VAUGHT'S THEOREM: THE FINITE SPECTRUM OF COMPLETE THEORIES IN $\aleph_0$

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Abstract. This expository paper introduces model theory with a focus on countable models of complete theories. Vaught's theorem concerns the possible finite values for the spectra of such theories. Properties of atomic and saturated models are also explored, with a focus on isomorphism classes for a given theory.

Contents

1. Preliminary Definitions

Model theory studies the relation between a language and its structures. This section aims to provide a working understanding for the reader unfamiliar with these terms.

1.1. Collections of Symbols.

Definition 1.1. A **signature** is a triple

$$\tau = \{S_F, S_R, \phi\}$$

where S_F is a set of function symbols, and S_R a set of relation symbols.

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The function $\phi : S_F \cup S_R \to \mathbb{N}$ associates each symbol with an arity. Here, we take $\mathbb{N}$ to include 0.

Example 1.2. The signature of rings is

$$\tau = \{\{+, *, -, 0, 1\}, \emptyset, \phi\}$$

with $\phi(+) = \phi(*) = 2$, $\phi(-) = 1$, $\phi(0) = \phi(1) = 0$.

Example 1.3. The signature of graphs is one binary relation symbol

$$\tau = \{\emptyset, \{E\}, \phi\}$$

with $\phi(E) = 2$. This is commonly used to signify the existence of an edge between two given vertices.

Definition 1.4. A **language** $\mathcal{L}$ is a collection of relation, function, and constant symbols

$$\mathcal{L} = \{r_1, \dots, r_i, f_1, \dots, f_j, c_1, \dots, c_k\}$$

with each relation and function symbol having an arity $n \in \mathbb{N}$.

A language distinguishes constants as a third set of symbols instead of allowing nullary functions and relations like in signatures. For example, the language of rings would be written similarly to its signature above, but with 0 and 1 understood to be constant symbols instead of nullary function symbols.

1.2. **Structures.**

Definition 1.5. Fix a language $\mathcal{L}$. A **model** of $\mathcal{L}$

$$\mathfrak{A} = \langle A, \mathfrak{I} \rangle$$

is a pair consisting of a universe A and an interpretation function $\mathfrak{I}$, which maps the symbols of $\mathcal{L}$ to their interpretation in $\mathfrak{A}$ in the following manner:

A function symbol f_i in $\mathcal{L}$ with arity n_i is mapped to a function $f : A^{n_i} \to A$, a relation symbol r_j in $\mathcal{L}$ with arity n_j to an n_j-placed relation in A, and a constant symbol c_k in $\mathcal{L}$ to an element $a \in A$.

It is common to write out the interpretations for the symbols in the language rather than defining the interpretation function itself. For example, a model for the language

$$\mathcal{L} = \{+, *, 0, 1\}$$

could be denoted as

$$\mathfrak{A} = \langle \mathbb{Z}, +^{\mathfrak{A}}, *^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}} \rangle$$

with $+^{\mathfrak{A}}, *^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}$ interpreted as addition, multiplication, and the identities in the ring of integers.

Note: models and corresponding universes will be denoted by the same letter (i.e. the models $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$ have universes A, A', B) unless stated otherwise, such as examples using $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, etc.

Definition 1.6. Suppose we have a model $\mathfrak{A} = \langle A, \mathfrak{I} \rangle$ of a language $\mathcal{L}$. We can expand $\mathcal{L}$ by taking a set of symbols S disjoint with $\mathcal{L}$ to form $\mathcal{L}' = \mathcal{L} \cup S$. Taking an interpretation function $\mathfrak{J}$ mapping S to interpretations in A, a natural extension of $\mathfrak{A}$ to a model of $\mathcal{L}'$ would be the model $\mathfrak{A}' = \langle A, \mathfrak{I} \cup \mathfrak{J} \rangle$. Suppose $S = \{s_1, \dots, s_n\}$. As convenient notation, we may write $\mathfrak{A}' = (\mathfrak{A}, s_1^{\mathfrak{A}'}, \dots, s_n^{\mathfrak{A}'})$. We say $\mathfrak{A}'$ is the **expansion** of $\mathfrak{A}$ to $\mathcal{L}'$.

Also, given a model $\mathfrak{A}' = \langle A, \mathfrak{I}\rangle$ of a language $\mathcal{L}'$ and a language $\mathcal{L} \subset \mathcal{L}'$, let $\mathfrak{J}$ be the restriction of $\mathfrak{I}$ to $\mathcal{L}$. Then we say $\mathfrak{A} = \langle A, \mathfrak{J}\rangle$ is the **reduct** of $\mathfrak{A}'$ to $\mathcal{L}$.

