RAMSEY THEORY

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Abstract. We give a proof to arithmetic Ramsey's Theorem. In addition, we show the proofs for Schur's Theorem, the Hales-Jewett Theorem, Van der Waerden's Theorem and Rado's Theorem, which are all extensions of the classical Ramsey's Theorem.

Contents

1. Introduction

Ramsey Theory is named after Frank Plumpton Ramsey(22 February 1903 - 19 January 1930), who was a British philosopher, mathematician and economist. This branch of mathematics concerns with "the preservation of properties under set partition", and is applied in areas such as number theory, algebra, topology and set theory. Some key results in Ramsey Theory are Van der Waerden's Theorem, the Hales-Jewett Theorem and Rado's Theorem, all of which state the existence of certain structure when a system is arbitrarily yet finitely partitioned.

2. Arithmetic Ramsey's Theorem

While it is possible to visualize and prove Ramsey's Theorem on graphs, in this paper we will mainly work on arithmetic Ramsey's Theorem-that is to say, we will interpret colorings as functions on natural numbers. We will start with infinite Ramsey's Theorem, which says that whenever $\mathbb{N}^{(r)}$ is finitely colored, there is an infinite monochromatic set. From there we can deduce finite Ramsey's Theorem, which claims the existence of a finite, monochromatic set in arbitrarily large size when $\mathbb{N}^{(r)}$ is finitely colored. Although graph theory's interpretation of Ramsey's Theorem may seem more direct and easier to comprehend, the arithmetic approach also bears its advantage, as we will see in the next section of this paper.

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We let $\mathbb{N}$ denote the set of all natural numbers. For $n \in \mathbb{N}$, $[n]$ denotes the set of all natural numbers less or equal to n. For $X \subset \mathbb{N}, r \in \mathbb{N}$, $X^{(r)}$ denotes the collection of all subsets of X of size r.

Definition 2.1. A *k-coloring* of $\mathbb{N}^{(r)}$ is a function from $\mathbb{N}^{(r)}$ to $[k]$. For $n \in \mathbb{N}$, a *k-coloring* of $[n]^{(r)}$ is a function $c : [n]^{(r)} \to [k]$.

Definition 2.2. For $m \in \mathbb{N}$, an *m-set* is a set with m elements.

Definition 2.3. When $\mathbb{N}^{(r)}$ is finitely colored, $M \subset \mathbb{N}$ is a *monochromatic set* if all elements in $M^{(r)}$ have the same color.

Example 2.4. When $\mathbb{N}^{(3)}$ is k-colored, we say that $M = \{1, 2, 3, 4\}$ is a 4-set monochromatic set if $c(\{1, 2, 3\}) = c(\{1, 2, 4\}) = c(\{1, 3, 4\}) = c(\{2, 3, 4\})$.

With the definitions given above, we start proving *Infinite Ramsey's Theorem* in r ($r \in \mathbb{N}$) dimensions. Notice that when $r = 1$, the *Infinite Ramsey's Theorem* is nothing but a paraphrase of our familiar *Pigeonhole Principle*. Once the base case is proven valid, we will induce on r to show that the statement holds true for all finite dimensions.

Theorem 2.5. *Let $r, k \in \mathbb{N}$ be given. Whenever $\mathbb{N}^{(r)}$ is k-colored, there exists an infinite monochromatic set.*

Proof. We will prove the statement by an induction on the dimension r. When $r = 1$, the pigeonhole principle implies the existence of an infinite monochromatic set. Therefore the base case holds true.

For the inductive hypothesis, suppose the statement is true for $r-1$ dimensions, and we will show that an infinite monochromatic set always exists in r dimensions. Let $c : \mathbb{N}^{(r)} \to [k]$ be a k-coloring on $\mathbb{N}^{(r)}$. Pick an arbitrary $a_1 \in \mathbb{N}$ and define a new k-coloring c' on $(\mathbb{N} \setminus \{a_1\})^{(r-1)}$ by giving all its elements the same color they have in the original coloring, when a_1 is added. That is to say, for every $E \in (\mathbb{N} \setminus \{a_1\})^{(r-1)}$, $c'(E) := c(E \cup \{a_1\})$. Let B_1 denote the infinite monochromatic set in $\mathbb{N} \setminus \{a_1\}$, suggested by the inductive hypothesis. And let c_1 denote its color.

Now let $a_2 \in B_1$ be given. Repeat the process above and we can get another infinite monochromatic set inside $B_1 \setminus \{a_2\}$, denoted by B_2, because $B_1 \setminus \{a_2\}$ is still infinite. Again, let c_2 denote the color of B_2.

The process can continue recursively and there will be a sequence of infinite monochromatic sets where $\mathbb{N} \supset B_1 \supset B_2 \supset \cdots$. Corresponding this sequence is a sequence of colors denoted by $c_1, c_2, \cdots$ and a sequence of selected elements $a_1, a_2, \cdots$.

Applying the pigeonhole principle, we can find a monochromatic subsequence of $c_1, c_2, \cdots$, where $c_{i(1)} = c_{i(2)} = \cdots = c_0$. Let $A = \{a_{i(n)} \mid n \in \mathbb{N}\}$ be the set of selected elements in corresponding positions. We want to show that A is the infinite monochromatic set in r dimensions.

Let $E \in A^{(r)}$ be given and let $i(m)$ denote the smallest index of elements in E. It follows that all elements of E, other than $a_{i(m)}$ are nested in $B_{i(m)}$, which implies $c(E) = c_{i(m)} = c_0$ and that A is monochromatic. The statement holds true for this specific r, and consequently for all finite dimensions. □

Definition 2.6. For $m, n, r \in \mathbb{N}$ with $m > n$, let $c_m : [m]^{(r)} \to [k]$ be a k-coloring of $[m]^{(r)}$ and $c_n : [n]^{(r)} \to [k]$ be a k-coloring of $[n]^{(r)}$. We say c_m *agrees on* c_n if for all $E \in [n]^{(r)}$, $c_m(E) = c_n(E)$.

It follows immediately from the *Infinite Ramsey's Theorem* that whenever $\mathbb{N}^{(r)}$ is finitely colored, there exists a finite monochromatic set of arbitrarily large size. However, it is always a concern whether there are finite sets satisfying this condition, and if there are, what their minimum size will be. Although the minimum size is yet to be determined, an interpretation of Ramsey problems on hypergraphs can show that such finite sets do exist. Here, alternatively, we will prove their existence in terms of arithmetic Ramsey's Theorem. To be more precise, we will show that if the following *Finite Ramsey's Theorem* is not true, there will be a coloring on $\mathbb{N}^{(r)}$ that generates no infinite monochromatic set, which contradicts the *Infinite Ramsey's Theorem.*

Theorem 2.7. *Let $m, r, k \in \mathbb{N}$ be given. There exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k-colored, there is always a monochromatic m-set.*

Proof. Suppose the statement is false for contradiction. Then for all $n \in \mathbb{N}$, there exists a k-coloring that does not have a monochromatic m-set. This particular coloring is denoted by $c_n : [n]^{(r)} \to [k]$.

