SOME PROPERTIES OF INTEGRAL APOLLONIAN PACKINGS

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ABSTRACT. Within the study of fractals, some very interesting number theoretical properties can arise from unexpected places. The integral Apollonian fractal, or integral Apollonian packing, is a circle fractal that can produce an infinite series of circles each with integer curvature. Not limited to that, some interesting relations between invariants, such as the Hausdorff dimension of the residual set of an Apollonian packing and the density of the Apollonian packing can be discovered.

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1. BACKGROUND

In this section, first, we will define an Apollonian packing and prove its existence. Next, we will describe and prove that there exists a form of map that transforms one Apollonian packing into another.

For any three circles with rational radii, such as $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{7}\}$, we can arrange them such that they are all mutually externally tangent. We can construct a unique circle which is tangent to the other three. Surprisingly, its radius will also be rational. In fact, if we continue to place circles in the same manner of tangency in the resulting curvilinear triangles formed by one of the new circles and two of the others, we will continue to see that the radii are rational.

At this point however, as a result of these smaller and smaller radii, it is convenient to work with their curvatures.

Definition 1.1. The **curvature** of a circle is the inverse of its radius up to a sign (based on orientation).

We shall later show that there exists a unique circle such that the original three circles lie within the interior of it. In Figure 1, the above process of inscribing circles in each curvilinear triangle has been repeated ad infinitum. A few of the

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circles' curvature is labeled. The outer circle's curvature is negative, whereas all circles in its interior have positive curvature.

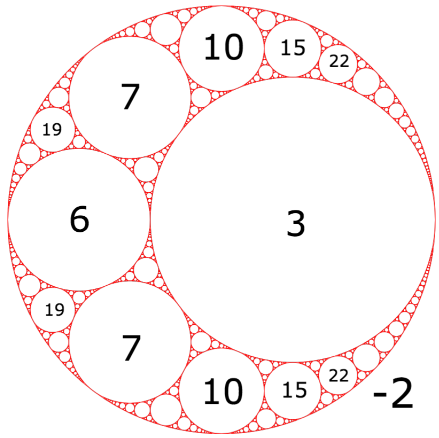

Figure 1, image of an Apollonian packing from a user on codegolf.stackexchange.com, [1]

Definition 1.2. An **Apollonian packing** is a fractal consisting of circles, constructed by triplets of circles. And, for each set of triplets in the packing there exists exactly two distinct circles in the fractal (or packing) that are tangent to all three circles in the triplet.

This above figure yields an Apollonian packing constructed initially by circles with curvatures $\{3, 6, 7\}$. In each curvilinear triangle, a unique circle that is tangent to all three sides of the curvilinear triangle is inscribed.

Definition 1.3. An Apollonian packing is an **integral** Apollonian packing if each circle in the packing has integer curvature.

For some notation, we shall let P denote an integral Apollonian packing and C_n be a circle in P. Let r_n be its radius and a_n be its curvature.

Theorem 1.4 (Apollonius' Theorem)**.** *Given three mutually tangent circles C_1, C_2, C_3, there exist exactly two circles C_4, C_5, tangent to all three.*

Our proof will rely on some other notions of circles.

Definition 1.5. The **circle inversion** of a point p with respect to a circle C with center O, radius r is the point p' lying on the ray $\overrightarrow{Op}$ such that $r^2 = d(O,p)d(O,p')$. Similarly, the circle inversion of a set of points in a plane with respect to the circle is the set of circle inversions of these points.

Lemma 1.6. *The circle inversion of two points p, q with respect to a circle C with center O, radius r preserves angles between the points. In other words, the triangles $\triangle Opq \sim \triangle Oq'p'$.*

The proof of this lemma relies on the definition of circle inversion and basic geometry.

Lemma 1.7. *The circle inversion of a circle (or the set of all points that make up a circle) about a circle will produce another circle (here, a line can be considered a circle with infinite radius).*

The proof of this can be found in [3]. The proof involves a bit of casework and basic geometry.

