an edge of the square S itself (not its subdivision), and an inside door otherwise. We call a triangular face a dead end if exactly one of its edges is a door; the only such faces have vertices (1,2,3) and (1,2,4); note that these are rainbow faces (but not the only types of rainbow faces). By the rules of the coloring, all outside doors must lie on the edge of S colored by 1 and 2, and by Lemma (2.1), there must be at least one outside door and in fact an odd number of them (figure 4).

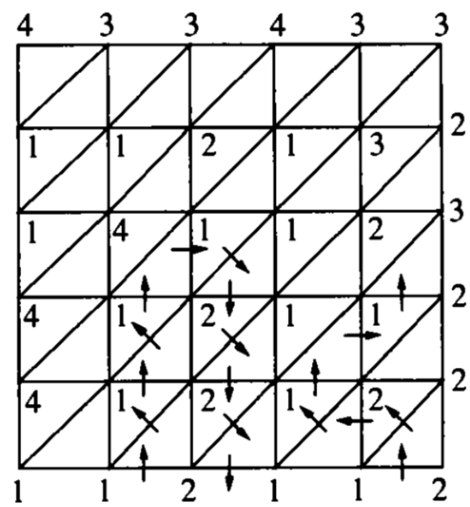

Figure 4. Walks through a triangulated square [3].

Any walk beginning from an outside door must either end at an outside door or end at a dead end. Since there is an odd number of doors, at least one walk must end at a dead end, so there is at least one dead end, and all dead ends are rainbow faces. ■

3. Brouwer's Theorem for a Square

We are now ready to prove the two-dimensional case of Brouwer's Theorem for a square.

Theorem 3.1. *(Brouwer's Theorem for a Closed Square.) Any continuous function f from a closed square to itself has at least one fixed point.*

PROOF: Note that throughout the proof, we think of f as shifting around the points of a closed square.

Consider a square S with corners (A_1, A_2, A_3, A_4) starting from the bottom left and going counterclockwise. We subdivide S with lines parallel to its edges (figure 5). If any one of the vertices of the subdivision is mapped to itself, then we are done. Otherwise, all vertices are displaced by the mapping. Now let p be a vertex of the subdivision, $q = f(p)$, and $\overrightarrow{pq}$ be the displacement vector from p to q. Finally, we assign a θ to each vertex, where θ is the angle formed by the positive horizontal axis (pointing from A_1 to A_2) and $\overrightarrow{pq}$. Then we color each vertex in the subdivision by the following rule:

Angle	**Color**
$\theta = 0$	1 or 4
$0 < \theta < \pi/2$	1
$\theta = \pi/2$	1 or 2
$\pi/2 < \theta < \pi$	2
$\theta = \pi$	2 or 3
$\pi < \theta < 3\pi/2$	3
$\theta = 3\pi/2$	3 or 4
$3\pi/2 < \theta < 2\pi$	4

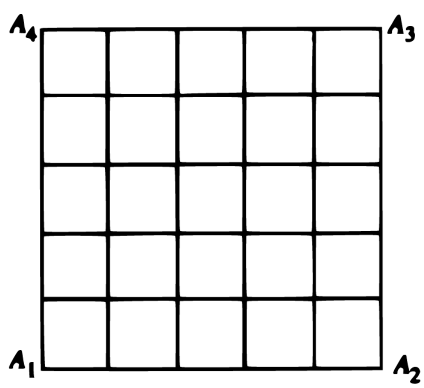

FIGURE 5. A subdivided square [3].

Note that by these rules, vertex A_1 may be colored by either 1, 2, or 4. To satisfy the conditions of Lemma (2.4), we choose the color 1. Similarly, we color A_2 with 2, A_3 with 3, and A_4 with 4. Furthermore, if p lies on the edge A_1A_2 of S but is not A_1 or A_2, then it may be colored by any of four colors depending on the direction of $\overrightarrow{pq}$, but a coloring of 1 or 2 will always be valid, so we color all vertices on A_1A_2 with 1 or 2 to satisfy the conditions of Lemma (2.4). We follow the same rule for the remaining three edges. Finally, vertices in the interior of S are colored freely by the rules above; the conditions of Lemma (2.4) are thus all satisfied. Therefore, for any subdivision of S, there exists at least one rainbow face.