For $[r]^{(r)}$, there are only k ways to color the set, which is finite. Therefore by the pigeonhole principle, infinitely many colorings of $c_{r+1}, c_{r+2}, \cdots$ agree on c_r. Similarly, for all $n \in \mathbb{N}$ with $n \geq r$, since the number of different colorings of $[n]^{(r)}$ is finite, infinitely many colorings of $c_{n+1}, c_{n+2}, \cdots$ agree on c_n. In this way we can form a subsequence $c_{i(1)}, c_{i(2)}, \cdots$, where for all $p, q \in \mathbb{N}$ with $q \leq p$, $c_{i(p)}$ agrees on $c_{i(q)}$.

Now define a coloring $d : \mathbb{N}^{(r)} \to [k]$ as follows: let $E \in \mathbb{N}^{(r)}$ be given and let n denotes the largest element of E. Then find $p \in \mathbb{N}$ where $i(p) \geq n$ and give E the same color as $c_{i(p)}$ does. As long as $i(p)$ is greater than n, it does not matter what value of p we take because according to the construction of the subsequence, the colorings will ultimately agree on the one corresponding to the smallest index. This implies that the coloring function is well-defined.

When $\mathbb{N}^{(r)}$ is colored by d, there exists no monochromatic m-set. Let an arbitrary m-set $M \subset \mathbb{N}$ be given. Find $p \in \mathbb{N}$ such that $i(p)$ is greater than the largest element of M. Let $E \in M^{(r)}$ be given and it is defined that $d(E) = c_{i(p)}(E)$. It follows that M being monochromatic under coloring d is equivalent to M being monochromatic under coloring $c_{i(p)}$, which contradicts to the initial definition of $c_{i(p)}$.

However, since $\mathbb{N}^{(r)}$ does not have a monochromatic m-set under coloring d, it cannot contain an infinite monochromatic set under the same coloring. This situation contradicts to the Infinite Ramsey's Theorem. Therefore for any $m, k, r \in \mathbb{N}$, there must exist $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is k-colored, there is a monochromatic m-set. □

3. An Extension of Ramsey's Theorem

Compared to graph theory's interpretation of *Ramsey's theorem*, arithmetic *Ramsey's Theorem* allows a further exploration of cases with special set structures. For example, *Schur's Theorem* claims that whenever $\mathbb{N}$ is finitely colored, there always exists a monochromatic 3-set where one element is the sum of the other two; *the Hales-Jewett Theorem* shows the existence of a monochromatic set with all its elements on a *combinatorial line*; *Van der Waerden's Theorem*, as a corollary of *the Hales-Jewett Theorem*, states that a monochromatic arithmetic progression always exists; and, finally, *Rado's Theorem* deals with monochromatic root for a certain matrix. To sum up, understanding

Ramsey's Theorem in terms of functions allows its applications in areas such as number theory, and linear algebra.

3.1. **Schur's Theorem.**

Theorem 3.1. *Whenever* $\mathbb{N}$ *is* k*-colored, there exists a monochromatic set* $\{x, y, z\}$ *such that* $x + y = z$*.*

Proof. Let $k \in \mathbb{N}$ be given. Let $c : \mathbb{N} \to [k]$ be a k-coloring on $\mathbb{N}$. Define $c' : \mathbb{N}^{(2)} \to [k]$ by letting $c'(\{p, q\}) := c(|q - p|)$, where p, q are two distinct natural numbers. Since $p - q \in \mathbb{N}$, the function is well-defined. By Theorem 2.7, there exists a monochromatic 3-set denoted by $\{p, q, r\}$. Suppose without losing generality that $p < q < r$. Then $c'(\{p, q\}) = c'(\{q, r\}) = c'(\{p, r\})$ implies $c(q - p) = c(r - q) = c(r - p)$. Let $x := q - p$, $y := r - q$ and $z := r - p$. It follows immediately that $x + y = z$ and that x, y, z have the same color . □

3.2. **The Hales-Jewett Theorem.**

Definition 3.2. For a finite $X \subset \mathbb{N}$, an *n-dimensional cube on* X, denoted by X^n, is the n-fold cartesian product of X. In other words, $X^n = \{(x_1, x_2, \cdots, x_n) \mid x_1, x_2, \cdots, x_n \in X\}$.

Definition 3.3. Let X be a finite subset of $\mathbb{N}$, n be an element of $\mathbb{N}$, and I be a non-empty subset of $[n]$. For each $i \in [n] \setminus I$, fix a number $a_i \in X$. A *line (combinatorial line)* is a set $L \subset X^n$ defined by:

$$L := \{(x_1, x_2, \cdots, x_n) \in X^n \mid x_i = a_i, for\ i \notin I; x_i = x_j, \forall\ i, j \in I\}.$$

Remark 3.4. I is also called the set of *active coordinates* because while the coordinates whose indexes are not elements of I are fixed by the chosen a_i, the coordinates with indexes in I are open to change.

Example 3.5. Let $X = \{1, 3, 5, 6\}, n = 3, I = \{1, 3\}, a_2 = 3$. The line thereby defined is:

$$L = \{(1, 3, 1), (3, 3, 3), (5, 3, 5), (6, 3, 6)\}.$$

Definition 3.6. Let L be a line in X^n. L is *monochromatic* if $\forall\ (x_1, x_2, \cdots, x_n) \in L$, the points have the same color.

Definition 3.7. The *Hales-Jewett number*, denoted by $HJ(m, k)$, is the smallest positive integer satisfying: for all $n \in \mathbb{N}, n \geq HJ(m, k)$, whenever $[m]^n$ is k-colored, there exists a monochromatic line.

We want to show that the Hales-Jewett number is always finite under all choices of $m, k \in \mathbb{N}$. In this paper, we will give a $color - focused$ proof for the statement. We will fix k, the number of different colorings, and do an induction on m, the "length" of the line.

Definition 3.8. Let L be a line in X^n with I being the set of active coordinates. The *first point* of L, denoted by L^-, is the point $(x_1, x_2, \cdots, x_n)$ such that $x_i = min(X)$, for $i \in I$. Similarly, let the *last point* of L, denoted by L^+, be the point $(x_1, x_2, \cdots, x_n)$ where $x_i = max(X)$, for $i \in I$.