Corollary 1.8. *Circle inversions preserve tangencies.*

This corollary is immediate from the lemmas. Now we can prove Apollonius' Theorem.

Proof. (Apollonius' Theorem). Let C_1, C_2, C_3 be three mutually tangent circles. Let p be the point of tangency between C_1 and C_2. If we invert the three circles in a circle E centered at p with radius r, the circles C_1 and C_2 will be mapped to circles through infinity, that is, parallel straight lines C_1' and C_2' (since C_1 and C_2 do not intersect other than at p) while circle C_3 is mapped to C_3', a circle tangent to both C_1' and C_2' (by Corollary 1.8).

In such a configuration of a circle tangent to two parallel straight lines, there are exactly two more circles, call them C_4' and C_5', that are tangent to all three inverted circles. When you take the circle inversion about the circle E once again, and since circle inversion preserves tangencies, there were be two new circles, C_4 and C_5 that are the circle inversions about E of C_4' and C_5' that are tangent to C_1, C_2, and C_3. ☐

Theorem 1.9 (Descartes' Theorem)**.** *Given four mutually tangent circles whose curvatures are a_1, a_2, a_3, a_4, then*

$$2a_1^2 + 2a_2^2 + 2a_3^2 + 2a_4^2 = (a_1 + a_2 + a_3 + a_4)^2.$$

A proof of this theorem is provided by Coxeter, in [4]. He uses some theorems of Euclidean geometry and provides a symmetry argument to prove the relation. In particular, in order to get symmetry in terms of signs of equations, Coxeter denotes the curvature of the outer circle by a negative sign, hence why in Figure 1, the outer circle had negative curvature. Note: Descartes' Theorem gives another proof of Apollonius' Theorem, by calculation.

Next, I shall introduce a special type of function that can be used to transform Apollonian packings.

Definition 1.10. Consider the plane $\mathbb{R}^2$ as the complex numbers $\mathbb{C}$ with (x, y) corresponding to $z = x + iy$, then a **linear fractional transformation** is a map $\mathbb{C} \cup \infty \to \mathbb{C} \cup \infty$ of the form:

$$z \mapsto \frac{a + bz}{c + dz} \qquad a, b, c, d \in \mathbb{C}.$$

Our goal will be to show that there exists a unique linear fractional transformation that maps one Apollonian packing into another.

Lemma 1.11. *A linear fractional transformation is equivalent to a composition of translations, scalings, and inversions.*

Proof. We can rewrite the general form of a linear fractional transformation like so:

$$\begin{aligned} \frac{b}{d} - \frac{bc - ad}{d^2 z + cd} &= \frac{b(dz + c) - bc + ad}{d^2 z + cd} \\ &= \frac{bdz + ad}{d^2 z + cd} \\ &= \frac{a + bz}{c + dz} \end{aligned}$$

$\frac{b}{d}$ represents a translation, and the other part of the expression represents a composition of scalings and inversions. □

Lemma 1.12. *A linear fractional transformation preserves angles.*

Proof. Consider how the real axis meets with a line through $(1, 0)$. Clearly, a translation will preserve angles, so any arbitrary line can be shifted to pass through $(1, 0)$. Let us verify that scalings and inversions will be angle preserving operations. Let our line follow the equation $z = 1 + re^{i\theta}$, with $r \in \mathbb{R}, \theta \in [0, 2\pi)$.

Translations Any translation will preserve angles.

Scalings Any scaling by a factor $k \in \mathbb{R}$ will still result in the same argument of θ. Thus, scalings preserve angles.

