MATHEMATICAL FOUNDATIONS OF TOPOLOGICAL QUANTUM COMPUTATION

CHIA-HSUN LEE

Abstract. We explore the mathematical foundations of topological quantum computation, a quantum computation model that is based on principles of topology which as a result is more resistant to quantum decoherence than existing models. From the generalization of the topological basis for the two common particle exchange statistics, we explore the possibility of particles that exhibit arbitrary exchange statistics called anyons. We will also look at the mathematical justification for the search for non-abelian anyons, which are instrumental in creating topological quantum computers.

Contents

1. Quantum Particle Statistics

A key principle in quantum mechanics is the concept of indistinguishable particles. We can tell particles apart by measuring their charge, mass, or by tracking the precise position of each individual particle as it moves. For example, to tell an electron from a proton, we might check its electric charge. However, it is impossible to tell an electron from an electron just by checking its charge, mass, spin, etc. The only way we can hope to tell two electrons apart is by tracking the precise trajectory of each of the electrons. However, that is against the principle of quantum theory that particles do not possess a specific position between measurements. Instead, their positions are described by a wave function.

The wave function $\Psi : M^n \to \mathbb{C}$ describes the state of an n indistinguishable particle system on a smooth manifold M. The square modulus of the wave function $|\Psi(v)|^2$ is interpreted as the probability density function of a system to be in a given state $v \in M^n$ [2]. That is, if we wish to know the probability of n particles to be

Date: August 29, 2016.

in a submanifold $R \subset M$, we can simply evaluate

$$\int_R |\Psi(v)|^2 \, \mathrm{d}v$$

Naturally, as the square modulus is viewed as a probability function, we get the normalization condition of wave functions:

$$\int_M |\Psi(v)|^2 \, \mathrm{d}v = 1$$

As the wave function describes the state of indistinguishable particles, one must expect there to be some amount of symmetry under the interchanging of particles. Exchange statistics arise from this expectation.

Exchange measures how a wave function would change under an exchange of particles. As explained in the previous section, we would like to preserve certain amounts of symmetry under exchange, since our particles are indistinguishable. Thus under an exchange, we expect the square modulus of our wave function, the probability function of our state, to remain unchanged and therefore respect the normalization condition,

$$\Psi(\sigma v) = \rho(\sigma)\Psi(v)$$

Observe that the phase factor $\rho(\sigma)$ has to be a unitary transformation under our restrictions.

In a two particle system in three dimensions, we can identify two possible exchange statistics. The Bose–Einstein statistics leave the wave function in a symmetric state,

$$\Psi(y, x) = \Psi(y, x)$$

and the Fermi–Dirac statistics leave the wave function in an anti-symmetric state.

$$\Psi(y, x) = -\Psi(x, y)$$

Particles that satisfy the Bose–Einstein statistics we call bosons and particles that satisfy the Fermi–Dirac statistics we call fermions.

In two dimensions, however, there can be particles that don't exhibit bosonic nor fermionic behavior but have arbitrary exchange statistics. Particles with such kind of exchange statistics are called anyons, as they can exhibit arbitrary exchange statistics. This is also where the term fractional exchange statistics is from. We will be exploring the mathematical theories that allow such a phenomenon to occur in two dimensions while restricting to exactly two possible exchange statistics in three, and in fact higher, dimensions.

See Baez p.p. 133 for a more detailed treatment of the basic principles of quantum mechanics.

2. Braid Groups

The Aharonov–Bohm effect tells us that the phase factor of a wave function under a particle exchange is affected by the path a particle takes [2]. The effect states that if a particle with charge q traveling along a path P in a zero magnetic field but a nonzero magnetic vector potential A (given by Maxwell's equations $\nabla \times A = B = 0$), the wave function describing the system gains a phase factor,

$$\frac{q}{\hbar} \int_P A \cdot \mathrm{d}x$$

Thus this tells us that the symmetric group action, as showed briefly before, isn't the only group at work here since the path that the particle takes affects the phase factor acquired under exchange.

Here we will define, algebraically, a group, called the Artin braid group, that is similar to the symmetric group but instead of permuting points, we twist strands of string. Thus in this group, swapping the same endpoints twice is a nontrivial element in the group, whereas in the symmetric group, we have $(i\ i+1)^2 = 1$.