Definition 3.9. Let $L_1, L_2, \cdots, L_k$ be different lines in X^n. $L_1, L_2, \cdots, L_k$ are *focused* at f if $L_i^+ = f$, for all $i \in [k]$.

Definition 3.10. Let $L_1, L_2, \cdots, L_k$ be different lines in X^n. $L_1, L_2, \cdots, L_k$ are *color-focused* at f if:

- $L_1, L_2, \cdots, L_k$ are focused at f;
- $L_1 \setminus \{L_1^+\}, L_2 \setminus \{L_2^+\}, \cdots, L_k \setminus \{L_k^+\}$ are monochromatic, and
- $L_1, L_2, \cdots, L_k$ have different color, respectively.

Lemma 3.11. *Let $m \in \mathbb{N}$ and suppose $HJ(m-1, k)$ is finite for all choices of $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. There exists $n \in \mathbb{N}$ such that whenever $[m]^n$ is k-colored, there is either*

- *a monochromatic line, or*
- *k color-focused lines*

Proof. We will prove the claim by induction. Let $P(r)(r \leq k)$ be the statement that there exists $n \in \mathbb{N}$ such that whenever $[m]^n$ is k-colored, there exists either

- a monochromatic line, or
- r color-focused lines

For the base case $r = 1$, let $n = HJ(m-1, k)$, which is finite by inductive hypothesis. A monochromatic line must lie in $[m-1]^n$, which is contained by $[m]^n$. Hence the monochromatic line is also a color-focused line in $[m]^n$. $P(1)$ holds true.

Suppose $P(r-1)$ is true, and let n_0 be the number suitable for $r-1$. Let $n = HJ(m-1, k^{m^{n_0}})$ and we want to show that $n_0 + n$ is suitable for r. Let $c : [m]^{n_0+n} \to [k]$ be an arbitrary k-coloring. If there is a monochromatic line in $[m]^{n_0+n}$, then we are done with the statement. Therefore we suppose there is no monochromatic line in $[m]^{n_0+n}$.

Note that $[m]^{n_0+n} = [m]^{n_0} \cdot [m]^n$. We can see each $[m]^{n_0}$ as a "block" and the entire space $[m]^{n_0+n}$ as an n-dimensional space where each block is a unit. There are k ways to color each point in a "block", so, in total, there are $k^{m^{n_0}}$ different ways to color a "block". By the definition of n, there exists a monochromatic line of length $m-1$ within the n-dimensional space of "blocks". Denote the line by L, and the corresponding set of active coordinates by I. Every two "blocks" on this line have exactly the same coloring. That is to say, all points in corresponding positions in these "blocks" are colored the same. Let B_1 denote the "block" represented by the first point, L^-, of the monochromatic line.

By the inductive hypothesis, there exists either a monochromatic line or $r-1$ color-focused lines in B_1. If there is a monochromatic line in B_1, it is also a monochromatic line in $[m]^{n_0+n}$ (we can describe a line in B_1 as a line in $[m]^{n_0+n}$ by maintaining the same set of active coordinates and adding n fixed coordinates that describe the relative position of B_1 in the larger space). Yet we assumed that there should be no monochromatic line in $[m]^{n_0+n}$. Hence B_1 has no monochromatic line.

This implies that there are $r-1$ color-focused lines in B_1, denoted by $L_1, L_2, \cdots, L_{r-1}$, and corresponding to sets of active coordinates denoted by $I_1, I_2, \cdots, I_{r-1}$. Furthermore, these lines focus at f.

Let $I_0 = \{i + n_0 \mid i \in I\}$. For $j \in [r-1]$, define L'_j by letting $L_j'^- := L_j^-, I'_j = I_j \cup I_0$. In other words, we are taking the first point of L_j as the first point of L'_j; and the point in the second "block" corresponding to the second point of L_j as the second point of L'_j $\cdots$. Then the newly defined lines satisfy:

- Each L'_j is monochromatic because
  (1) each L_j is monochromatic;

(2) each "block" has exactly the same color.

- Two different L'_j have different colors because two different L_j have different colors;
- All L'_j are focused at (f, L^+).

Now define L'_r by letting $L'^-_r := f, I'_r := I_0$. Then

- L'_r is monochromatic because all its point are in corresponding positions in "blocks";
- L'_r has a different color from $L'_1, L'_2, \cdots L'_{r-1}$ because f has a different color from $L_1, L_2, \cdots, L_{r-1}$;
- L'_r also focuses on (f, L^+).

In conclusion, $L'_1, L'_2, \cdots, L'_r$ are r color-focused lines and $P(r)$ is true. By induction, $P(k)$ is also true. □

Theorem 3.12. *$HJ(m, k)$ is finite for all $m, k \in \mathbb{N}$.*

Proof. We will prove the statement by an induction on m. For the base case when $m = 1$, let $n \in \mathbb{N}$ be given. It follows immediately that $[1]^n$ is a single point and is always monochromatic. Hence the base case is true.

Suppose the statement also holds for $m - 1$. We then prove $HJ(m, k)$ is also finite for all choices of k.

Let $k \in \mathbb{N}$ be given. By Lemma 3.11, there exists $n \in \mathbb{N}$ such that whenever $[m]^n$ is k-colored, there exists either

- a monochromatic line, or
- k color-focused lines.

If it is the first case, we are done with the inductive statement. If it is the second case, by the pigeonhole principle,the focus point shares color with one of the color-focused lines. Therefore there is also a monochromatic line.

By induction, the statement is true for all choices of $m, k \in \mathbb{N}$. That is to say, $HJ(m, k)$ is finite for all $m, k \in \mathbb{N}$. □

Remark 3.13. The theorem can be understood as that when k people are playing tic-tac-toe with m coordinates, as long as the dimension is large enough, there cannot be a draw.

3.3. **Van der Waerden's Theorem.**

Definition 3.14. Let $a, m, d \in \mathbb{N}$ be given. Then $a, a + d, \cdots a + (m - 1)d$ is a *monochromatic arithmetic progression* if for all $i \in \{0, 1, \cdots, m - 1\}$, $a + id$ has the same color.

Definition 3.15. Let $m, k \in \mathbb{N}$ be given. The *Van der Waerden Number*, $W(m, k)$, is the least positive number satisfying: $\forall\ n \in \mathbb{N}, n \geq W(m, k)$, whenever $[n]$ is k-colored, there is a monochromatic arithmetic progression of length m.

Although *Van der Waerden's Theorem* can be proven independently, it is easier to derive it from *the Hales-Jewett Theorem*. In the next proof, we use $\sum L^j$ to denote the sum of coordinates of the j^{th} point on a line. That is to say, if $L^j = (x_1, x_2, \cdots, x_n)$, $\sum L^j := \sum_{i=1}^n x_i$.

