SPECTRAL THEOREM AND APPLICATIONS

JINGJING (JENNY) LI

Abstract. This paper is dedicated to present a proof of the Spectral Theorem, and to discuss how the Spectral Theorem is applied in combinatorics and graph theory. In this paper, we also give insights into the ways in which this theorem unveils some mysteries in graph theory, such as expander graphs and graph coloring.

Contents

1. Introduction

The topic of this paper is a fundamental theorem of mathematics: **The Spectral Theorem**. This theorem concerns symmetric transformations on finite dimensional Euclidean spaces. It specifies a condition for matrix diagonalization, which is widely used in discrete mathematics and other fields including physical sciences (for example, quantum mechanics). In order to understand the Spectral Theorem, first we need to look at some definitions:

Definition 1.1. Let V be a Euclidean vector space. A set of vectors $\{b_1, \ldots, b_k\} \subseteq V$ forms an orthonormal basis if

$$\langle b_i, b_j \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

and the set spans V.

Definition 1.2. Let V be as above and let ϕ be a linear transformation from V to itself. ϕ is a symmetric transformation if $(\forall x, y \in V)(\langle \phi(x), y \rangle = \langle x, \phi(y) \rangle)$.

If we represent ϕ with respect to an orthonormal basis then we get a symmetric matrix, i.e. $A_{ij} = A_{ji}$. The converse is also true: if A is a symmetric matrix, then it defines a symmetric transformation on $\mathbb{R}^n$. This justifies the use of the same adjective for both notions.

For example, $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ defines a symmetric transformation on $\mathbb{R}^2$.

Date: September 23, 2016.

Now let us state the Spectral Theorem:

Theorem 1.3. *In a finite-dimensional Euclidean space, every symmetric transformation has an orthonormal eigenbasis.*

In this paper, we will prove this theorem in several steps that will be presented as a series of separate lemmas. We will conclude by including specific examples of the theorem's application to graph theory, such as the theory of expander graphs.

2. Proof of the Spectral Theorem

To prove the Spectral Theorem, the first step is to prove that there exists one eigenvector. Then, we want to show that assuming the theorem holds true for a subspace, we can prove the theorem for the orthogonal complement of that space. Lastly, combining the first and second steps, we can prove the entire Spectral Theorem by induction, because we know there always exists one eigenvector.

Definition 2.1. For every $v \neq 0$, define the Rayleigh quotient as $\dfrac{\langle \phi v, v\rangle}{\langle v, v\rangle}$. Furthermore, define

$$\lambda = \sup_{v\neq 0} \frac{\langle \phi v, v\rangle}{\langle v, v\rangle}.$$

Note that the function $v \mapsto \langle \phi v, v\rangle$ is a continuous function on the compact set $\{v \mid \|v\| = 1\}$. Therefore, by the Extreme Value Theorem, the number λ is attained.

To prove the Spectral Theorem, we need to find an eigenbasis. So the first step is to find one eigenvector. The next lemma will show that a vector that maximizes the Rayleigh quotient is actually an eigenvector.

Lemma 2.2. *Suppose $\|v_0\| = 1$ satisfies $\langle \phi v_0, v_0\rangle = \lambda$, then $\phi v_0 = \lambda v_0$.*

Proof. Since λ is the supremum of all Rayleigh quotients, for all v and w such that $v + w \neq 0$, we have

$$\lambda \geq \frac{\langle \phi(v+w), v+w\rangle}{\langle v+w, v+w\rangle} \implies \lambda\langle v+w, v+w\rangle \geq \langle \phi(v+w), v+w\rangle.$$

We notice that this last inequality holds even if $v + w = 0$. Expanding both sides of the inequality, we have

$$\lambda(\langle v, v\rangle + \langle w, w\rangle + 2\langle v, w\rangle) \geq \langle \phi v, v\rangle + \langle \phi w, w\rangle + \langle \phi v, w\rangle + \langle \phi w, v\rangle.$$

Since ϕ is symmetric, $\langle \phi w, v\rangle = \langle w, \phi v\rangle = \langle \phi v, w\rangle$, and thus we get

$$\lambda\langle v, v\rangle + \lambda\langle w, w\rangle + 2\langle \lambda v, w\rangle \geq \langle \phi v, v\rangle + \langle \phi w, w\rangle + 2\langle \phi v, w\rangle.$$