Now consider the sequence of subdivisions $\{I_1, I_2, \dots, I_n, \dots\}$ where the n^{th} subdivision divides each edge of S into 2^n equal subintervals and straight lines are drawn through these points parallel to the sides of S to form 2^{2n} square faces. As $n \to \infty$, the side lengths of each face tend to zero.

If any one of the vertices in any subdivision I_n is mapped to itself by f, then we are done. Otherwise, suppose all vertices of every subdivision are displaced by f. By Lemma (2.4), there exists at least one rainbow face in each subdivision; call the rainbow face of the n^{th} subdivision R_n, and suppose it has vertices (w_n, x_n, y_n, z_n). Because a closed square is compact, there exists a subsequence $\{w_{n_k}\}$ of $\{w_1, w_2, \dots, w_n, \dots\}$ that converges to a point in S. Furthermore, since $\{w_{n_k}\}$, $\{x_{n_k}\}$, $\{y_{n_k}\}$, and $\{z_{n_k}\}$ become arbitrarily close as $k \to \infty$, the subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$, and $\{z_{n_k}\}$ also converge to the same point, call it p_0.

We claim that p_0 is a fixed point. Suppose not. Then p_0 and $q_0 = f(p_0)$ are two distinct points. By the table above, there are eight possible cases.

Case 1 ($\theta = 0$): Suppose that q_0 lies directly to the right of p_0, so $\theta = 0$ for p_0. Draw a vertical line L separating p_0 and q_0. Since f is continuous, we can choose an ϵ-neighborhood around q_0 and a δ-neighborhood around p_0 such that all points within the δ-neighborhood map to points in the ϵ-neighborhood; choose small enough neighborhoods so that they do not intersect with L (let the radii of the δ- and ϵ-neighborhoods be less than than the perpendicular distance between p_0 or q_0, respectively, and L). Then the neighborhoods share no common points. Furthermore, all points in the δ-neighborhood must be shifted to the right to the ϵ-neighborhood; that is, for all points p in the δ-neighborhood, the corresponding θ must satisfy $0 < \theta < \pi/2$, $\theta = 0$, or $3\pi/2 < \theta < 2\pi$. Then p must be colored by 1 or 4.

Note that for sufficiently large k, the vertices $\{w_{n_k}\}$, $\{x_{n_k}\}$, $\{y_{n_k}\}$, and $\{z_{n_k}\}$ lie within the δ-neighborhood of p_0, so all must be colored by 1 or 4. But this contradicts the fact that R_{n_k} is a rainbow face.

Case 2 ($0 < \theta < \pi/2$): Suppose that q_0 lies above and to the right of p_0, so that $0 < \theta < \pi/2$ for p_0. Draw a horizontal line L_1 and a vertical line L_2 such that p_0 and q_0 lie in the third and first quadrants, respectively, formed by L_1 and L_2. Since f is continuous, we can choose an ϵ-neighborhood around q_0 and a δ-neighborhood around p_0 such that all points within the δ-neighborhood map to points in the ϵ-neighborhood; choose small enough neighborhoods so that they do not intersect with L_1 or L_2. (To see that this is possible, consider the perpendicular distances between the points p_0 and q_0 and the lines L_1 and L_2. Let the radii of the δ- and ϵ-neighborhoods, respectively, be less than the minimum of these two distances.) Then the neighborhoods share no common points. Furthermore, all points in the δ-neighborhood must be shifted to the right and up to the ϵ-neighborhood; that is, for all points p in the δ-neighborhood, the corresponding θ must satisfy $0 < \theta < \pi/2$ (figure 6). Then p must be colored by 1.