Theorem 3.16. *Let $m, k \in \mathbb{N}$ be given. $W(m, k)$ is finite and $W(m, k) \leq m \cdot HJ(m, k)$.*

Proof. Let $m, k \in \mathbb{N}$ be given and let $n = HJ(m,k)$. Give $[mn]$ a k-coloring $c : [mn] \to [k]$ and define $c' : [m]^n \to [k]$ by letting $c'(x_1, x_2, \cdots, x_n) := c(x_1 + x_2 + \cdots x_n)$, for $x_1, x_2, \cdots, x_n \in [m]$. Since $x_1 + x_2 + \cdots x_n \leq mn$, the function is well-defined.

By the definition of n, there exists a monochromatic line in $[m]^n$. Let I be the set of active coordinates and $d := |I|$. Let $a := \sum L^-$. Note that $L^- = \{(x_1, x_2, \cdots, x_n) \mid x_i = a_i, for\ i \notin I; x_i = 1, for\ i \in I\}$, and the j^{th} point $L^j = \{(x_1, x_2, \cdots, x_n) \mid x_i = a_i, for\ i \notin I; x_i = j, for\ i \in I\}$. This means $a + (j-1)d = \sum L^j$. The line is monochromatic then implies $a, a+d, \cdots a+(m-1)d$ are colored the same, and is thus a monochromatic arithmetic progression of length m. This further implies $W(m,k) \leq m \cdot HJ(m,k)$. □

3.4. **Rado's Theorem.**

Definition 3.17. Let $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{N}^n$ be given and let $\mathbf{x}$ be finitely colored. $\mathbf{x}$ is *monochromatic* if all its coordinates have the same color.

Definition 3.18. Let A be an $m \times n$ matrix with rational entries. A is *partition regular (PR)* if whenever $\mathbb{N}$ is finitely colored, there is a monochromatic $\mathbf{x} \in \mathbb{N}^n$ such that $A\mathbf{x} = \mathbf{0}$.

Example 3.19. $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$ is partition regular.

Proof. Let $\mathbb{N}$ be k-colored. By *Schur's Theorem*, there exists monochromatic $a, b, c \in \mathbb{N}$ such that $a + b = c$. Now let $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Then $A\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a + b - c = 0.$ □

Example 3.20. For $\lambda \in \mathbb{N}$, $A = \begin{bmatrix} \lambda & -1 \end{bmatrix}$ is PR if and only if $\lambda = 1$.

Proof. If $\lambda = 1$, Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\mathbf{x}$ is monochromatic and $A\mathbf{x} = 0$. Suppose $\lambda \neq 1$. Define a 2-coloring $c : \mathbb{N} \to \{1, 2\}$ by letting

- $c(a) := 1$, if $\lambda \nmid a$;
- $c(a) := 2$, if $\lambda \mid a$ and $c(\frac{a}{\lambda}) = 1$;
- $c(a) := 1$, if $\lambda \mid a$ and $c(\frac{a}{\lambda}) = 2$.

Then let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ be given such that $A\mathbf{x} = 0$. Then it follows $\lambda a = b$. However, if $\lambda a = b$, by the definition of the coloring, $c(a) \neq c(b)$ and $\mathbf{x}$ cannot be monochromatic. Therefore A is not PR. □

Lemma 3.21. *A is PR if and only if λA is PR for all $\lambda \in \mathbb{Q} \setminus \{0\}$.*

Proof. To prove the forward direction, suppose A is PR. It follow that there exists a monochromatic $\mathbf{x} \in \mathbb{N}^n$ such that $A\mathbf{x} = \mathbf{0}$. Associative law implies that $(\lambda A)\mathbf{x} = \lambda(A\mathbf{x}) = \mathbf{0}$. Therefore λA is PR. For the backward direction, we only need to replace multiplication with division and show the statement is also true. □

Other than seeing an $m \times n$ matrix A in terms of single entries, we also view it as a set of columns. That is, $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$,

where $\mathbf{c}_i = \begin{bmatrix} a_{1,i} \\ a_{2,i} \\ \vdots \\ a_{m,i} \end{bmatrix}$

Definition 3.22. Let A be an $m \times n$ matrix with rational entries. A has the *columns property (CP)* if there is a partition $B_1 \cup B_2 \cup \cdots \cup B_r$ of $[n]$ such that

- $\sum_{i \in B_1} \mathbf{c}_i = \mathbf{0}$;
- $\sum_{i \in B_s} \mathbf{c}_i \in span(\mathbf{c}_j \mid j \in B_1 \cup \cdots \cup B_{s-1})$, for all $2 \leq s \leq r$.

Remark 3.23. In this definition, "span" means "span over $\mathbb{R}$".

Example 3.24. $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$ has CP.

Proof. Let $B_1 = \{1, 3\}; B_2 = \{2\}$. Then

- $\sum_{i \in B_1} \mathbf{c}_i = 1 + (-1) = 0$;
- $\sum_{i \in B_2} \mathbf{c}_i = 1 = 1 + 0 \cdot (-1)$.

Therefore A has CP. □

Example 3.25. Let $\lambda \in \mathbb{N}$. $A = \begin{bmatrix} \lambda & -1 \end{bmatrix}$ has CP if and only if $\lambda = 1$.

Proof. If $\lambda = 1$, let $B_1 = \{1, 2\}$. It can be easily checked that A has CP. If $\lambda \neq 1$, no matter how the partition is formed, it cannot satisfy the first condition for having a CP. □

From the examples given above, we may conjecture that there are certain relationships between being PR and having CP. *Rado's Theorem* gives an elegant description of the relationship by claiming that A is PR if and only if it has CP.

To prove *Rado's Theorem*, we will start with the special, and relatively easy, case when A is a $1 \times n$ matrix. We call this *Rado's Theorem for a Single Equation*. Instead of proving the relationship directly, we will find a "medium" statement and show that it is equivalent to both "A is PR" and "A has CP". The "medium" statement is as follows: Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ be a matrix with rational entries. There exists $I \subset [n]$ such that $\sum_{i \in I} a_i = 0$.

Lemma 3.26. *Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ be given with $a_1, a_2, \cdots, a_n \in \mathbb{Q}$. A has CP if and only if there exists $I \subset [n]$ such that $\sum_{i \in I} a_i = 0$.*

Proof. To prove the forward direction, suppose A has CP. It follows that there exists a partition $B_1 \cup \cdots \cup B_r$ of $[n]$ such that $\sum_{i \in B_1} a_i = 0$. Let $I = B_1$ and the statement is proved.