Definition 2.1. The braid group with n strands, B_n is given by the presentation,

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2\rangle$$

Each generator σ_i can be thought of as entangling the i-th and $(i+1)$-th strand by taking the i-th strand over the $(i+1)$-th strand. Naturally, we have an inverse element σ_i^{-1} by doing the reverse action.

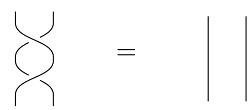
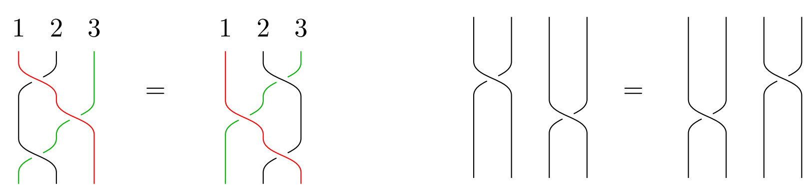


Figure 1. The braid generators and its inverse

The first set of relations mean that taking the strand on your left and crossing it over then taking the strand on your right and crossing it under is the same as crossing the right strand under first then the left strand over. Note in the following diagram that the first strand is always on top and the third strand is always on the bottom. The second set of relations means that the generators commute when the strands they are working on are sufficiently far apart.



(a) First set (b) Second set

Figure 2. Relations of the braid group

Observe that we have a natural homomorphism π into the symmetric group $B_n \to S_n$ by sending the braid to the permutation of the end points,

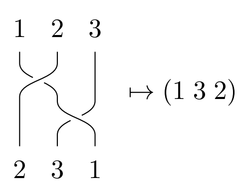


Figure 3. Homomorphism into the symmetric group

We define the pure braid group P_n as the kernel of the homomorphism π of the braid group into the symmetric group. Note that the pure braid group has generators given by wrapping a strand around another strand and returning its original endpoint.

$$\alpha_{i,j} = \sigma_{j-1}\sigma_{j-2}\dots\sigma_i^2\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\dots\sigma_{j-1}^{-1} \text{ for } i < j$$

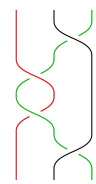


FIGURE 4. The generator $\alpha_{1,3}$ of the pure braid P_3

We are able to obtain the following relations for the pure braid group from the relations for the braid group [3],

$$\alpha_{r,s}\alpha_{i,j}\alpha_{r,s}^{-1} = \begin{cases} \alpha_{i,j} & s < i \text{ or } j < r \\ \alpha_{i,s}^{-1}\alpha_{i,j}\alpha_{i,s} & i < j = r < s \\ \alpha_{i,j}^{-1}\alpha_{i,r}^{-1}\alpha_{i,j}\alpha_{i,r}\alpha_{i,j} & i < r < j = s \\ \alpha_{i,s}^{-1}\alpha_{i,r}^{-1}\alpha_{i,s}\alpha_{i,r}\alpha_{i,j}\alpha_{i,r}^{-1}\alpha_{i,s}^{-1}\alpha_{i,r}\alpha_{i,s} & i < r < j < s \end{cases}$$

These relations for the pure braid group and the braid group all correspond to the Reidemeister moves in knot theory. This is more apparent in the braid group where $\sigma_i\sigma_i^{-1} = 1$ corresponds to the second Reidemeister move and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ corresponds to the third Reidemeister move.

By definition we get the following short sequence where i is the inclusion of P_n into B_n.

$$1 \longrightarrow P_n \overset{i}{\hookrightarrow} B_n \overset{\pi}{\twoheadrightarrow} S_n \longrightarrow 1 \tag{2.2}$$

Lemma 2.3. *There is a homomorphism $p_i : P_i \to P_{i-1}$ given by removing the last strand,*

$$p_n(\alpha_{i,j}) = \begin{cases} 1 & j = n \\ \alpha_{i,j} & \text{otherwise} \end{cases}$$

Let $F_{n-1} = \ker(p_n)$. Then the following is an exact sequence where i is the inclusion of F_{n-1} into P_n,

$$1 \longrightarrow F_{n-1} \overset{i}{\hookrightarrow} P_n \overset{p_n}{\twoheadrightarrow} P_{n-1} \longrightarrow 1 \tag{2.4}$$

and F_{n-1} is the free group on $n-1$ letters.

The proof is left as an exercise for the reader. Note that F_{n-1} is generated by the generators $\alpha_{1,n}, \dots, \alpha_{n-1,n}$ of P_n as no matter how you wrap around the n-th strand, they all get untangled once you take that strand away.