Definition 1.7. Two models $\mathfrak{A}, \mathfrak{A}'$ of a language $\mathcal{L}$ are **isomorphic**, denoted $\mathfrak{A} \cong \mathfrak{A}'$, if there exists a bijection $g : \mathfrak{A} \to \mathfrak{A}'$ such that

$$r_i^{\mathfrak{A}}(a_1, \dots, a_n) \iff r_i^{\mathfrak{A}'}(g(a_1), \dots, g(a_n))$$
$$g(f_j^{\mathfrak{A}}(a_1, \dots, a_n)) = f_j^{\mathfrak{A}'}(g(a_1), \dots, g(a_n))$$
$$g(c_k^{\mathfrak{A}}) = c_k^{\mathfrak{A}'}$$

for all relation symbols r_i, function symbols f_j, and constant symbols c_k in $\mathcal{L}$.

1.3. **Formulas and consistency.** The set $\mathcal{S}$ of logical symbols consists of the following:

$$), (\text{ parentheses}$$
$$v_0, \dots, v_n, \dots \text{ variables}$$
$$\wedge, \neg \text{ connectives (and, not)}$$
$$\forall \text{ quantifier (for all)}$$
$$\equiv \text{ binary equivalence relation}$$

Definition 1.8. The **cardinal of a language** is

$$||\mathcal{L}|| = \max(\omega, |\mathcal{L}|) = \omega \cup |\mathcal{L}|$$

i.e. the cardinal of a language is never finite. This is because the formulas in $\mathcal{L}$ are constructed from $\mathcal{L} \cup \mathcal{S}$, where $|\mathcal{S}| = \omega$ because $\mathcal{S}$ contains countably many variables.

A language is **countable** if its cardinal is countable. This paper deals primarily with countable languages.

The **cardinal of a model** $\mathfrak{A}$ is the cardinal of its universe $|A|$.

Definition 1.9. A **string** of a set X is a finite ordered tuple of elements $x \in X$.

A string of $\mathcal{S} \sqcup \mathcal{L}$ is a **term** of $\mathcal{L}$ iff it can be constructed inductively by applying the following steps a finite number of times:

- A variable is a term.
- A constant symbol is a term.
- Let f be a function symbol with arity n, and $t_1, \dots, t_n$ be terms of $\mathcal{L}$. Then $f(t_1, \dots, t_n)$ is a term.

An **atomic formula** of $\mathcal{L}$ can be constructed from terms in two ways:

- Let t_1, t_2 be terms of $\mathcal{L}$. Then $t_1 \equiv t_2$ is an atomic formula.
- Let r be a function symbol with arity n, and $t_1, \dots, t_n$ be terms of $\mathcal{L}$. Then $r(t_1, \dots, t_n)$ is an atomic formula.

A **formula** of $\mathcal{L}$ is a string of symbols that can be constructed inductively through a finite number of applications of the following steps:

- All atomic formulas are formulas.
- Let ϕ, ψ be formulas of $\mathcal{L}$. Then $\phi \wedge \psi$ and $\neg\phi$ are formulas.
- Let ϕ be a formula of $\mathcal{L}$ and x a variable. Then $(\forall x)\phi$ is a formula.

Remark 1.10. We often use other logical connectives $\vee, \rightarrow, \iff$ (or, implies, iff) and the quantifier $\exists$ (there exists). These are just abbreviations of the logical connectives and quantifier above:

$$\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$$
$$\phi \rightarrow \psi = \neg\phi \vee \psi$$
$$\phi \iff \psi = (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$$
$$(\exists x)\phi(x) = \neg(\forall x)\neg\phi(x)$$

Definition 1.11. A variable that occurs in a formula that is not restricted by a quantifier is a **free variable**. We use the notation $\phi(x_1, \ldots, x_n)$ to denote a formula ϕ with (at most) n free variables.

For example, $\phi = (\forall x)(\exists y)(x = y)$ has no free variables, while $\phi(x, y) = (\exists z)((x = z) \wedge \neg(y = z))$ is free in the variables x, y. A formula with no free variables is a **sentence**.

Definition 1.12. An $\mathcal{L}$-sentence ϕ is **true** in a model $\mathfrak{A}$ of $\mathcal{L}$ iff the sentence holds when the quantifiers of ϕ range over the universe A. We write this as $\mathfrak{A} \models \phi$, read "$\mathfrak{A}$ models ϕ".