Inversions That is, $(1 + re^{i\theta}) \mapsto (1 + re^{i\theta})^{-1}$. Let us calculate the tangent vector at $r = 1$ for the inverted expression.

$$\lim_{r\to 0} \frac{1}{r}\Big(\frac{1}{1+re^{i\theta}} - 1\Big) = \lim_{r\to 0} \Big(\frac{1}{r}\frac{-re^{i\theta}}{1+re^{i\theta}}\Big) = \lim_{r\to 0} \frac{-e^{i\theta}}{1+re^{i\theta}} = -e^{i\theta}$$

Thus, the angle formed between the real axis and the equation still forms the same angle θ, and inversions thus preserve angles. □

Theorem 1.13. *A linear fractional transformation maps circles to circles.*

Proof. A translation and scaling will clearly map a circle to a circle. An inversion in this sense is equivalent to a circle inversion about a circle with radius 1. A circle inversion, by Lemma 1.7, maps a circle to another circle. □

Theorem 1.14. *Given 6 points: $z_1, z_2, z_3, \zeta_1, \zeta_2, \zeta_3 \in \mathbb{C} \cup \infty$, then there exists a unique linear fractional transformation f such that $f(z_i) = \zeta_i$ for $i = 1, 2, 3$.*

Proof. Let us define two new functions,

$$h(z) = \frac{(z_2 - z_3)}{(z_2 - z_1)}\frac{(z - z_1)}{(z - z_3)},$$

and

$$g(z) = \frac{(\zeta_2 - \zeta_3)}{(\zeta_2 - \zeta_1)}\frac{(z - \zeta_1)}{(z - \zeta_2)}.$$

Note that $h(z_1, z_2, z_3) = (0, 1, \infty)$ and $g(\zeta_1, \zeta_2, \zeta_3) = (0, 1, \infty)$. Thus,

$$f(z) = g^{-1}(h(z)),$$

sends $z_i \to \zeta_i$ for $i = 1, 2, 3$.

To show that this $f(z)$ is unique, suppose p is another linear fractional transformation that satisfies the requirements. Then gph^{-1} will map $(0, 1, \infty) \to (0, 1, \infty)$. Let us investigate the form of gph^{-1}. Since

$$gph^{-1}(z) = \frac{a + bz}{c + dz},$$

and $0 \to 0$, and $a = 0$ must be true. Additionally, $\infty \to \infty$, so, $d = 0$ as well. Finally, because $1 \to 1$, $b = c + d$ and the fact that $d = 0$ means that $b = c$. Thus, $gph^{-1}(z) = z, \forall z$. As a result, we see that $p = g^{-1}h \equiv f$. □

Now we are ready to relate linear fractional transformations to Apollonian packings.

Lemma 1.15. *If P is an Apollonian packing and f is a linear fractional transformation, then $f(P)$ is also an Apollonian packing.*

Proof. To prove this, we will show another fact first: that any three distinct points can generate an Apollonian packing. If we pick three distinct points, there exists one unique circle that passes through all three of the points. Furthermore, we can draw the tangent lines to the circle at each of the distinct points, and produce an inscribing triangle.

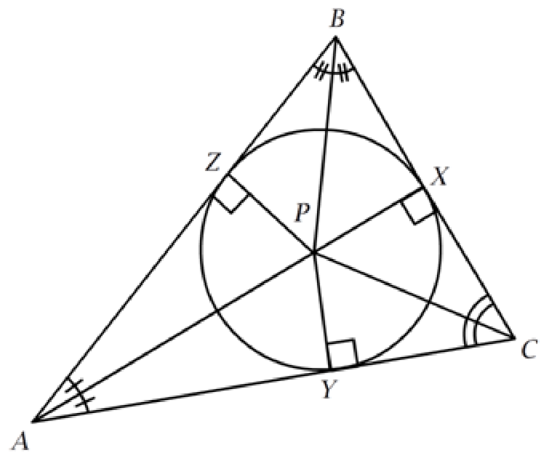

Figure 2, from [2]

Here, X, Y, Z represent the distinct points, A, B, C are the vertices of our triangle, and P is the center of our circle. $PZ = XP$ because they are both radii of P and $BP = BP$. Thus, by HL congruence, $\triangle BPZ \cong \triangle BPX$. Repeating this argument, we have that $\triangle APZ \cong \triangle APY$ and $\triangle CPY \cong \triangle CPX$.