For the backward direction, let $I \subset [n]$ be the set with the property that

$$\sum_{i \in I} a_i = 0.$$

Let $B_1 = I$ and $B_2 = [n] \setminus I$. We have

- $\sum_{i \in B_1} a_i = 0$.
- $\sum_{i \in B_2} a_i \in span(a_i \mid i \in B_1)$ because all columns are just scalars.

Hence A has CP. □

To prove the other half of the statement (that is, the "medium" statement is equivalent to "A is PR"), we need to define a new coloring law relying on different base system.

Definition 3.27. Let $x_{(p)} = d_r \cdot p^r + d_{r-1} \cdot p^{r-1} + \cdots + d_1 \cdot p + d_0$ be a natural number in base p. $L(x)$ denotes the position of the last non-zero digit. That is, $L(x) := min\{i \mid d_i \neq 0\}$.

Definition 3.28. Let $x_{(p)} = d_r \cdot p^r + d_{r-1} \cdot p^{r-1} + \cdots + d_1 \cdot p + d_0$ be a natural number in base p. $d(x)$ denotes the last non-zero digit. That is, $d(x) := d_{L(x)}$.

Example 3.29. Let $x_{(7)} = 1026500$. Then $d(x) = 5$ and $L(x) = 2$.

In the following discussions, we assume without stating that no entry in the matrix is zero because if there exists $a_i = 0$, we can simply let $I := \{i\}$ and find the claim true.

Lemma 3.30. *Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ be a matrix with all rational entries. If A is PR, then there exists $I \subset [n]$ such that $\sum_{i \in I} a_i = 0$.*

Proof. By Lemma 3.21, we can multiply each entry with the least common multiple of the denominators and form a matrix with only integer entries without changing its PR property. Therefore in the following discussion, we take A as a matrix with integer entries exclusively.

Let p be an arbitrary prime number. Define a $(p-1)$ coloring $c : \mathbb{N} \to [p-1]$ by letting $c(x) := d(x), \forall\, x \in \mathbb{N}$. Since A is PR, $A\mathbf{x} = 0$ has a monochromatic solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$. $A\mathbf{x} = 0$ means

$$\sum_{i=1}^{n} a_i x_i = 0.$$

Let $L = min\{L(x_i) \mid i \in [n]\}$, and let $I = \{i \mid L(x_i) = L\}$. We now show that

$$\sum_{i \in I} a_i = 0.$$

Since

$$\sum_{i=1}^{n} a_i x_i = 0,$$

it follows

$$p^{L+1} \mid \sum_{i=1}^{n} a_i x_i.$$

That is to say,

$$p^{L+1} \mid \sum_{i=1}^{n} a_i(d_{r,i} \cdot p^r + d_{r-1,i} \cdot p^{r-1} + \cdots + d_{L+1,i} \cdot p^{L+1} + d_{L,i} \cdot p^L),$$

which further implies

$$p \mid \sum_{i \in I} a_i d_{L,i}$$

because all the terms before $a_i \cdot d_{L,i}$ are already multiple of p^{L+1}.

Since $x_1, x_2, \cdots, x_n$ are monochromatic, $d = d_{L,i}$ for all $i \in I$. Then

$$p \mid \sum_{i \in I} a_i d.$$

Since p is prime and $d < p$,

$$p \mid \sum_{i \in I} a_i.$$

Since the argument holds for infinitely many primes and the choices of I is finite, infinitely many primes share a single I. That means for this I, $p \mid \sum_{i \in I} a_i$ for infinitely many p. It follows immediately that

$$\sum_{i \in I} a_i = 0.$$

$\square$

Lemma 3.31. *Let* $A = \begin{bmatrix} 1 & \lambda & -1 \end{bmatrix}$ *with* $\lambda \in \mathbb{Q}$. *Then for any* $k \in \mathbb{N}$, *there exists* $n \in \mathbb{N}$ *such that whenever* $[n]$ *is* k*-colored, there is monochromatic* $\{x, y, z\}$ *satisfying*

$$x + \lambda y = z.$$

Proof. When $\lambda = 0$, let $x = y = z$. Then $\{x, y, z\}$ is monochromatic and $x + \lambda y = z$.

When $\lambda \neq 0$, suppose without losing generality that $\lambda > 0$ (if $\lambda < 0$, we can first prove for $-\lambda$, which is positive, and then switch the positions of x and z in the equation). Then we let $\lambda = \frac{r}{s} (r, s \in \mathbb{N})$.

We prove the statement by an induction on k. To prove the base case $k = 1$, let $n = max\{r, s\} + 1$. Let $x = 1, y = s, z = r + 1$. Then $\{x, y, z\}$ is monochromatic and $1 + s \cdot \frac{r}{s} = r + 1$.

Suppose the statement is true for $k - 1$ and n_0 is the number suitable for $k - 1$. Let $t = max\{r, s\}$ and $n = W(n_0 t + 1, k)$. We claim that n is suitable for k.

Let $c : [n] \to [k]$ be an arbitrary coloring; there exists a monochromatic arithmetic progression of length $n_0 t + 1$, denoted by

$$a, a + d, \cdots, a + n_0 t d.$$

Suppose the progression has color i. Now consider the progression

$$sd, 2sd, \cdots, n_0 sd.$$

There are two possibilities:

- There exists $p \in [n_0]$ such that $c(psd) = i$. Let $x = a, y = psd, z = a + prd$. As $psd \leq n_0sd \leq n_0td \leq a+n_0td \leq n$ and $a+prd \leq a+n_0rd \leq a+n_0td \leq n$, the coloring function is well-defined on y and z and $\{x, y, z\}$ is monochromatic. In addition, $x + \lambda y = a + psd \cdot \frac{r}{s} = a + prd = z$.

- There does not exist $p \in [n_0]$ with $c(psd) = i$. Then the progression $s, 2sd, \cdots, n_0sd$ is at most $(k-1)$-colored. Define $c_0 : [n_0] \to [k-1]$ by letting $c_0(p) := c(psd)$. By the inductive hypothesis, there exists $x, y, z \in [n_0]$ such that $c_0(x) = c_0(y) = c_0(z)$ and $x + \lambda y = z$. Then $c(xsd) = c(ysd) = c(zsd)$ and $xsd + \lambda ysd = zsd$. In other words, there will be a monochromatic 3-set with the anticipated property.