3. Configuration Spaces

Observe that in the previous section, the strands in each braid look like the trajectory of an exchange over time. Each strand cannot loop back on itself, representing the inability for a particle to travel back in time, nor can it pass through another strand, the inability for two particles to occupy the same space at a time. In this section, we will show that the trajectories of particles in an exchange correspond to braids in the braid group. But first, we will need a way to track the trajectories of particles in a particle system.

Definition 3.1. The ordered configuration space of n points on M is a sub-manifold of M^n defined as,

$$F_n(M) = \{(x_1, x_2, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

Intuitively, each point in $F_n(M)$ represents a possible configuration of n particles on our manifold and we can track the positions of n particles as they evolve over time by looking at paths in $F_n(M)$. While in reality, we can't expect particles to follow a definite continuous path like this due to quantum mechanics, but in a loose sense, the "fuzziness" cancels out in a way we are able to calculate the phase factor gained through the Aharonov–Bohm effect using continuous paths. We won't delve into the details as that would require some measure theory and is out of the scope of this paper, you can find more details of this in [2]. Note that we disallow particle creation or annihilation in our particle system under our model. Also note that the paths the particles trace out over time cannot overlap as that would mean two particles will be occupying the same position at a time. Both properties are what we expect in an exchange.

However, the ordered configuration space is able to distinguish between particles, which is undesirable. The following configurations of particles are different in the ordered configuration space but are in principle indistinguishable through measurement.

$$(v_1, v_2, v_3) \quad \neq \quad (v_3, v_2, v_1)$$

As we are interested in indistinguishable particles, we will need a way to identify configurations in $F_n(M)$ that are the same under observation. Therefore instead of looking at n-tuples in M^n, we need to look at sets of size n in M^n. Thus we will need to identify permutations of our n particles in the ordered configuration space. We can do that by defining an S_n-action, $\varphi : S_n \times F_n(M) \to F_n(M)$, on the space, which will allow us to get another indistinguishable configuration by permuting the particles.

$$\varphi(\sigma, x_1, x_2, \dots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Identity and compatibility of the group action should be clear from our definition. $\varphi(\sigma, v)$ will be denoted σv from now on for convenience. We will call the space obtained by identifying a configuration v with gv for all $g \in S_n$ the unordered configuration space.

Definition 3.2. The unordered configuration space of n particles on a manifold M is defined as all the possible configurations of n indistinct points in M,

$$C_n(M) = F_n(M)/S_n$$

Note that this is essentially saying the following,

$$C_n(M) = \{\{x_1, x_2, \dots, x_n\} \subset M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

Now that we've set up a space that allows us to track n indistinguishable particles, it should be clear that paths in our ordered configuration space get turned into loops since we've identified the endpoints. Thus loops in this space represent the trajectories of our n particles under a particle exchange. Furthermore, Stokes theorem suggests that we look at homotopic loops, loops that can be continuously deformed to one another, as the Aharonov–Bohm effect tells us that the phase difference $\Delta\varphi$ of particles taking two different paths is determined by the magnetic flux of the area S enclosed by the two paths (again given by $B = \nabla \times A$ from Maxwell's equations),

$$\Delta\varphi = \frac{q\Phi_B}{\hbar} = \frac{q}{\hbar}\iint_S B \cdot \mathrm{d}s = \frac{q}{\hbar}\oint_{\partial S} A \cdot \mathrm{d}r$$

If the two paths are homotopic, the boundary of the enclosed area must be nullhomotopic. Therefore the phase difference $\Delta\varphi$ must be 0. In our configuration space, homotopic loops (in the unordered space) and homotopic paths (in the ordered space) represent particle trajectories being homotopic. Thus we will be studying homotopic loops of the configuration space. The fundamental group, which identifies homotopic loops and gives them a group structure [6], will provide us with the tool to see how different loops in the space will affect the wave function of the system.