Since the inequality holds for any v and w, it holds particularly for $v = v_0$ and $w = tu$, where $t \in \mathbb{R}$ and $u \in V$ is any vector. Since $\|v_0\| = 1$ and $\langle \phi v_0, v_0\rangle = \lambda$, the previous inequality becomes

$$\begin{aligned}\lambda + \lambda t^2\langle u, u\rangle + 2t\langle \lambda u, u\rangle &\geq \lambda + t^2\langle \phi u, u\rangle + 2t\langle \phi v_0, u\rangle \\ &\implies t^2(\lambda\langle u, u\rangle - \langle \phi u, u\rangle) + 2t(\langle \lambda v_0 - \phi v_0, u\rangle) \geq 0.\end{aligned}$$

As t varies we see that the left-hand side of the inequality traces a parabola in t. At $t = 0$ the parabola vanishes, and the inequality demonstrates that $t = 0$ is in fact a global minimum, so the derivative at 0 vanishes. Thus, for all $u \in U$, we deduce that $\langle \lambda v_0 - \phi v_0, u\rangle = 0$. In particular for $u = \lambda v_0 - \phi v_0$, we get $0 = \langle \lambda v_0 - \phi v_0, \lambda v_0 - \phi v_0\rangle = \|\lambda v_0 - \phi v_0\|^2$. So $\phi v_0 = \lambda v_0$. □

Now that we have one eigenvector, we need to continue this process and produce additional eigenvectors inside $\{v_0\}^\perp$.

Lemma 2.3. *Let U be a subspace of V. If U is ϕ-invariant, then $U^\perp$ is also ϕ-invariant.*

Proof. Let $u' \in U^\perp$. We want to show that $\phi u' \in U^\perp$, i.e. for every $u \in U$, it holds that $\langle \phi u', u\rangle = 0$. Since ϕ is symmetric, $\langle u', \phi u\rangle = \langle u, \phi u'\rangle = 0$, because the ϕ-invariance of U implies $\phi u \in U$ and $u' \in U^\perp$. □

Lemma 2.4. *Let V be a Euclidean vector space, and let $U \subseteq V$ be a linear subspace with inner product induced from V. As before, let ϕ be a symmetric transformation. If U is ϕ-invariant, then $\phi|_U$ is symmetric.*

Proof. By definition, $\phi u = \phi|_U(u)$. So for all $u, u' \in U$, the symmetry of ϕ implies $\langle \phi u, u'\rangle_V = \langle u, \phi u'\rangle_V$. Since the inner product on U is the restriction of the inner product on V, we have $\langle \phi|_U(u), u'\rangle_U = \langle u, \phi|_U(u')\rangle_U$. Therefore $\phi|_U$ is symmetric. □

With the lemmas proven above, we can now proceed to prove the Spectral Theorem.

Proof of Theorem 1.3. Base case: $\dim V = 1$. Take any nonzero vector v in this one-dimensional vector space. Divide v by its norm so that it becomes an orthonormal basis. The vector ϕv is a scalar multiple of the given vector v.

Induction hypothesis: for $k = n-1$, a symmetric $k \times k$ matrix has an orthonormal eigenbasis. Then since $\dim U^\perp = n-1$, U is ϕ-invariant.

Inductive step: Let $\lambda = \max_{\|x\|=1} \langle x, \phi x\rangle = \langle v_0, \phi v_0\rangle$. By Lemma 2.2, we have $\phi v_0 = \lambda v_0$. Define $U = \mathrm{span}\{v_0\}$ as a ϕ -invariant subspace of V. Since $\dim U = 1$, it follows that $\dim U^\perp = n-1$. By Lemma 2.3, $U^\perp$ is ϕ-invariant, and by Lemma 2.4, $\phi|_{U^\perp}$ is symmetric. By the induction hypothesis, $\phi|_{U^\perp}$ has orthonormal eigenbasis $\{v_1, \ldots, v_{n-1}\}$ on $U^\perp$. Therefore, we conclude $\{v_0, v_1, \ldots, v_{n-1}\}$ is an orthonormal eigenbasis of ϕ on V. □

3. Consequences and Applications – Spectral Graph Theory

Now, we will look at the specific ways in which the Spectral Theorem and its consequences are applied to solve problems in graph theory, including how eigenvalues illuminate some "mysteries" of graphs.