In both cases, n is suitable for k. Hence the statement is true for all $k \in \mathbb{N}$. □

Lemma 3.32. *Suppose $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$. If there exists $I \subset [n]$ such that $\sum_{i \in I} a_i = 0$, then A is PR.*

Proof. Let a k-coloring on $\mathbb{N}$ be given. Let $i_0 \in I$ be given and let $\lambda = \frac{\sum_{i \notin I} a_i}{a_{i_0}}$. By Lemma 3.31, there exists a monochromatic $\{x, y, z\}$ such that $x + \lambda y = z$. Define

- $x_i := x,\ if\ i = i_0;$
- $x_i := y,\ if\ i \notin I;$
- $x_i := z,\ if\ i \in I \setminus \{i_0\}.$

Then we can rewrite

$$\begin{aligned}
\sum_{i=1}^{n} a_i x_i &= a_{i_0} x_{i_0} + \sum_{i \notin I} a_i x_i + \sum_{i \in I \setminus \{i_0\}} a_i x_i \\
&= x_{i_0} + \frac{\sum_{i \notin I} a_i}{a_{i_0}} x_i + \frac{\sum_{i \in I \setminus \{i_0\}} a_i}{a_{i_0}} x_i \\
&= x + \lambda y + \frac{\sum_{i \in I \setminus \{i_0\}} a_i}{a_{i_0}} z \\
&= z + \frac{\sum_{i \in I \setminus \{i_0\}} a_i}{a_{i_0}} z \\
&= \frac{\sum_{i \in I \setminus \{i_0\}} a_i + a_{i_0}}{a_{i_0}} z \\
&= \frac{\sum_{i \in I} a_i}{a_{i_0}} z \\
&= 0.
\end{aligned}$$

This implies that the $\mathbf{x}$ thereby defined is a monochromatic solution and A is PR. □

Theorem 3.33. *Let $A = [a_1, a_2, \cdots, a_n]$ with $a_1, a_2, \cdots, a_n \in \mathbb{Q}$ be given. A is PR if and only if A has CP.*

Proof. Suppose there does not exist $i \in [n]$ such that $a_i = 0$. By Lemma 3.30 and Lemma 3.32, A is PR if and only if there exists $I \subset [n]$ such that $\sum_{i \in I} a_i = 0$. By Lemma 3.26, A has CP if and only if there exists $I \subset [n]$ such that $\sum_{i \in I} a_i = 0$. It follows that A is PR if and only if A has CP. □

Finally, we are going to prove the complete version of Rado's Theorem, which says that an $m \times n$ matrix A is PR if and only if it has CP. We will first show the direction of "A has CP if it is PR". To do so, we need the assistance of "base coloring" again.

In the following proof, for a vector $\mathbf{v} \in \mathbb{Z}^m$, we use $\mathbf{v} \equiv 0(mod\ p)$ to denote that $v_i \equiv 0(mod\ p)$ for all $i \in [m]$. We will first prove as a lemma that if a vector $\mathbf{v}$ is not in the span of a collection of other vectors, there is always a vector orthogonal to every vector in the collection, but not to $\mathbf{v}$.

Lemma 3.34. *Let $\mathbf{v}_1, \cdots, \mathbf{v}_n, \mathbf{v} \in \mathbb{Z}^m$. If $\mathbf{v} \notin span(\mathbf{v}_1, \cdots, \mathbf{v}_n)$, then there exists $\mathbf{u} \in \mathbb{Z}^m$ such that $\mathbf{u} \cdot \mathbf{v}_j = 0$ for all $j \in [n]$ but $\mathbf{u} \cdot \mathbf{v} \neq 0$.*

Proof. Let $S = span(\mathbf{v}_1, \cdots, \mathbf{v}_n)$. We can reduce $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ to a list that are both linearly independent and span S. Applying Gram-Schmidt Orthogonalization we can form an orthonormal basis, denoted by $\{\mathbf{e}_1, \cdots, \mathbf{e}_k\}, k \leq n$.

Since $\mathbf{v} \notin S$, $\mathbf{v} \notin span(\mathbf{e}_1, \cdots, \mathbf{e}_k)$. Yet we can extend the list until it spans $\mathbf{v}$. Denote the new list by $\{\mathbf{e}_1, \cdots, \mathbf{e}_{k'}\}$. Then

$$\mathbf{v} = \sum_{i=1}^{k'} \alpha_i \mathbf{e}_i.$$

There exists $i \in \{k+1, \cdots, k'\}$ such that $\alpha_i \neq 0$; otherwise $\mathbf{v} \in S$. Let $\mathbf{u} = \mathbf{e}_i$. Since $\mathbf{e}_i$ is orthogonal to $\mathbf{e}_r$ for all $r \neq i$, $\mathbf{u} \cdot \mathbf{e}_r = 0$ for all $r \neq i$. That is to say, $\mathbf{u} \perp S$ and therefore $\mathbf{u} \perp \mathbf{v}_j$ for all $j \in [n]$, which also means $\mathbf{u} \cdot \mathbf{v}_j = 0$ for all $j \in [n]$. Mean, we have $\mathbf{u} \cdot \mathbf{v} = \alpha_i ||\mathbf{e}_i||^2 = \alpha_i \neq 0$. The statement holds true. $\square$

Theorem 3.35. *Let A be an $m \times n$ matrix. If A is PR, A has CP.*

Proof. Let $A = \{\mathbf{c}_1, \cdots, \mathbf{c}_n\}$ be given. Assume without losing generality that all the entries of A are integers. Let p be an arbitrary prime, and define a $(p-1)$-coloring $c : \mathbb{N} \to [p-1]$ by letting $c(x) := d(x), \forall\ x \in \mathbb{N}$, where $d(x)$ denote the last non-zero digit of x when it is written in base p. Since A is PR, there exists a monochromatic solution to $A\mathbf{x} = 0$ under this coloring, which we denote as $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Note that $d(x_1) = d(x_2) = \cdots = d(x_n) = d$.

Let $L = min\{L(x_1), \cdots, L(x_n)\}$ and $B_1 = \{i \mid L(x_i) = L\}$. For the other indexes, group them based on the position of their last non-zero digits. Precisely, let $i, j \in B_s$ when $L(x_i) = L(x_j)$, and $i \in B_s, j \in B_t$ with $s < t$ if $L(x_i) < L(x_j)$. Thereby we form a partition $B_1 \cup B_2 \cdots \cup B_r$ of $[n]$. As there are finitely different partitions of $[n]$, infinitely many prime numbers share a single partition, denoted by $B_1 \cup \cdots \cup B_r$. We now show that this partition satisfies columns property.