We can construct three distinct circles each centered about A, B, C such that their radii are of length BZ, AY, and CY. These three circles will all be mutually externally tangent to each other at X, Y, and Z, and from there, we can repeatedly apply Apollonius' theorem and produce an Apollonian packing as done in the beginning of the section.

Now, consider three distinct points that can construct P. Then, the linear fractional transformation f on these three points will be sent to three new points, which can generate three mutually externally tangent circles that can construct an Apollonian packing themselves. Thus, $f(P)$ is also an Apollonian packing. □

Theorem 1.16. *Given two Apollonian packings P_1, P_2, there exists a unique linear fractional transformation f such that $f(P_1) = P_2$.*

Proof. We can take any three mutually tangent circles in P_1 and any three mutually tangent circles in P_2. If we look at the points of tangency, there will be a unique linear fractional transformation by Theorem 1.14 that sends the points from P_1 to those in P_2. Combining this with Lemma 1.15 completes the proof. □

2. Descartes Quadruples

Let us examine again the implications of Descartes' Theorem and how it relates to an Apollonian packing.

Recall, Descartes' Theorem states that given four mutually tangent circles with curvatures a_1, a_2, a_3, a_4, then

$$2a_1^2 + 2a_2^2 + 2a_3^2 + 2a_4^2 = (a_1 + a_2 + a_3 + a_4)^2.$$

In particular, if a_1, a_2, a_3 are given, then there are two possible curvatures for a_4. Descartes' Theorem gives rise to a quadratic equation in a_4, with

$$a_4 = a_1 + a_2 + a_3 \pm \sqrt{a_1a_2 + a_2a_3 + a_1a_3}.$$

Let

$$a_{4+} = a_1 + a_2 + a_3 + \sqrt{a_1a_2 + a_2a_3 + a_1a_3},$$
$$a_{4-} = a_1 + a_2 + a_3 - \sqrt{a_1a_2 + a_2a_3 + a_1a_3}.$$

Note that $a_{4+} = 2(a_1 + a_2 + a_3) - a_{4-}$. In particular, we have just seen that if the initial starting four circles in an Apollonian packing have integer curvature, then the other circle generated in that packing will have integer curvature. This is one of the beautiful facts of integral Apollonian packings.

We will now discuss about Descartes quadruples.

Definition 2.1. An **integer Descartes quadruple** $(a, b, c, d) \in \mathbb{Z}^4$ is any integer representation of zero of the **Descartes integral form**,

$$Q_D(w, x, y, z) := 2(w^2 + x^2 + y^2 + z^2) - (w + x + y + z)^2$$

If we consider an arbitrary column vector $\vec{v} = (w, x, y, z)^T$, then

$$Q_D(w, x, y, z) = \vec{v}^T Q_D \vec{v},$$

with $Q_D = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$.

Definition 2.2. The size of any integer Descartes quadruple $(a, b, c, d) \in \mathbb{Z}^4$ is measured by the **Euclidean norm** $H(a, b, c, d)$, which is:

$$H(a, b, c, d) := (a^2 + b^2 + c^2 + d^2)^{1/2}$$

Let $N_D(T)$ be the number of integer Descartes quadruples with Euclidean norm at most T. In the works of Graham in [5], it is worked out that $N_D(T)$ can be approximated as

$$N_D(T) = \frac{\pi^2}{4L(2, \chi_{-4})} T^2 + O(T(logT)^2),$$

as $T \to \infty$, with

$$L(2, \chi_{-4}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159.$$

The idea is to compare $N_D(T)$ with $N_L(T)$, the number of integer Lorentz quadruples with Euclidean norm at most T where

$$Q_L(w, x, y, z) = -w^2 + x^2 + y^2 + z^2.$$

Lorentz quadruples are more commonly worked with by super relativity in physics. There exists a mapping that gives a bijection from the set of integer Descartes quadruples to the set of integer Lorentz quadruples, with details provided in [5]. The Lorentz quadratic form is easier to work with and thus it is easier calculate $N_L(T)$. The full proof is omitted here but is available under [5].