Note that for sufficiently large k, the vertices $\{w_{n_k}\}$, $\{x_{n_k}\}$, $\{y_{n_k}\}$, and $\{z_{n_k}\}$ lie within the δ-neighborhood of p_0, so all must be colored by 1. But this contradicts the fact that R_{n_k} is a rainbow face.

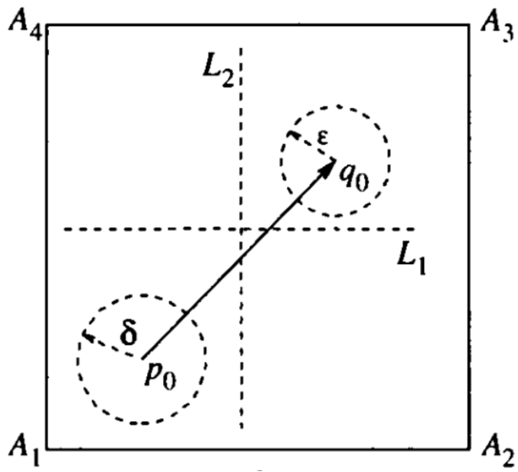

FIGURE 6. Case 2 (case 1 is similar) [3].

Cases 3-8: Cases 3, 5, and 7 are analogous to case 1, and cases 4, 6, 8 are analogous to case 2. Therefore, $p_0 = q_0 = f(p_0)$, which is our fixed point. ■

Remarkably, we have been able to prove Brouwer's Theorem in two dimensions from a lemma about walking through rooms in a house! In order to prove Brouwer's Theorem in the n-dimensional case, we must prove Sperner's lemma in n dimensions. To do so, we require the notion of simplices.

4. Simplices and Sperner's Lemma in Higher Dimensions

Sperner's Lemma in one dimension was concerned with a line subdivided into intervals (smaller lines), while Sperner's Lemma in two dimensions was concerned with a triangle subdivided into faces (smaller triangles). Naturally, the three-dimensional analogue would be concerned with a tetrahedron subdivided into smaller tetrahedra. In general, the n-dimensional notion of a triangle or tetrahedron is called a simplex.

Definition 4.1. Consider $n+1$ points $v_1, \ldots, v_{n+1} \in \mathbb{R}^N$, where $N \geq n$, which are affinely independent, that is, the vectors $v_2 - v_1, v_3 - v_1, \ldots, v_{n+1} - v_1$ are linearly independent. Then the ***n*-dimensional simplex** with vertices $v_1, \ldots, v_{n+1}$ is the set of points

$$\Delta^n = \left\{ \sum_{i=1}^{n+1} \alpha_i v_i \middle| \alpha_i \geq 0, \sum_{i=1}^{n+1} \alpha_i = 1 \right\}.$$

Remark 4.2. A zero-dimensional simplex is a point, a one-dimensional simplex is a closed interval, a two-dimensional simplex is a closed triangle, and a three-dimensional simplex is a closed tetrahedron.

Remark 4.3. Note that if the points $v_1, v_2, \ldots, v_{n+1}$ form an n-dimensional simplex, any subset of those points of size $k \leq n$ form a $(k-1)$-dimensional simplex. That is, if we consider the subset of points $v_{m_1}, \ldots, v_{m_k}$, they form the $(k-1)$-dimensional simplex with vertices $v_{m_1}, \ldots, v_{m_k}$.

Definition 4.4. A ***k*-face** of Δ^n is a $(k-1)$-dimensional simplex formed by k of the $n+1$ vertices of Δ^n.

Example 4.5. A 2-face of T is simply an edge of the triangle T, while a 1-face is one of the vertices of T. A 3-face of a tetrahedron is one of its four triangular faces.

Definition 4.6. A **simplicial subdivision** of Δ^n is a set of points in Δ^n subdividing it into smaller n-dimensional simplices called **cells** such that all cells cover Δ^n and any two cells either share a full face of dimension less than n or no points at all.