We will assume that we are working in $\mathbb{R}^m$ where $m \geq 2$, as in one dimension we can't make nontrivial exchanges because no particles can cross over each other without occupying the same space. For $m \geq 2$ then, both $F_n(\mathbb{R}^m)$ and $C_n(\mathbb{R}^m)$ are path-connected as it is possible to go to an arbitrary configuration given an initial configuration. Since the spaces we will be working with are path-connected, the fundamental groups of the space at any base point are all isomorphic to each other through the change of base point map [6]. Thus we will be using a fixed base point in the following proofs for convenience. We shall define the following,

$$e_i = (0, \ldots, \underbrace{1}_{i\text{-th}}, \ldots 0)$$

Then we will be using the following as the basepoint for the ordered configuration space $F_n(\mathbb{R}^m)$,

$$\widetilde{x}_0 = (e_1, e_2, \ldots, e_n)$$

and for the unordered configuration space $C_n(\mathbb{R}^m)$,

$$x_0 = \{e_1, e_2, \ldots, e_n\}$$

First we will look at a result that will be useful in understanding the fundamental group of the configuration space.

Theorem 3.3. *The projection $F_n(M) \to F_{n-1}(M)$ removing the last coordinate is a fiber bundle with fiber $M - P$ where P is a set of $n-1$ points in M.*

Proof. The proof is in Fadell–Neuwirth [4]. □

The above fiber bundle gives as a long exact sequence of homotopy groups, which will be useful in proving the following theorems. We will now relate the braid group with the fundamental group of the unordered configuration space at the base point x_0, with the following homomorphism into the fundamental group $h_i : B_i \to \pi_1(C_i(\mathbb{R}^m), x_0)$,

$$\sigma_k \mapsto [\gamma : t \mapsto \{e_1, \ldots, e_{k-1}, (1-t^2)e_k + (2t-t^2)e_{k+1}, t^2 e_k + (1-t^2)e_{k+1}, e_{k+2}, \ldots, e_n\}]$$

sending each generator σ_k to the action of exchanging the particles at e_k and e_{k+1}.

Lemma 3.4. *The fundamental group of $F_n(\mathbb{R}^2)$ is the pure braid group P_n.*

Proof. To prove this we will be using the short five lemma, which states that given the following diagram in the category of groups, if the rows are short exact sequences and g, h are group isomorphisms, then f is a group isomorphism as well.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \\ 1 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 1 \end{array}$$

Recall that we have the short exact sequence in (2.4), we also have a short exact sequence of fundamental groups given by the fiber bundle $\mathbb{R}^2 \setminus Q \to F_n(\mathbb{R}^2) \to F_{n-1}(\mathbb{R}^2)$. Let h_i be a homomorphism between P_i and $\pi_1(F_i(\mathbb{R}^2))$ as defined previously, then clearly the following diagram commutes,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_{n-1} & \hookrightarrow & P_n & \xrightarrow{p_n} & P_{n-1} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle h_n|_{F_{n-1}}} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n-1}} & & \\ 1 & \longrightarrow & \pi_1(\mathbb{R}^2 \setminus Q, \widetilde{x}_0) & \hookrightarrow & \pi_1(F_n(\mathbb{R}^2), \widetilde{x}_0) & \xrightarrow{\widetilde{p_n}} & \pi_1(F_{n-1}(\mathbb{R}^2), x_0) & \longrightarrow & 1 \end{array}$$

Recall that F_{n-1} is the free group on $n-1$ letters. Since $\mathbb{R}^2$ minus $n-1$ points is homotopy equivalent to the wedge sum of $n-1$ circles, thus the fundamental group of $\mathbb{R}^2 \setminus Q$ is also the free group on $n-1$ letters. Each of the generators of F_{n-1}, $\alpha_{i,n}$, is mapped to the loop that encircles the i-th hole once. Thus $h_n|_{F_{n-1}}$ must be an isomorphism.

Now we proceed by induction. Note that $\pi_1(F_1(\mathbb{R}^2), \widetilde{x}_0)$ is the trivial group as $F_1(\mathbb{R}^2) = \mathbb{R}^2$ is contractible, and P_1 is also trivial as you can't make a nontrivial braid with just one strand. Thus $\pi_1(F_1(\mathbb{R}^2), \widetilde{x}_0) \cong P_1$. Now suppose $h_{i-1} : P_{i-1} \to \pi_1(F_{i-1}(\mathbb{R}^2), \widetilde{x}_0)$ is an isomorphism. Then by the five lemma h_i must also be an isomorphism. This proves the lemma by induction. □

Theorem 3.5. *The fundamental group of $C_n(\mathbb{R}^2)$ is the braid group B_n.*

Proof. We will prove this theorem again by using the short five lemma. But first we will need to show that the quotient map $F_n(\mathbb{R}^2) \to F_n(\mathbb{R}^2)/S_n$ is a covering projection, as that would mean there is a natural surjective homomorphism from $F_n(\mathbb{R}^2)/S_n = C_n(\mathbb{R}^2) \to S_n$ given by the unique path lifting property. Intuitively, the homomorphism sends a braid to its permutation of endpoints. Proposition 1.40 in [6] tells us that it suffices to show that the S_n-action is topologically free (properly discontinuous) for the quotient map to be a covering projection.