We have defined λ to be the supremum of all Rayleigh quotients. The next lemma shows that λ is the maximum of all eigenvalues.

Lemma 3.1. *For any eigenvalue λ_i of ϕ, we have $\lambda_i \leq \lambda$, where $\lambda = \max_{\|x\|=1} \langle x, \phi x\rangle$.*

Proof. If λ_i is an eigenvalue of V and v_i is a corresponding eigenvector, without loss of generality, we may assume that $\|v_i\| = 1$. So we deduce

$$\lambda_i = \lambda_i \langle v_i, v_i \rangle = \langle v_i, \lambda_i v_i \rangle = \langle v_i, \phi v_i \rangle \leq \max_{\|x\|=1} \langle x, \phi x \rangle = \lambda.$$

□

In the same way that we find λ, we can find all eigenvalues. Throughout this section, we denote orthonormal eigenvectors $v_1, \ldots, v_n$, where for all $i \in [n]$, v_i has corresponding eigenvalue λ_i. Additionally, the λ's are decreasing, and so $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

One of the most widely used consequences of the Spectral Theorem is the Courant-Fischer min-max principle.

Theorem 3.2 (**Courant-Fischer Principle**). *The eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of the symmetric matrix A are given by*

$$\lambda_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{x \in S, x \neq 0} \frac{x^t A x}{x^t x} = \min_{\substack{S' \subseteq \mathbb{R}^n \\ \dim(S')=n-k+1}} \max_{x \in S', x \neq 0} \frac{x^t A x}{x^t x}.$$

Proof. First we prove that λ_k is achievable.

By the Spectral Theorem, there exists an orthonormal eigenbasis for A. Suppose $v_1, \ldots, v_n$ form such an orthonormal eigenbasis. Let $S_k =$ span $\{v_1, \ldots, v_k\}$. For all $x \in S_k$, we can expand and express $x = \sum_{i=1}^{k} ({v_i}^t x) v_i$. Considering the Rayleigh quotient

$$\frac{x^t A x}{x^t x} = \frac{\sum_{i=1}^{k} \lambda_i ({v_i}^t x)^2}{\sum_{i=1}^{k} ({v_i}^t x)^2} \geq \frac{\sum_{i=1}^{k} \lambda_k ({v_i}^t x)^2}{\sum_{i=1}^{k} ({v_i}^t x)^2} = \lambda_k.$$

Next we prove that λ_k is the maximum over all such expressions.

Let $S'_k =$ span $\{v_k, \ldots, v_n\}$. Since $\dim S'_k = n - k + 1$, for all S such that $\dim S = k$ we have $S \cap S'_k \neq \emptyset$. Then we can deduce that $\min_{x \in S} \frac{x^t A x}{x^t x} \leq \min_{x \in S \cap S'_k} \frac{x^t A x}{x^t x}$. Similar to previous expressions, for all $x \in S \cap S'$, we can write $x = \sum_{i=k}^{n} ({v_i}^t x) v_i$. Since for all $i \geq k$ we have $\lambda_k \leq \lambda_i$, it follows that

$$\frac{x^t A x}{x^t x} = \frac{\sum_{i=k}^{n} \lambda_i ({v_i}^t x)^2}{\sum_{i=k}^{n} ({v_i}^t x)^2} \leq \frac{\sum_{i=k}^{n} \lambda_k ({v_i}^t x)^2}{\sum_{i=k}^{n} ({v_i}^t x)^2} = \lambda_k.$$

We have thus proven for all $S \subseteq \mathbb{R}^n$ with dimension k that $\lambda_k \leq \min_{x \in S} \frac{x^t A x}{x^t x}$. The proof for the equality $\lambda_k = \min_{\substack{S' \subseteq \mathbb{R}^n \\ \dim(S')=n-k+1}} \max_{x \in S'} \frac{x^t A x}{x^t x}$ is analogous.

□

Now, we will look at the specific ways in which the Spectral Theorem and its consequences are applied to solve problems in combinatorics and graph theory. For all the following examples, let G be a finite graph, whose vertices are labeled 1 through n.