Remark 4.7. In the case of a triangle T, a simplicial subdivision is analogous to a triangulation of T. A cell of T is a subtriangle of T.

Definition 4.8. A **proper coloring** of a simplicial subdivision of the simplex Δ^n is an assignment of $n+1$ colors to the points of the subdivision by the following rules:

(1) the $n+1$ vertices of Δ^n are colored by different colors;
(2) if a point of the subdivision lies on a k-face of Δ^n, it must be colored by one of the k numbers coloring the vertices of that face.

We are now ready to prove the general version of Sperner's Lemma.

Theorem 4.9. *(Sperner's Lemma.) Any proper coloring of a simplicial subdivision of a simplex Δ^n contains a rainbow cell, i.e., a cell colored by $n+1$ different colors. In fact, the number of rainbow cells is odd.*

Proof: We prove by induction on n. The zero-dimensional case is trivial (any subdivision of a single point must be that point itself, and it must be colored by one color), and the one- and two-dimensional cases have been proven in Lemmas (2.1) and (2.3). Now suppose the theorem holds for any $(n-1)$-dimensional simplex, so that it contains an odd number of rainbow cells. Then consider the cells of a subdivision of Δ^n. Let R denote the number of rainbow cells (cells which are colored by $n+1$ colors). Let C denote the number of cells colored by exactly n colors $1, 2, \ldots, n$, so that one of these colors is used twice and the rest exactly once. Let D be the number of n-faces of cells of the subdivision colored by all the colors $1, 2, \ldots, n$. Let D_O be the number of such faces that are subsets of the n-faces of Δ^n itself (not its cells), and D_I be the rest of such faces. Intuitively, faces of type D_O are on the "outside" of Δ^n, while faces of type D_I are on the "inside" of Δ^n. Note that R and C are counting n-dimensional simplices, while D_O and D_I are counting $(n-1)$-dimensional simplices.

We wish to count the number of n-faces colored by $1, 2, \ldots, n$ (that is, faces of type D). Note that each rainbow cell has exactly one face colored by $1, 2, \ldots, n$. Each cell of type C has exactly two faces colored by $1, 2, \ldots, n$. Then we count a total of $R + 2C$ n-faces colored by $1, 2, \ldots, n$.

However, note that every face of type D_O (which is an $(n-1)$-dimensional simplex) is an n-face of exactly one cell in the subdivision of Δ^n, while every face of type D_I is an n-face shared between exactly two cells in the subdivision. Therefore, $R+2C$ actually counts the number of D_I faces twice, so we get $R+2C = D_O+2D_I$.

Now consider the faces of the cells of Δ^n. The only n-faces of the cells of type D_O are on the n-face of Δ^n colored by $1, 2, \ldots, n$. By the inductive hypothesis, this face has an odd number of rainbow faces. Then D_O is odd, so R is also odd. ■

Remark 4.10. Note that cells of type C correspond to communicating rooms in the house analogy of Proposition (2.2), faces of type D_O correspond to outside doors, and faces of type D_I correspond to inside doors.

5. Brouwer's Theorem in Higher Dimensions

Before we prove Brouwer's Theorem for a simplex, we state the following lemma.

Lemma 5.1. *Consider the simplex $\Delta^n \in \mathbb{R}^{n+1}$ with the vertices* $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$.
(a) For all $x = (x_1, \ldots, x_{n+1}) \in \Delta^n$, $\sum x_k = 1$.
(b) All points of Δ^n have nonnegative coordinates.