A group action is topologically free when every point admits an open neighborhood such that their orbits do not overlap. That is, for every point $p \in F_n(\mathbb{R}^2)$ there exists an open neighborhood U such that if $g(U) \cap U \neq \varnothing$ then $g = e$. Te see this fact, let $(x_1, \ldots, x_n) \in F_n(M)$ with Hausdorff M. Find pairwise disjoint open U_i such that $x_i \in U_i$ and define $U = \prod_{i=1}^n U_i$. Then suppose $(x_{g(1)}, \ldots, x_{g(n)}) \in g(U) \cap U$. Then it is the case that $x_i \in U_i$ and $x_{g(i)} \in U_i$. However, the U_i are pairwise disjoint and therefore for all $1 \leq i \leq n$, we have that $g(i) = i$. Thus $g = e$. [5].

Thus we have our short exact sequence now that we've shown that the quotient map is a covering projection,

$$1 \to \pi_1(F_n(\mathbb{R}^2), \widetilde{x}_0) \to \pi_1(C_n(\mathbb{R}^2), x_0) \xrightarrow{\widetilde{\pi}} S_n \to 1$$

Now recall that we have the short exact sequence (2.2). Let f_n be a homomorphism from B_n to $\pi_1(C_n(\mathbb{R}^2))$, clearly the following diagram commutes,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_n & \overset{i}{\hookrightarrow} & B_n & \overset{\pi}{\longrightarrow} & S_n & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \pi_1(F_n(\mathbb{R}^2), \widetilde{x}_0) & \overset{p}{\longrightarrow} & \pi_1(C_n(\mathbb{R}^2), x_0) & \overset{\widetilde{\pi}}{\longrightarrow} & S_n & \longrightarrow & 1
\end{array}$$

Since h_n is an isomorphism, by the short five lemma, f_n must also be an isomorphism. □

Theorem 3.6. *The fundamental groups of $F_n(\mathbb{R}^m)$ and $C_n(\mathbb{R}^m)$ for $m > 2$ are, respectively, the trivial group 0 and the symmetric group S_n.*

Proof. From the fiber bundle we get the long exact sequence on homotopy groups,

$$\begin{aligned}
\cdots \to &\pi_n(\mathbb{R}^m \setminus Q) \to \pi_n(F_n(\mathbb{R}^m)) \to \pi_n(F_{n-1}(\mathbb{R}^m)) \to \pi_{n-1}(\mathbb{R}^m \setminus Q) \\
&\to \pi_{n-1}(F_n(\mathbb{R}^m)) \to \pi_{n-1}(F_{n-1}(\mathbb{R}^m)) \to \cdots \to \pi_1(F_{n-1}(\mathbb{R}^m)) \to 1
\end{aligned}$$

Note that when $m > 2$, then the space $\mathbb{R}^m \setminus P$, obtained by taking finitely many points out of $\mathbb{R}^m$, is simply connected, therefore $\pi_1(\mathbb{R}^m \setminus Q) = 0$. Thus we have this short exact sequence:

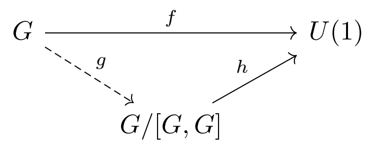
$$0 = \pi_1(\mathbb{R}^m \setminus P) \to \pi_1(F_n(\mathbb{R}^m)) \to \pi_1(F_{n-1}(\mathbb{R}^m)) \to 0$$

Then we can induct on $\pi_1(F_1(\mathbb{R}^m)) = \pi_1(\mathbb{R}^m) = 0$ to obtain $\pi_1(F_n(\mathbb{R}^m)) = 0$. This proves the statement for $F_n(\mathbb{R}^m)$.