Example 1.13. Consider the language $\mathcal{L} = \{<, 0\}$ and the formula $\phi = (\exists x)(x < 0)$. Two models of $\mathcal{L}$ are $\mathfrak{A} = \langle \mathbb{N}, <, 0\rangle$ and $\mathfrak{B} = \langle \mathbb{Q}, <, 0\rangle$, with both models interpreting the binary relation symbol $<$ with its common use and the constant symbol 0 to the number 0 in both universes (here, $0 \in \mathbb{N}$). Then we say ϕ holds in $\mathfrak{B}$ but not in $\mathfrak{A}$ because there is an element $q \in \mathbb{Q}$ such that $q < 0$ but there is no element $n \in \mathbb{N}$ with $n < 0$. Note that we restrict the quantifiers to the universe of each model. We write this as

$$\mathfrak{B} \models \phi$$
$$\mathfrak{A} \not\models \phi$$

Note that $\mathcal{L}$ does not have a direct judgement of ϕ as it does not include any type of interpretation.

Definition 1.14. The notation $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$ means that by substituting the n-tuple $a_1, \ldots, a_n \in A$ for the free variables in ϕ, ϕ is now a true sentence in $\mathfrak{A}$. We say that $a_1, \ldots, a_n$ **satisfies**, or **realizes**, ϕ in $\mathfrak{A}$.

Example 1.15. Take the language $\mathcal{L}$ and models $\mathfrak{A}$ and $\mathfrak{B}$ from the last example, but now with an additional constant symbol 1, interpreted as the number 1 in both universes. Consider the formula $\phi(x) = (x < 1) \wedge (x > 0)$. There does not exist an integer $n \in \mathbb{N}$ that satisfies this formula, so $\mathfrak{A} \not\models \phi$. There do exist values $q \in \mathbb{Q}$ that satisfy this formula. For example, $\mathfrak{B} \models \phi[0.5]$, or $\mathfrak{B} \models \phi[0.99]$.

Definition 1.16. Two models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{L}$ are **elementarily equivalent** ($\mathfrak{A} \equiv \mathfrak{B}$) iff for every $\mathcal{L}$-sentence ϕ

$$\mathfrak{A} \models \phi \iff \mathfrak{B} \models \phi$$

1.4. **Sets of formulas.**

Definition 1.17. Given a model $\mathfrak{A}$ and a set of sentences Σ, we say $\mathfrak{A}$ is **a model of** Σ iff $\mathfrak{A}$ is a model of every $\sigma \in \Sigma$

$$\mathfrak{A} \models \Sigma \iff (\forall \sigma \in \Sigma)(\mathfrak{A} \models \sigma)$$

Σ is **satisfiable** iff there exists a model of Σ.

Definition 1.18. Consider a set Σ of formulas and a formula ϕ of $\mathcal{L}$. We say ϕ is **deducible** from Σ ($\Sigma \vdash \phi$) if there is a finite list of formulas $\psi_1, \ldots, \psi_n = \phi$ such that each ψ_k is some $\sigma \in \Sigma$, a logical axiom, or follows from the logical rules of deduction applied to previous formulas in the list:

- Detachment: From ϕ and $\phi \to \psi$, deduce ψ.
- Generalization: From ϕ deduce $(\forall x)\phi$.

This list of formulas is a **proof of** ϕ from Σ. We say ϕ is a **consequence** of Σ ($\Sigma \models \phi$) if every model of ϕ is a model of Σ.

Definition 1.19. Σ is **inconsistent** iff there exists an $\mathcal{L}$-formula σ such that $\Sigma \vdash (\sigma \wedge \neg\sigma)$. Otherwise Σ is **consistent**. Σ is **maximally consistent** if for every consistent set of $\mathcal{L}$-formulas Γ with $\Sigma \subset \Gamma$, $\Sigma = \Gamma$ (i.e. Σ is not properly contained in any consistent set of $\mathcal{L}$-formulas).

Definition 1.20. A **theory** T of a language $\mathcal{L}$ is a collection of sentences of $\mathcal{L}$. Suppose K is the set of all consequences of T. We say T is **complete** if K is maximally consistent, i.e. for every $\mathcal{L}$-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$. A **set of axioms** of T is a set of sentences with the same set of consequences K.

Note that T is always a set of axioms of T.

Definition 1.21. Given a model $\mathfrak{A}$ of a language $\mathcal{L}$, we can always construct the consistent theory $T = Th(\mathfrak{A})$, called the **theory of** $\mathfrak{A}$, which consists of all sentences of $\mathcal{L}$ that hold in $\mathfrak{A}$.

1.5. **Completeness and Compactness.** The Completeness and Compactness Theorems are essential tools for model theoretic proofs. Rather than significantly deviating from the topic of this paper for a thorough proof, we instead just state the completeness theorem for reference and show how compactness follows.