3. Hausdorff Dimension

Hausdorff dimension is based on a measure known as the Hausdorff measure. Hausdorff dimension is commonly used to measure the dimension of fractals. These ideas are needed later on in Section 4 of the paper, where we relate the Hausdorff dimension of the residual set of an Apollonian packing to the circle counting function of an Apollonian packing (definitions will be provided later).

Definition 3.1. For any non-empty subset of $\mathbb{R}^n, U$, the **diameter** is defined as:

$$|U| := \sup\{|x-y| : x, y \in U\}$$

If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, then we say $\{U_i\}$ is a δ-cover of F. We define $\mathcal{H}^s_\delta(F)$ by taking a set $F \subset \mathbb{R}^n$, s non-negative, and $\delta > 0$, then:

$$\mathcal{H}^s_\delta(F) = \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\Big\} \tag{3.2}$$

In other words, we are looking at all covers of F by sets of diameter at most δ and we are looking to minimize the sum of the sth powers of the diameters. As δ decreases, the number of such covers of F is a subset of the original number of covers with original δ. Therefore, we find that the infimum increases and approaches a limit as $\delta \to 0$.

Definition 3.3. The **s-dimensional Hausdorff measure** of a set F is defined as,

$$\mathcal{H}^s(F) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(F).$$

Note that because of the monotonicity of the infimum and the lower bound of 0, the limit exists for any subset F of $\mathbb{R}^n$. Note that the limiting value can be 0 and ∞ (for $s > 0$, the s-dimensional Hausdorff measure of a set of discrete points has value 0 and for $s = 0$, we have that the Hausdorff measure is infinite).

We can show that $\mathcal{H}^s$ is a measure with some effort.

Hausdorff measures generalize the ideas of length, area, volume. When scaling dimensions by a certain factor k, the length of a curve is multiplied by k, while the area is multiplied by k^2, and so on. We would like when dimensions are scaled by k, that the s-dimensional Hausdorff measure is equivalently scaled by a factor k^s.will attempt to make this scaling idea rigorous.

Definition 3.4. A transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ is a **similarity** of **scale** $c > 0$ if

$$|S(x) - S(y)| = c|x - y|,$$

for all $x, y \in \mathbb{R}^n$.

Lemma 3.5. *Let S be a similarity transformation of scale $k > 0$. If $F \subset \mathbb{R}^n$, then*

$$\mathcal{H}^s(S(F)) = k^s \mathcal{H}^s(F).$$

The proof relies on the definition and the countability of the unions of measurable sets.

Theorem 3.6. *Let $F \in \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a mapping such that*

$$|f(x) - f(y)| \le c|x - y|^\alpha, \qquad (x, y \in F)$$

for constants $c > 0$ and $\alpha > 0$. Then for each s,

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F).$$

Proof. If $\{U_i\}$ is a δ-cover of F, then

$$|f(F \cap U_i)| \leq c|F \cap U_i|^\alpha \leq c|U_i|^\alpha \leq c|U_i|^\alpha.$$

Let $\epsilon = c\delta^\alpha$. Then $\{f(F \cap U_i)\}$ is an ϵ-cover of $f(F)$. Thus,

$$\sum_i |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \sum_i |U_i|^s.$$

We see that as $\delta \to 0$, $\epsilon \to 0$ and that by taking the infimum of the last expression we have that

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F).$$

□

In particular, when $\alpha = 1$, this is when f is a Lipschitz mapping. Furthermore, if f is an isometry, that is, $|f(x) - f(y)| = |x - y|$, then $\mathcal{H}^s(f(F)) = \mathcal{H}^s(F)$ by Theorem 3.6. Thus, the Hausdorff measure is invariant under translations and rotations.

Let us refer back to definition 3.1. Since $\mathcal{H}^s_\delta(F)$ is non-increasing with s, so is $\mathcal{H}^s(F)$. Additionally, if $t > s, \delta < 1, F \subset \mathbb{R}^n, \{U_i\}$ is a δ-cover of F, then:

$$\sum_i |U_i|^t \leq \sum_i |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_i |U_i|^s.$$

In other words, $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$. Taking the limit as $\delta \to 0$, we see that if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$ for $t > s$. Thus there exists some critical value of s from which $\mathcal{H}^s(F)$ drops from ∞ to 0.