Since the S_n action on $F_n(\mathbb{R}^m)$ is topologically free, thus from proposition 1.40 in [6], we have that S_n is isomorphic to $\pi_1(C_n(\mathbb{R}^m))/\pi_1(F_n(\mathbb{R}^m))$. Since we've already shown that $\pi_1(F_n(\mathbb{R}^m))$ is trivial, the fundamental group of $C_n(\mathbb{R}^m)$ must be S_n. □

4. Abelian Exchange Statistics

The wave function we're working with is a complex valued function (quantum states with no degeneracy), thus we are looking at the representations (homomorphisms into vector spaces) of $\pi_1(C_n(\mathbb{R}^m))$ in $U(1) \hookrightarrow \mathbb{C}$. Each different representation of the fundamental group of our configuration space gives us a different possible exchange statistics. The unitary group of dimension 1 is abelian, hence the name abelian exchange statistics, thus any representation of the fundamental group into $U(1)$ must factor uniquely through the abelianization map [1] (for any $G \to H$ where H is abelian, there exists a unique map from $G/[G,G] \to H$ such that the following diagram commutes).



Note that since $U(1) \cong \mathbb{S}^1$ embedded in $\mathbb{C}$ is simply the unit circle, any phase factor gained in a particle exchange in a nondegenerate particle system can only be in the form of $e^{\theta i}$.

Proposition 4.1. *The abelianization of S_n is isomorphic to the cyclic group of order 2, $\mathbb{Z}/2\mathbb{Z}$ and the abelianization of B_n is isomorphic to the infinite cyclic group $\mathbb{Z}$.*

Sketch of Proof. We can think of the abelianization of a group as forcing all elements to commute, that is, for any $g, h \in S_n$, we make $ghg^{-1}h^{-1} = 1$. Thus to abelianize a group, we add the above relation to the group. If we force the braid group to commute, we get from the following relation, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, which implies $\sigma_i = \sigma_{i+1}$. Thus the abelianization of the braid group,

$$B_n^{\text{ab}} = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i = \sigma_{i+1}, \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1 \rangle = \langle \sigma_1 \rangle \cong \mathbb{Z}$$

is the group generated by one element of infinite order, which is the infinite cyclic group $\mathbb{Z}$. Recall that we have the homomophism of the braid group into the symmetric group by sending each braid to the permutation of the end points. Thus clearly, we have that each generator σ_i of the braid group is mapped to the generator $(i\ i+1)$ of the symmetric group. Furthermore, since for the symmetric group, $(i\ i+1)^2 = 1$, we have that the abelianization of the symmetric group is simply,

$$S_n^{\text{ab}} = \langle \tau = h(\sigma_1) \mid \tau^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

the cyclic group of order 2, $\mathbb{Z}/2\mathbb{Z}$. □

Remark 4.2. The abelianization of a group is the group quotient its commutator subgroup, the subgroup generated by all its commutators. For the symmetric group S_n, its commutator subgroup is the alternating group A_n as, it contains all 3-cycles,

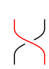
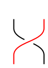
$$(a\ b\ c) = (a\ c\ b)^2 = ((a\ b)(a\ c))^2 = [(a\ b), (a\ c)] \in [S_n, S_n]$$

and the alternating group contains precisely all permutations generated by 3-cycles. The alternating group contains all even permutations, thus the abelianization map is the signature of a permutation $\text{sgn} : S_n \to \mathbb{Z}/2\mathbb{Z}$, which sends each permutation to 0 or 1 depending on its parity.

As for the braid group, its commutator subgroup is the group containing all braids with writhe 0. The writhe of a braid is the sum of exponents of generators,

$$\text{Wr}\,(\sigma_1^{a_1}\sigma_2^{a_2}\ldots\sigma_i^{a_i}) = a_1 + a_2 + \ldots a_i$$

or, equivalently and more intuitively, the number of left-handed coressings minus the number of right-handed crossings,

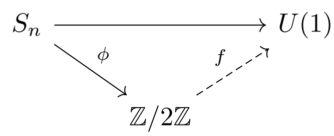
$\text{Wr}(\gamma) = \#$ of  $- \#$ of 

Then clearly, $\text{Wr} : B_n \to \mathbb{Z}$ is exactly the abelianization map of the braid group.

Now that we've found the abelianization of the symmetric group and the braid group, we can now find all the representations of those groups into $U(1)$ by just looking at representation of their abelianization.