- First we show that
$$\sum_{i \in B_1} \mathbf{c}_i = 0.$$
Since
$$\sum_{i=1}^{n} \mathbf{c}_i x_i = \mathbf{0},$$
it follows
$$\sum_{i=1}^{n} \mathbf{c}_i x_i \equiv \mathbf{0}(mod\ p^{L+1}).$$

That is to say,
$$\sum_{i\in B_1} \mathbf{c}_i d \equiv \mathbf{0}(mod\ p).$$
As p is prime and $d < p$,
$$\sum_{i\in B_1} \mathbf{c}_i \equiv \mathbf{0}(mod\ p).$$
And since the argument holds for infinitely many prime number p,
$$\sum_{i\in B_1} \mathbf{c}_i = 0.$$

- Next we show that for any $2 \le s \le r$,
$$\sum_{i\in B_s} \mathbf{c}_i \in span(\mathbf{c}_i \mid i \in B_1 \cup \cdots \cup B_{s-1}).$$
Let $s \in \{2, \cdots, r\}$ be given; suppose the statement is not true for contradiction. Let $L_s = L(x_i)$ with $i \in B_s$. We know that
$$\sum_{i=1}^{n} \mathbf{c}_i x_i \equiv \mathbf{0}(mod\ p^{L_s+1}).$$
That is to say,
$$\sum_{i\in B_s} \mathbf{c}_i d + \sum_{i\in B_1\cup\cdots\cup B_{s-1}} \mathbf{c}_i x_i \equiv \mathbf{0}(mod\ p).$$
By lemma 3.34, there exists $\mathbf{u} \in \mathbb{Z}^m$ satisfying that $\mathbf{u}\cdot\mathbf{c}_j = 0$ for all $j \in B_1 \cup \cdots \cup B_{s-1}$ and $\mathbf{u}\cdot\sum_{i\in B_s} \mathbf{c}_i \ne 0$. Then
$$\mathbf{u}\cdot(\sum_{i\in B_s} \mathbf{c}_i d + \sum_{i\in B_1\cup\cdots\cup B_{s-1}} \mathbf{c}_i x_i) \equiv \mathbf{0}(mod\ p),$$
which also means
$$\mathbf{u}\cdot\sum_{i\in B_s} \mathbf{c}_i d + \sum_{i\in B_1\cup\cdots\cup B_{s-1}} \mathbf{u}\cdot\mathbf{c}_i x_i \equiv 0(mod\ p).$$
As $\mathbf{u}\cdot\mathbf{c}_i = 0$ for all $i \in B_1 \cup \cdots \cup B_{s-1}$,
$$\sum_{i\in B_1\cup\cdots\cup B_{s-1}} \mathbf{u}\cdot\mathbf{c}_i x_i = 0.$$
It follows that
$$\mathbf{u}\cdot\sum_{i\in B_s} \mathbf{c}_i d \equiv 0(mod\ p).$$
Since p is prime and $d < p$,
$$\mathbf{u}\cdot\sum_{i\in B_s} \mathbf{c}_i \equiv 0(mod\ p).$$
And as this works for infinitely many prime p,
$$\mathbf{u}\cdot\sum_{i\in B_s} \mathbf{c}_i = 0,$$
which is a contradiction.

Therefore A has CP. □

In the following discussions, we use $[-p,p]$ to denote $\{-p,-(p-1),\cdots,p-1,p\}$, where $p\in\mathbb{Z}$.

Definition 3.36. For $m,p,c\in\mathbb{N}$, an *(m,p,c)-set* contains elements in the form of:

$$cx_1+\lambda_2x_2+\cdots+\lambda_mx_m$$
$$cx_2+\lambda_3x_3+\cdots+\lambda_mx_m$$
$$\cdots$$
$$cx_{m-1}+\lambda_mx_m$$
$$cx_m$$

Where $x_1,x_2,\cdots,x_m$ are arbitrary elements in $\mathbb{N}$ and for each $j\in[m]$, for all $i>j$, $\lambda_i\in[-p,p]$.

That is to say, an *(m,p,c)-set* is a set

$$S=\{\sum_{i=1}^{m}\lambda_ix_i \mid for\ some\ j\in[m],\lambda_i=0\ if\ i<j;\lambda_i=c\ if\ i=j;\lambda_i\in[-p,p]\ if\ i>j\}.$$

We call $x_1,x_2,\cdots,x_m$ *generators*.

Again, a monochromatic $(m,p,c)-set$ is an $(m,p,c)-set$ where all its elements have the same color.

Lemma 3.37. *Let $m,p,c\in\mathbb{N}$. Whenever $\mathbb{N}$ is finitely colored, there exists a monochromatic $(m,p,c)-set$.*

Proof. Let $m,p,c,k\in\mathbb{N}$ be given, and let $\mathbb{N}$ be k-colored. Let $M=k(m-1)+1$. We first show that there exists an $(M,p,c)-set$ that is row monochromatic.

Let n to be large enough and let

$$A=\{c,2c,\cdots,\frac{n}{c}\cdot c\}.$$

Let n_1 to be large enough. By Van der Waerden's Theorem, there exists a monochromatic arithmetic progression of length $2n_1+1$ in A. This progression has color k_1 and we name it

$$R_1=\{x_1c-n_1d_1,\cdots,x_1c,\cdots,x_1c+n_1d_1\},$$

where d_1 is the common difference and $d_1>0$. Let

$$B_1=\{d_1,2d_1,\cdots,\frac{n_1}{(M-1)p}d_1\}.$$

We can always let n_1 to be large enough so that it is divisible by $(M-1)p$. Let $b_2,\cdots,b_M\in B_1$ be given. Then for $\lambda_2,\cdots,\lambda_M\in[-p,p]$,

$$(-p)\cdot\frac{n_1}{(M-1)p}d_1\cdot(M-1)\le\lambda_2b_2+\cdots+\lambda_Mb_M\le p\cdot\frac{n_1}{(M-1)p}d_1\cdot(M-1).$$

That is to say,

$$-n_1d_1\le\sum_{i=2}^{M}b_i\le n_1d_1.$$

Then

$$cx_1+\sum_{i=2}^{M}b_i\in R_1,$$

and is monochromatic. Let $A_2 \subset B_1$ and

$$A_2 = \{cd_1, 2cd_1, \cdots, \frac{n_1}{(M-1)pc} \cdot cd_1\}.$$

Let n_2 to be large enough. There exists a monochromatic arithmetic progression with length $2n_2 + 1$, which we name as

$$R_2 = \{x_2' cd_1 - n_2 d_2, \cdots, x_2' cd_1, \cdots, x_2' cd_1 + n_2 d_2\},$$

where d_2 is the common difference and the progression has color k_2. Now let

$$B_2 = \{d_2, 2d_2, \cdots \frac{n_2}{(M-2)p} d_2\}.$$

Let

$$x_2 = x_2' d_1 \leq \frac{n_1}{(M-1)pc} \cdot cd_1 = \frac{n_1}{(M-1)p} d_1.$$

This means $x_2 \in B_1$.

Let $b_3, \cdots, b_M \in B_2$ be given. For the same reason, when $\lambda_3, \cdots, \lambda_M \in [-p, p]$,

$$cx_2 + \sum_{i=3}^{M} \lambda_i b_i \in R_2$$

and has the same color. Continue this process, we can generate an $(M, p, c) - set$ that is row monochromatic.