Example 3.3 (**Adjacency matrix**). We define A to be the adjacency matrix such that

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

We note that the matrix A_{ij} is symmetric, and this will provide the connection between the Spectral Theorem and its application to graph theory.

We can use adjacency matrices in a lot of different ways. For example, denote

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

where 1 is in the i-th place. Then

$$[1, \ldots, 1]Ae_i = \text{the number of vertices adjacent to the } i\text{-th vertex.}$$

We define this number to be the degree of the i-th vertex, denoted by $\deg(v_i)$.

Example 3.4 (**Laplacian matrix**). The Laplacian matrix is a matrix that averages over the neighbors of every vertex, and it is defined as follows:

Let D be the degree matrix whose i-th entry is equal to the degree of the i-th vertex. The Laplacian matrix of G is defined as $L = D - A$. More formally,

$$L_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian matrix plays an important role in random walks on graphs.

Definition 3.5. We let λ denote the largest absolute value of any eigenvalue other than the largest one, that is, $\lambda = \dfrac{\max_{\lambda > 1} |\lambda_i|}{d}$. The spectral gap is the difference between the largest eigenvalue and the eigenvalue whose absolute value is the second largest, which is equal to $\lambda_1 - \lambda$.

We are interested in the spectral gap because it is related to the randomness of the graph in a subtle way, which we will later see more clearly in the Expander Mixing Lemma. Further, in order to get a quantity that can be compared across different graphs, we proceed to normalize the spectral gap.

Definition 3.6. A graph is d-regular if all of its vertices have degree d, which represents the regularity and equals the largest eigenvalue. We define

$$\text{spectral expansion} = \frac{\text{spectral gap}}{d}.$$

Remark 3.7. Consider a d-regular graph $G = (V, E)$ whose adjacency matrix A has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We let $d = \lambda_1$, which is the largest eigenvalue of A. Let S and T be sets of vertices from G, i.e. $S, T \subseteq V$. The number of possible pairs of vertices from S and T is $|S| \cdot |T|$. The number of edges between pairs of vertices from S and T is expected to be $|S| \cdot |T| \cdot \dfrac{d}{n-1}$, because for an arbitrary pair of vertices i, j of G, the probability that there is an edge between i and j is $\dfrac{\text{number of edges}}{\binom{n}{2}} = \dfrac{\frac{dn}{2}}{\binom{n}{2}} = \dfrac{\frac{dn}{2}}{\frac{n(n-1)}{2}} = \dfrac{d}{n-1}$.

Before we state the Expander Mixing Lemma, let us take a look at the following lemma that concerns the relationship between the largest eigenvalue of S and degree of vertices in the subgraph of S.

Lemma 3.8 (**Average Degree Lemma**). *Let $S \subseteq V$ be a subset of vertices, and consider the subgraph of V spanned by S. Let $\deg_{ave}(S)$ denote the average degree of vertices in the subgraph S (whose edges are all between vertices of S). Then $\deg_{ave}(S) \leq \lambda_1$.*

Proof. Let $v(S)$ and $v(T)$ be the characteristic vectors of S and T. Considering the Rayleigh quotient, we can deduce

$$\lambda_1 = \max_{x \neq 0} \frac{x^t A x}{x^t x} \geq \frac{v(S)^t A v(S)}{v(S)^t v(S)}.$$

Then we get

$$v(S)^t v(S) = \|v(S)^2\| = |S|.$$

By the definition of $v(S)$, we have

$$v(S)^t A v(S) = \sum_{(u,v) \in S \times S} A(u, v).$$

We know

$$\sum_{(u,v) \in S \times S} A(u, v) = \sum_{u \in S} \sum_{v \in S} A(u, v) = \deg(u),$$

because here we are counting the number of neighboring vertices of u, which is the definition of degree. Let n be the size of S. Then we get $v(S)^t v(S) = n$.

In conclusion,

$$\lambda_1 = \max_{x \neq 0} \frac{x^t A x}{x^t x} \geq \frac{v(S)^t A v(S)}{v(S)^t v(S)} = \frac{\sum\limits_{(u,v) \in S \times S} A(u, v)}{n} = \frac{\sum\limits_{u \in S} \deg(u)}{n} = \deg_{ave}(S).$$

□

One final ingredient for the Expander Mixing Lemma is the expander graph.