If we look at the Hausdorff measure of a curve in $\mathbb{R}^n$ with respect to s, the Hausdorff measure will usually start from infinity, and then at some positive finite value, jump to 0 where it stays as s goes to infinity. The only case where this is not true is when the curve is made up of finite points, then the s-dimensional Hausdorff measure starts out at a finite value and jumps to 0.

Definition 3.7. The **Hausdorff dimension** of F, written $\dim_H F$, is defined for any set $F \subset \mathbb{R}^n$:

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

As a result, $\mathcal{H}^s(F) = \infty$ if $0 < s < \dim_H F$ and $\mathcal{H}^s(F) = 0$ if $s > \dim_H F$. If $s = \dim_H F$, then $\mathcal{H}^s(F)$ may be $0, \infty$, or $0 < \mathcal{H}^s(F) < \infty$.

There are some properties of Hausdorff dimension that will be mostly stated without proof.

- **Monotonicity** If $E \subset F$, then $\dim_H E \leq \dim_H F$.
- **Countable stability** If $F_1, F_2, \ldots$ is a countable sequence of sets, then $\dim_H \bigcup_{i=1}^\infty F_i = \sup_{i \leq i < \infty}\{\dim_H F_i\}$.

  Proof. $\dim_H \bigcup_{i=1}^\infty F_i \geq \dim_H F_j$ for each j from monotonicity. Furthermore, if $s > \dim_H F_i$ for all I, then $\mathcal{H}^s(F_i) = 0$, so $\mathcal{H}^s(\bigcup_{i=1}^\infty F_i) = 0$, giving the opposite inequality. □

- **Countable sets** If F is countable then $\dim_H F = 0$. (Note that if F_i is a single point, then $\mathcal{H}^0(F_i) = 1 \implies \dim_H F_i = 0$.

- **Open sets** If $F \subset \mathbb{R}^n$ is open, then $\dim_H F = n$.

To help understand this notion of Hausdorff dimension, we can calculate the Hausdorff dimension of the Cantor set.

Example 3.8 (Hausdorff dimension of Cantor set)**.** Let F be the middle third Cantor set, that is, the numbers in $[0, 1]$ whose base-3 expansion does not contain the digit 1, i.e. all of the numbers $a_1 3^{-1} + a_2 3^{-2} + a_3 3^{-3} + \cdots$, with $a_i = 0$ or 2 for each i. If $s = \log 2/\log 3$, then $\dim_H F = s$ and $\frac{1}{2} \leq \mathcal{H}^s(F) \leq 1$.

Proof. The Cantor set F splits into a left part $F_L = F \cap [0, \frac{1}{3}]$ and a right part $F_R = F \cap [\frac{2}{3}, 1]$. Both parts are similar to F but scaled by a ratio of $\frac{1}{3}$, with $F = F_L \cup F_R$ being a disjoint union. Thus, for any s,

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = (\frac{1}{3})^s \mathcal{H}^s(F) + (\frac{1}{3})^s \mathcal{H}^s(F),$$

by scaling of Hausdorff measures (Lemma 3.5). Thus, we have that

$$1 = 2(\frac{1}{3})^s \implies s = \log 2/\log 3 = \dim_H(F).$$

□

Our goal is to show that the Hausdorff dimensions of two Apollonian packings are equal. To do this, we need a few more theorems.

Theorem 3.9. *Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies*

$$|f(x) - f(y) \leq c|x - y|^\alpha, \qquad (x, y \in F)$$

then $\dim_H f(F) \leq (1/\alpha) \dim_H F$.