Theorem 4.3. *There can only be two exchange statistics in three and higher dimensions, bosons and fermions.*

Proof. Since $\pi_1(C_n(\mathbb{R}^m)) = S_n$ for $m \geq 3$, thus we will be looking at representations of the form $\mathbb{Z}/2\mathbb{Z} \to U(1)$.



A representation of a cyclic group is completely determined by where 1 is sent. For $\mathbb{Z}/2\mathbb{Z}$ there are only 2 possible places to send it to, namely, 1 and -1.

If 1 is sent to 1, we get the trivial map and the resulting exchange statistics are the bosonic statistics that leave the particle system in a symmetric state, $\Psi(\sigma v) = \Psi(v)$. Otherwise, we get the fermionic particle statistics, that leave the system in a anti-symmetric state if the exchange is an odd permutation of particles, $\Psi(\sigma v) = \text{sgn}(\sigma)\Psi(v)$. ☐

Theorem 4.4. *Particles in two dimensions can exhibit arbitrary exchange statistics.*

Proof. In two dimensions, since the fundamental group of $C_n(\mathbb{R}^2)$ is B_n whose abelianization is $\mathbb{Z}$, we will be looking at representations of $\mathbb{Z}$ in $U(1)$. However, since 1 can now be sent to any place on the unit circle in $\mathbb{C}$ there are infinitely many possible representations of $\mathbb{Z}$ and therefore there are infinitely many possible exchange statistics in $\mathbb{R}^2$. ☐

5. Topological Quantum Computation

By requiring there to be no degeneracy and for exchanges to be abelian we are losing a lot of information that is encoded in the braid that is created from our exchange. Thus it is desireable to be able to retain that information. Therefore motivating non-abelian exchange statistics, which arise from degeneracy in the particle system, where the same particle configuration exhibits different wavefunctions. It is represented by having a vector space of states that all correspond to the same configuration. Thus instead of representations of the braid group into $U(1)$, we are now dealing with representations into $U(n)$, which is non-abelian, hence the name. This allows for much more types of statistics that retain much more information of the exchange and is in fact essential to realizing topological quantum computation. Topological quantum computation relies on the arbitrariness of exchange statistics of such anyons and since it is based on topological attributes of particle exchanges, small disturbances would not affect the state of the quantum system. By employing these principles in quantum computation, this model effectively solves the problem of quantum decoherence that plagues many existing models and is the biggest hurdle to overcome in creating reliable quantum computers [8]. Topological quantum computers are also compatible with existing quantum computers so current developments in quantum algorithms and such can be adopted without modification.

Furthermore, there are algorithms that make use of the braiding of particles, further harnessing the topological nature of topological quantum computation [7].

As of the time of writing, abelian anyons have been observed through the Fractional Quantum Hall Effect and there is strong evidence to believe in the existence of non-abelian anyons, which is permissible in the spin-statistics theorem.

Acknowledgements

I would like to thank my mentor, Claudio Gonzales, for introducing me into this deeply interesting topic and giving me a succinct overview of the prerequisite mathematical knowledge. I would also like to thank, Agnès Beaudry, and Peter May, for inspiring my interest in algebraic topology and without whom I would not have the opportunity to participate in the REU. Lastly, I would like to thank Joe Klafka and Matthew Morgado for providing me with feedback and memes.

Revisions

Jan 17th, 2017. A previous version of this article used a metric approach to proving the fact that the symmetric group action on $F_n(\mathbb{R}^2)$ is topologically free. My mentor, Claudio, mentioned that an easier and more elegant proof is possible using the fact that $\mathbb{R}^n$ is Hausdorff.

You may contact me at chl [at] uchicago.edu for corrections and inquiries.

References

[1] *Algebra, Graduate Texts in Mathematics*. Springer, 2002.
[2] John Baez and Javier P. Munian. *Gauge Fields, Knots and Gravity*. 1st ed. Vol. 4. Series on Knots and Everything. World Scientific, 1994.
[3] Joan S. Birman and Tara E. Brendle. *Braids: A Survey*. 2005.
[4] Edward Fadell and Lee Neuwirth. "Configuration Spaces". In: *Mathematica Scandinavica* 10 (1962).
[5] Claudio Gonzales. "Topological Quantum Computation". Private communication.
[6] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.
[7] Michael A. Levin and Xiao-Gang Wen. *String-net condensation: A physical mechanism for topological phases*. 2004.
[8] Jiannis Pachos. *Introduction to Topological Quantum Computation*. 2012.