Definition 3.9. A λ-expander graph is a d-regular graph whose spectral expansion is $1 - \lambda$ (which is equal to $\dfrac{\lambda_1}{d} - \lambda$).

Intuitively, on an expander graph random walks converge quickly to the stationary distribution. The speed of convergence is a consequence of the Expander Mixing Lemma, which we will prove next. It is important to note that given the size of the spectral expansion, this lemma allows us to know how close the graph is from being in the random situation.

Theorem 3.10 (**Expander Mixing Lemma**). *Let $G = (V, E)$ be a d-regular, n-vertex graph with spectral expansion $1 - \lambda$, such that its adjacency matrix A has eigenvalues $\lambda_1 = d \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $S, T \subseteq V$ be two subsets, and let $E(S,T) = \{s \in S, t \in T : \{s,t\} \in E\}$ be the set of edges between the elements of S and T. Then we have*

$$\left| |E(S,T)| - \frac{d|S| \cdot |T|}{n} \right| \leq \lambda d \sqrt{|S| \cdot |T|}.$$

Proof. Let $\{v_1, \ldots, v_n\}$ be an eigenbasis corresponding to the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. As we saw in the proof of the Average Degree Lemma, we can expand $v(S)$ and $v(T)$ into $v(S) = \sum_{i=1}^{n} \alpha_i v_i$ and $v(T) = \sum_{i=1}^{n} \beta_i v_i$. For all $i \in [n]$, let λ_i be the eigenvalue corresponding to eigenvector v_i. Then we have

$$|E(S,T)| = v(S)^t \cdot A \cdot v(T) = \left(\sum_{i=1}^{n} \alpha_i v_i \right)^t A \left(\sum_{i=1}^{n} \beta_i v_i \right) = \sum_{i=1}^{n} \lambda_i \alpha_i \beta_i.$$

We know the graph is d-regular, and so every vertex has d edges coming out of it. Thus when we apply A to the vector $\underline{1} = (1, 1, 1, \ldots)$, the result is d times the vector $\underline{1}$. To normalize this quantity, we divide it by its norm $\sqrt{n}$, and the result we get is v_1. Then, since $v_1 = \frac{1}{\|\underline{1}\|} = \frac{1}{\sqrt{n}}$, we conclude $\alpha_1 = \langle v(S), v_1 \rangle = \frac{|S|}{\sqrt{n}}$.

In [1] it is shown that the largest eigenvalue $\lambda_1 \leq d$. Thus we know the eigenvalue of $\underline{1}$ is d, which proves that $\lambda_1 = d$. Then we get

$$|E(S,T)| = \lambda_1 \alpha_1 \beta_1 + \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i = d \cdot \frac{|S| \cdot |T|}{n} + \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i.$$

Given $\lambda = \frac{\max_{\lambda > 1} |\lambda_i|}{d}$, we know $|\lambda_i| \leq \lambda \cdot d$, for all $i > 1$. Then we deduce

$$\left| |E(S,T)| - \frac{d \cdot |S| \cdot |T|}{n} \right| \leq \sum_{i=2}^{n} |\lambda_i \alpha_i \beta_i| \leq \lambda \cdot d \cdot \sum_{i=2}^{n} |\alpha_i \beta_i|.$$

By Cauchy-Schwartz Inequality, we can deduce

$$\left| |E(S,T)| - \frac{d \cdot |S| \cdot |T|}{n} \right| \leq \lambda \cdot d \cdot \|v(S)\| \cdot \|v(T)\| = \lambda \cdot d \cdot \sqrt{|S| \cdot |T|}.$$

□

To grasp Wilf's Theorem, which allows us to determine elusive properties of graphs with the tool of eigenvalues, we need the following definition.

Definition 3.11. A legal k-coloring of a graph G is a map $c : V \to \{1, 2, \ldots, k\}$ such that for all $\{i, j\} \in E$ we have $c(i) \neq c(j)$. The chromatic number of G, denoted $\chi(G)$, is the smallest value of k for which a legal k-coloring exists. In simpler terms, the chromatic number of a graph is the smallest number of colors needed to assign colors to the vertices of a graph without assigning the same color to any pair of adjacent vertices.