Proof. If $s > \dim_H F$, then by Theorem 3.6,

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F) = 0.$$

This implies that $\dim_H f(F) \leq s/\alpha$ for all $s > \dim_H F$. □

Corollary 3.10. *If $f : F \to \mathbb{R}^m$ is bi-Lipschitz , i.e.*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|, \qquad (x, y \in F)$$

where $0 < c_1 \leq c_2 < \infty$*, then* $\dim_H f(F) = \dim_H F$.

Proof. The first inequality that $\dim_H f(F) \leq \dim_H F$ follows from Theorem 3.9 where $\alpha = 1$. Apply the same inequality to $f^{-1} : f(F) \to F$ (which exists since f is Lipschitz) and we get that $\dim_H f(F) \geq \dim_H F$. □

Finally, we shall relate the Hausdorff dimensions of two Apollonian packings.

Theorem 3.11. *Given two Apollonian packings P_1 and P_2,* $\dim_H P_1 = \dim_H P_2$.

Proof. There exists a unique linear fractional transformation f such that $f(P_1) = P_2$ by Theorem 1.16. Let us write

$$P_1 = \bigcup_{n \in \mathbb{N}} (P_1 \cap \{U_n\} \cup 0,$$

if $0 \in P_1$, where

$$\{U_n\} := \{z \in \mathbb{R}^2 : |z| > 1/n\}.$$

By countable stability,

$$\dim_H P_1 = \sup_n \dim_H(P_1 \cap \{U_n\}).$$

Since we can write f as a composition of translations, scalings, and inversions, as long as the inversion function $g(z) = 1/z$ is bi-Lipschitz on the $\{U_n\}$, then we can apply Corollary 3.10. On $\{U_n\}$, $g(z)$ is clearly Lipschitz since n is finite and $g^{-1}(z) = z$ is also clearly Lipschitz. As a result,

$$\dim_H f(P_1) = \sup_n \dim_H(P_1 \cap \{U_n\}) = \sup_n \dim_H(f(P_1 \cap \{U_n\})) = \dim_H P_2.$$

□

4. The Hausdorff Dimension of the Residual Set of an Apollonian Packing and the Density of Circles in an Apollonian Packing

The next section will rely on the results of Boyd, from [7], [8], and [9]. We will talk about the relationship between the Hausdorff dimension of the residual set of an Apollonian packing, and the packing constant of an Apollonian packing. In Theorem 4.9, we see that these constants are found in a relationship that asymptotically counts the number of circles in an Apollonian packing.

In this section, an open circle is an open ball and a closed circle is a closed ball.

Definition 4.1. If U is an open set in 2 dimensions, which has finite Lebesgue measure $||U||$, then a **complete packing** of U by open circles is a collection $S = \{C_n\}$ of pairwise disjoint open circles contained in U and such that

$$\sum_{n=1}^{\infty} ||C_n|| = ||U||.$$

Definition 4.2. The **residual set** of an Apollonian packing P with interior circles C_n is denoted by R and is given by:

$$R = P \setminus \bigcup_{n=1}^{\infty} \overline{C_n}$$

where $\overline{C_n}$ denotes the closure of C_n.

We can find an f that is a linear fractional transformation that sends the residual set of an Apollonian packing to another residual set of an Apollonian packing. By a similar argument to Theorem 3.11, we can find that the Hausdorff dimension of these residual sets are equal. In fact, computations by Thomas and Dhar estimate the Hausdorff dimension of the residual set of any Apollonian circle packing to be 1.30568673 with a possible error of 1 in the last digit, as seen in [10].

Definition 4.3. The **packing constant** e of an Apollonian packing P is defined to be

$$e(P) := \sup\{e : \sum_{C \in P} r(C)^e = \infty\} = \inf\{e : \sum_{C \in P} r(C)^e < \infty\}$$

where $r(C)$ is the radius of a circle C.

Boyd has shown that the Hausdorff dimension of the residual set of an Apollonian packing is equal to the packing constant of an Apollonian packing. He shows that $\dim_H P \leq e(P)$ in [7]. The other direction is proven in [8].

We write $C(a,r), \overline{C(a,r)}$ for the open and closed circles with center a and radius r.