By the pigeonhole principle, since there are only k colors, at least m rows have the same color k'. Denote the rows by $l_1, l_2, \cdots, l_m$, then

$$cx_{l_1} + \lambda_{l_1 - 1} x_{l_1 - 1} + \cdots + \lambda_M x_M$$

$$\cdots$$

$$cx_{l_m} + \lambda_{l_m - 1} x_{l_m - 1} + \cdots + \lambda_M x_M$$

is monochromatic. Let $x_{l_1}, \cdots, x_{l_m}$ be the generators of the $(m, p, c) - set$ and it is monochromatic. □

Lemma 3.38. *Let A be an $m \times n$ matrix with CP. Then there exists $m, p, c \in \mathbb{N}$ such that every $(m, p, c) - set$ contains a solution to $A\mathbf{x} = \mathbf{0}$.*

Proof. Let $A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$ be given. Assume without losing generality that all the entries of A are integers. Since A has CP, there exists a partition $B_1 \cup \cdots \cup B_m$ of $[n]$ such that for all $2 \leq s \leq m$,

$$\sum_{i \in B_s} \mathbf{c}_i \in span(\mathbf{c}_j \mid j \in B_1 \cup \cdots \cup B_{s-1}).$$

That is, for $2 \leq s \leq m, j \in B_1 \cup \cdots \cup B_{s-1}$, there exists $q_{s,j}$ such that

$$\sum_{i \in B_s} \mathbf{c}_i = \sum_{j \in B_1 \cup \cdots \cup B_{s-1}} q_{s,j} \mathbf{c}_j.$$

Also, $q_{s,j} = \frac{a_{s,j}}{b_{s,j}}$, $a_{s,j} \in \mathbb{N} \cup \{0\}, b_{s,j} \in \mathbb{Z} \setminus \{0\}$.

Let c to be the least common multiple of $b_{s,j}$, where $2 \leq s \leq m$, $j \in B_1 \cup \cdots \cup B_{s-1}$. That is,

$$c := [b_{s,j} \mid 2 \leq s \leq m, j \in B_1 \cup \cdots \cup B_{s-1}].$$

Let

$$p := c \cdot max\ \{a_{s,j} \mid 2 \leq s \leq m, j \in B_1 \cup \cdots \cup B_{s-1}\}.$$

Let $x_1, \cdots, x_m \in \mathbb{N}$ be given. For each $s \in \{2, \cdots, m\}$, we have

$$\sum_{i \in B_s} \mathbf{c}_i - \sum_{j \in B_1 \cup \cdots \cup B_{s-1}} q_{s,j} \mathbf{c}_j = \mathbf{0}.$$

When $s = 1$,

$$\sum_{i \in B_1} \mathbf{c}_1 = \mathbf{0}$$

because A has CP. Let

$$d_{s,i} = 0 \ if \ i \notin B_1 \cup \cdots \cup B_s;$$

$$d_{s,i} = 1 \ if \ i \in B_s;$$

$$d_{s,i} = -q_{s,i} \ if \ i \in B_1 \cup \cdots \cup B_{s-1}.$$

For all s,

$$\sum_{i=1}^{n} d_{s,i} \mathbf{c}_i = \mathbf{0}.$$

Let $i \in n$, and

$$y_i = \sum_{s=1}^{m} d_{s,i} x_s = d_{1,i} x_1 + d_{2,i} x_2 + \cdots + d_{r,i} x_m.$$

Let $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum_{s=1}^{r} d_{s,1} x_s \\ \vdots \\ \sum_{s=1}^{r} d_{s,n} x_s \end{bmatrix}$

Let $i \in [n]$ be given.

$$c y_i = c \cdot \sum_{s=1}^{m} d_{s,i} x_s.$$

Suppose $i \in S'$, then

$$\begin{aligned} c y_i &= c(0 + \cdots + 0 + x_{S'} - q_{S'+1,i} x_{S'+1} - \cdots - q_{r,i} x_m) \\ &= c \cdot x_{S'} - c \cdot \frac{a_{S'+1,i}}{b_{S'+1,i}} x_{S'+1} - \cdots - c \cdot \frac{a_{r,i}}{b_{r,i}} x_m. \end{aligned}$$

For $j \in \{S'+1, \cdots, m\}$,

$$\begin{aligned} |c \cdot \frac{a_{j,i}}{b_{j,i}}| &= a_{j,i} \cdot |\frac{c}{b_{j,i}}| \\ &\leq max\{a_{s,i} \mid for \ all \ s, i\} |\frac{c}{b_{j,i}}| \\ &= |\frac{p}{b_{j,i}}| \\ &\leq |p|. \end{aligned}$$

Also, $\frac{p}{b_{j,i}}$ is an integer. That is to say, for $j \in \{S'+1, \cdots, m\}$, $-q_{j,i} \in [-p, p]$. Then $c \cdot y_i \in (m, p, c) - set$.

$$
\begin{aligned}
A\mathbf{y} &= \begin{bmatrix}\mathbf{c}_1 & \cdots & \mathbf{c}_n\end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
&= \sum_{i=1}^{n} \mathbf{c}_i y_i \\
&= \sum_{i=1}^{n} (\sum_{s=1}^{m} d_{s,i} x_s \mathbf{c}_i) \\
&= \sum_{s=1}^{m} x_s (\sum_{i=1}^{n} d_{s,i} \mathbf{c}_i) \\
&= \mathbf{0}.
\end{aligned}
$$

Then $\mathbf{y}$ is a solution in the given (m, p, c)-set. □

Theorem 3.39. *Let A be an $m \times n$ matrix. A is PR if and only if A has CP.*

Proof. By Theorem 3.35, if A is PR, A has CP. By Lemma 3.38, if A has CP, then there exists $m, p, c \in \mathbb{N}$ such that every (m, p, c)-set contains a solution. By Lemma 3.37, for this fixed m, p, c, there is a monochromatic (m, p, c)-set. Then in this set, there must be a monochromatic solution. In other words, A is PR. □

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References

[1] Weisstein, Eric W. “Ramsey Number.” From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RamseyNumber.html

[2] I.B. Leader. Ramsey Theory. http://tartarus.org/gareth/maths/notes/iii/Ramsey_Theory.pdf

[3] Matthew Steed. Some Theorems and Applications of Ramsey Theory. http://math.uchicago.edu/ may/REUDOCS/Steed.pdf

[4] Simon Fraser University. Ramsey Theory. http://www.sfu.ca/ mdevos/notes/graph/445_extremal-3.pdf

[5] https://en.wikipedia.org/wiki/Frank_P._Ramsey