Proof: The claims follow immediately from Definition (4.1). ■

Theorem 5.2. *(Brouwer's Theorem for a Simplex.) Let $\Delta^n \in \mathbb{R}^{n+1}$ be the simplex from Lemma (5.1). Then any continuous function $f : \Delta^n \to \Delta^n$ has a fixed point.*

Proof: Consider a sequence of simplicial subdivisions $\{I_1, I_2, \ldots, I_m, \ldots\}$ such that each subsequent subdivision subdivides every cell of the previous subdivision, so that $I_m \subset I_{m+1}$. Then the distances between the vertices of each cell I_m tend to zero as $m \to \infty$.

Suppose there are no fixed points, so that for all $x \in \Delta^n$, $f(x) \neq x$. Let x_i denote the i^{th} coordinate of a point $x \in \Delta^n$. We claim that for all $x \in \Delta^n$, there exists an i^* such that $f(x)_{i^*} < x_{i^*}$. To see that this is true, first note that since f maps Δ^n to itself, $f(x) \in \Delta^n$ for all $x \in \Delta^n$. Then by part (a) of Lemma (5.1), $\sum x_i = \sum f(x)_i = 1$. Since $f(x) \neq x$, there must be at least two $i_1, i_2 \in \{1, 2, \ldots, n+1\}$ such that $f(x)_{i_1} < x_{i_1}$ and $f(x)_{i_2} > x_{i_2}$. For any point of any subdivision of Δ^n, we color it by some such i^* (there may be more than one).

We claim that such a coloring is a proper coloring of a simplicial subdivision. By part (b) of Lemma (5.1), all points of Δ^n have nonnegative coordinates. Then for the l^{th} vertex, the only i for which $f(x)_i < x_i$ is $i = l$, so we color the l^{th} vertex by l. Therefore, each of the $n+1$ vertices of Δ^n are colored by a different color, satisfying the first criterion of a proper coloring.

Now suppose that a vertex of the subdivision lies on a k-face of Δ^n formed by all but $n+1-k$ of the vertices of Δ^n; call these unused vertices $v_{\alpha_1}, \ldots, v_{\alpha_{n+1-k}}$ Then the $\alpha_l{}^{\text{th}}$ coordinate of any point on the n-face must be 0, so it is possible that $f(x)_i < x_i$, as long as $i \neq \alpha_1, \ldots, \alpha_{n+1-k}$. Therefore, any point of a subdivision on a k-face of Δ^n must be colored by one of the k numbers coloring the vertices of that face, satisfying the second criterion of a proper coloring.

Sperner's Lemma is satisfied, so there must exist a rainbow cell R_m in every subdivision I_m. Let the j^{th} vertex of R_m be denoted by $x^{(m,j)}$. Since Δ^n is compact, the sequence $\{x^{(m,1)}\}$ has a convergent subsequence; for cleanliness of notation, let us assume that $\{x^{(m,1)}\}$ itself converges to some $x^* \in \Delta^n$. Note that as $m \to \infty$, the vertices of each cell (and in particular, each rainbow cell) grow arbitrarily close to another, so for all j, the sequence $\{x^{(m,j)}\}$ converges to the same point x^*. That is, for all j, $\lim_{m\to\infty} x^{(m,j)} = x^*$.

By our coloring rule, for any j, there is a color i_j^* such that $f(x^{(m,j)})_{i_j^*} < x_{i_j^*}^{(m,j)}$. Since $x^{(m,j)}$ is a the set of all vertices of a rainbow cell for any m, i_j^* takes on all the values $1, 2, \ldots, n+1$. Furthermore, since f is continuous,

$$f(x^{(m,j)})_{i_j^*} < x_{i_j^*}^{(m,j)}$$
$$\lim_{m\to\infty} f(x^{(m,j)})_{i_j^*} \leq \lim_{m\to\infty} x_{i_j^*}^{(m,j)}$$
$$f\left(\lim_{m\to\infty} x^{(m,j)}\right)_{i_j^*} \leq \lim_{m\to\infty} x_{i_j^*}^{(m,j)}$$
$$f(x^*)_{i_j^*} \leq x^*_{i_j^*}$$

So we have $f(x^*)_{i_j^*} \leq x^*_{i_j^*}$ for all $i_j^* = 1, 2, \ldots, n+1$. But equality is necessarily true, because $\sum x_{i_j^*} = \sum f(x)_{i_j^*} = 1$. We thus have a contradiction, so f must have a fixed point. ■

Note that we have only proved Brouwer's theorem for a particular Δ^n. To extend the argument to *any* Δ^n and, in fact, to a large class of subsets of Euclidean space, we use the topological notion of homeomorphism.