Definition 4.4. Let U be a non-empty open set in $\mathbb{R}^2$. A **simple osculatory packing** of P is a sequence (possibly finite) of circles $\{C_n\}$ with radii $\{r_n\}$ such that

(1) $C_1 \subset U$ and $r_1 = \sup_{y\in U} d(y, U^C)$,
(2) for $n \geq 1, C_{n+1} \subset R_n = U \setminus \bigcup_{k=1}^n \overline{C_k}$, and $r_{n+1} = \sup_{y\in R_n} d(y, R_n^C)$.

Definition 4.5. An **osculatory packing** of U is a sequence of pairwise disjoint circles $\{C_n\}$ (each with radius r_n) contained in U such that for some $m \geq 1, \{C_n\}_{n\geq m}$ is a simple osculatory packing of R_m, where R_m is defined in the same fashion as in Definition 4.4. Similarly, $e(S)$ where S is an osculatory packing of U is defined

$$e(S) = \inf\{a : \sum_{n=1}^{\infty} r_n^a < \infty\}.$$

In particular, an Apollonian Packing is an osculatory packing (the recursion in the definition of simple osculatory packings is equivalent to the recursion that generates an Apollonian Packing, though worded differently).

Lemma 4.6. *Let U be a non-empty open set of finite measure and let $\{C_n\}$ be an osculatory packing of U, with $C_n = C(a_n, r_n)$ and m as in Definition 4.5. Then for any $t = m, m+1, \ldots,$*

$$U \subset \bigcup_{n=1}^{t-1} \overline{C(a_n, r_n)} \cup \bigcup_{n=t}^{\infty} \overline{C(a_n, 2r_n)},$$

where $C(a,r)$ denotes a circle centered at a with radius r.

The proof of this lemma is left in [7]. With this lemma, we can see the connection between the Hausdorff dimension of the residual set and the packing constant.

Theorem 4.7. *Let $S_0 = \{C_n\}$ be an osculatory packing of U. Then $\dim_H S_0 \leq e(S_0)$.*

Proof. By the lemma,

$$\begin{aligned} R &= \bigcap_{t=m}^{\infty} (U \setminus \bigcup_{n=1}^{t-1} \overline{C_n}) \\ &\subset \bigcap_{t=m}^{\infty} (\bigcup_{n=t}^{\infty} \overline{C(a_n, 2r_n)}) \\ &\subset \bigcup_{n=t}^{\infty} \overline{C(a_n, 2r_n)} \end{aligned}$$

for any $t \geq m$. Given $\delta > 0$, there is a t such that $2r_n < \delta$ for $n \geq t$. We can choose a such that $e(S_0) \leq a \leq N$ such that $\sum_{n=1}^{\infty} r_n^a < \infty$. Then,

$$\sum_{n=t}^{\infty} \{|\overline{C(a_n, 2r_n)}|\}^a \leq 4^a \sum_{n=1}^{\infty} r_n^a < \infty.$$

This implies that $\dim_H S_0 \leq e(S_0)$, as desired. □

Theorem 4.7 shows the inequality $\dim_H S_0 \leq e(S_0)$ holds. As mentioned earlier, the Apollonian packing is an osculatory packing of the largest circle in any given Apollonian packing. Thus, $\dim_H P \leq e(P)$ is true. Again, the other direction is proven in [8].

Definition 4.8. The **circle counting function** $N_P(T)$ is the number of circles in a given Apollonian packing P whose curvature is no larger than T.

Boyd was able to compare the circle counting function with the packing constant, and in other words, the Hausdorff dimension of the residual set. In particular, by [9], Boyd showed that

Theorem 4.9. *For a bounded Apollonian circle packing P, the circle counting function $N_P(T)$ satisfies*

$$\lim_{T\to\infty} \frac{log(N_P(T))}{log(T)} = \dim_H R$$

That is, $N_P(T) = T^{\dim_H R + o(1)}$ as $T \to \infty$.

This theorem tells us that for integral Apollonian packings, the distribution of circles according to their curvature is asymptotically approximately the same.

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