Theorem 3.12 (**Wilf's Theorem**). *$\chi(G) \leq 1 + \lambda_1$.*

Wilf's Theorem reveals how a graph is controlled by eigenvalues. Determining coloring is laborious, and there is no quick and easy way to do so without going through and checking all possible colorings. Thus, Wilf's Theorem is very powerful because it guarantees that there exists one coloring and an upper bound of the number of colors needed to color the graph, without having to know any specific coloring of the graph.

Proof. To prove Wilf's Theorem, it is sufficient to show there exists an ordering for the vertices $v_1, v_2, \ldots, v_n$ such that for all $i \in [n]$, the number of neighbors that v_i has among the ones that come before it is at most λ_1. If we can order the vertices this way, then we can color the vertices one by one with $\lambda_1 + 1$ many colors.

Now we want to show such an ordering exists. We prove by induction on i that we can construct v_{n-i}. We begin with $i = 0$ and show that we can choose v_n with at most λ_1 many neighboring vertices. Let V be the set of all vertices $v_1, v_2, \ldots, v_n$. We know such v_n exists because by Lemma 3.8, $\deg_{ave} V \leq \lambda_1$, which implies that there exists $v_n \in V$ such that $\deg_V(v_n) \leq \lambda_1$.

For the induction hypothesis assume that we have already chosen $v_{n-k+1}, \ldots, v_n$. We want to show that we can choose v_{n-k}. Let S be the collection of all vertices except for $v_{n-k+1}, \ldots, v_n$. Then we get a subgraph with $n - k$ vertices. Applying Lemma 3.8 to S, we know there exists $v_{n-k} \in S$ such that $\deg_S(v_{n-k}) \leq \lambda_1$.

By reverse induction, we have thus proven that we can construct an ordering of vertices such that every vertex v_i has at most λ_1 many neighbors among the previous vertices.

□

In conclusion, the Spectral Theorem can reveal deep insights into a graph, as demonstrated by its applications in graph theory such as Wilf's Theorem. This is exactly the beauty of the Spectral Theorem, which is unveiling rather inaccessible aspects of the graph by providing easily computable invariants of a graph, in the form of eigenvalues.

Acknowledgments. It is a pleasure to thank my mentor, Nir Gadish, who has dedicated significant amount of time and effort to my research endeavor. I sincerely appreciate his generous support and help with formalizing the proof of the Spectral Theorem, and exploring enlightening consequences and interesting applications in spectral graph theory.

References

[1] Daniel A. Spielman. Chapter 16 – Spectral Graph Theory. `http://www.cs.yale.edu/homes/spielman/PAPERS/SGTChapter.pdf`
[2] Daniel A. Spielman. Spectral Graph Theory – Lecture 3: The Adjacency Matrix and Graph Coloring. `http://www.cs.yale.edu/homes/spielman/561/lect03-15.pdf`
[3] Daniel A. Spielman. Spectral Graph Theory – Lecture 4: Courant-Fischer and Graph Coloring. `http://www.cs.yale.edu/homes/spielman/561/2009/lect04-09.pdf`
[4] Fan Chung Graham. Chapter 1: Eigenvalues and the Laplacian of a graph. `http://www.math.ucsd.edu/~fan/research/cb/ch1.pdf`
[5] Laszlo Babai. Linear algebra and applications to graphs Part II. `http://people.cs.uchicago.edu/~laci/reu01/linalg2.pdf`
[6] Madhur Tulsiani. REU 2013: Apprentice Program – Lecture 7: July 8, 2013. `http://ttic.uchicago.edu/~madhurt/courses/reu2013/lecture7.pdf`
[7] Madhur Tulsiani. REU 2013: Apprentice Program – Lecture 15: July 22, 2013. `http://ttic.uchicago.edu/~madhurt/courses/reu2013/class722.pdf`

[8] Salil Vadhan. CS225: Pseudorandomness – Lecture 7: Expander Graphs. http://people.seas.harvard.edu/~salil/cs225/spring07/lecnotes/lec7.pdf
[9] Thomas Sauerwald and He Sun. Spectral Graph Theory and Applications – Lecture 3: Expander Mixing Lemma. https://resources.mpi-inf.mpg.de/departments/d1/teaching/ws11/SGT/Lecture3.pdf