Definition 5.3. Let X and Y be two nonempty subsets of $\mathbb{R}^n$. Suppose there exists a continuous bijective function $f : X \to Y$ such that the inverse f^{-1} is also continuous. Then f is called a **homeomorphism**, and X and Y are said to be **homeomorphic**.

Intuitively, we may consider X and Y as two geometric objects made of clay in $\mathbb{R}^n$. Then if the clay can be continuously stretched and formed from shape X to shape Y without tearing or puncturing the clay, or sealing any "holes," then X and Y are homeomorphic. To illustrate, the common mathematical joke that a topologist cannot tell the difference between a doughnut and a coffee cup is because the two are homeomorphic subsets of $\mathbb{R}^3$. By creating a dimple anywhere in a doughnut-shaped piece of clay and making it deeper, one can form a coffee cup without tearing or puncturing the clay.

It can be shown that the simplex described in Theorem (5.1) is homeomorphic to any Δ^n. We now present the following lemma without proof:

Lemma 5.4. *All simplices Δ^n are homeomorphic to another. Furthermore, any simplex Δ^n is homeomorphic to the closed n-ball B^n, as well as any n-cell. In fact, all convex compact subsets of $\mathbb{R}^n$ are homeomorphic to another.*

We say that a subset X of $\mathbb{R}^n$ has the fixed point property if any continuous function $f : X \to X$ has a fixed point. To complete the proof of Brouwer's Theorem, it remains to show that homeomorphism preserves the fixed point property.

Lemma 5.5. *If $Y \subset \mathbb{R}^n$ is homeomorphic to $X \subset \mathbb{R}^n$ and X has the fixed point property, then so does Y.*

Proof: Consider the homeomorphism $g : X \to Y$, and let $f : Y \to Y$ be continuous. Now consider $h : X \to X$, where $h = g^{-1}fg$. Then since h is the composition of continuous functions, it is also continuous, so h has a fixed point x_0 in X. Now let $y_0 = g(x_0)$. We claim that y_0 is a fixed point under f in Y. To see this,

$$f^{-1}(y_0) = f^{-1}g(x_0) = f^{-1}gh(x_0) = g(x_0) = y_0,$$

so of course, $f(y_0) = y_0$. Since f was an arbitrary continuous mapping to begin with, we have shown that any function $f : Y \to Y$ has a fixed point, so Y has the fixed point property. ■

We finally have the tools needed to prove Brouwer's Theorem in its most general form.

Theorem 5.6. *(Brouwer's Theorem.) Let X be any convex compact subset of $\mathbb{R}^n$. Then $f : X \to X$ has a fixed point.*

Proof: This follows immediately from Theorem (5.2), Lemma (5.4), and Lemma (5.5). Any convex compact subset of $\mathbb{R}^n$ is homeomorphic to the simplex Δ^n of Lemma (5.1), which has the fixed point property. It follows that any convex compact subset $X \in \mathbb{R}^n$ has the fixed point property. ■

We conclude with an interesting real-world illustration.

Example 5.7. Consider a martini glass filled with some liquid. The space occupied by the liquid X may be considered a convex compact subset of $\mathbb{R}^3$. If we stir the liquid gently enough, then we may regard the stirring as a continuous function $f : X \to X$ that displaces the molecules of liquid throughout the glass, so long as none of the liquid splashes out of the glass. Then by Brouwer's Theorem, once the liquid has come to rest, at least one point of the liquid will end up in the same spot where it began.

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