Theorem 1.22. ***Completeness Theorem:*** *Let Σ be a set of sentences of $\mathcal{L}$. Then Σ is consistent iff Σ is satisfiable.*

In first order logic, this is equivalent to the following:

An $\mathcal{L}$-formula ϕ is a consequence of a set of $\mathcal{L}$-formulas Σ iff ϕ is deducible from Σ.

Theorem 1.23. ***Compactness Theorem:*** *Let Σ be a set of sentences of $\mathcal{L}$. Then Σ is satisfiable iff every finite subset of Σ is satisfiable.*

Proof. Assuming the completeness theorem, compactness follows easily. Let Σ be a satisfiable set of $\mathcal{L}$-formulas. Then by completeness, Σ is consistent. Every finite subset of Σ is then consistent, so they are satisfiable.

Now let every finite subset of Σ be satisfiable. By completeness, every finite subset of Σ is consistent. Suppose for contradiction that Σ is inconsistent. Then there exists a proof of $\phi = \gamma \wedge \neg\gamma$ for some $\mathcal{L}$-formula γ. But proofs only contain finitely many formulas, so the finite subset of Σ used in the proof of ϕ is inconsistent. Therefore, Σ is consistent, so Σ is satisfiable.

$\square$

2. Omitting types

This section details the relation between certain types of theories and their models.

2.1. **Locally Realizing and Omitting.** We use the notation $\Sigma(x_1, \dots, x_n)$ to denote a set of formulas where the free variables among all σ in Σ are in $x_1, \dots, x_n$.

Definition 2.1. A model $\mathfrak{A}$ **realizes** a set of formulas $\Sigma = \Sigma(x_1, \dots, x_n)$ iff there exists an n-tuple $a_1, \dots, a_n \in A$ that satisfies every σ in Σ:

$$\mathfrak{A} \models \Sigma[a_1, \dots, a_n] \iff (\forall \sigma \in \Sigma)(\mathfrak{A} \models \sigma[a_1, \dots, a_n])$$

We may also say that $a_1, \dots, a_n$ **realizes** Σ in $\mathfrak{A}$. The phrase Σ is **satisfiable in** $\mathfrak{A}$ is equivalent to $\mathfrak{A}$ realizing Σ. If $\mathfrak{A}$ does not realize Σ, then $\mathfrak{A}$ **omits** Σ.

Definition 2.2. Let T be a theory in a language $\mathcal{L}$ and $\Sigma = \Sigma(x_1, \dots, x_n)$ a set of $\mathcal{L}$-formulas. We say that T **locally realizes** Σ iff there is an $\mathcal{L}$-formula $\psi = \psi(x_1, \dots, x_n)$ with the following properties:

1. ψ is consistent with T (i.e. $T \cup \{\psi\}$ is consistent)
2. For all σ in Σ, $T \models \psi \to \sigma$

Note that the second condition is equivalent to all σ in Σ being consequences of $T \cup \{\psi\}$. We can then connect this definition with the models of T in the following manner: given any model $\mathfrak{A}$ of T, for any n-tuple $a_1, \dots, a_n \in \mathfrak{A}$, if $\mathfrak{A} \models \psi[a_1, \dots, a_n]$, then $\mathfrak{A} \models \Sigma[a_1, \dots, a_n]$ (any n-tuple of a model of T that satisfies ψ also satisfies Σ).

T **locally omits** Σ if it does not locally realize Σ.

2.2. **Omitting Types and Extended Omitting Types Theorems.** We now introduce the connection between omitting models and locally omitting theories.

Theorem 2.3. ***Omitting Types Theorem:*** *Suppose T is a consistent theory in a countable language $\mathcal{L}$, and $\Sigma = \Sigma(x_1, \dots, x_n)$ a set of $\mathcal{L}$-formulas. Then*

1. *If T locally omits Σ, there is a countable model of T that omits Σ.*
2. *If T is complete and has a model that omits Σ, then T locally omits Σ.*

Proof. (1) By assumption, T locally omits $\Sigma(x_1, \dots, x_n)$. Let $C = \{c_0^1, \dots, c_0^n, c_1^1, \dots\}$ be a countable set of constant symbols not in $\mathcal{L}$. Consider the extension $\mathcal{L}' = \mathcal{L} \cup C$ of $\mathcal{L}$. Note that because $\mathcal{L}$ and C are countable, $\mathcal{L}'$ is countable. Then we can enumerate the sentences of $\mathcal{L}'$ as $\psi_0, \psi_1, \dots$. We construct an increasing sequence of theories

$$T = T_0 \subset T_1 \subset \dots$$

with the following properties:

1. $(\neg\sigma(c_m^1, \dots, c_m^n)) \in T_{m+1}$ for some $\sigma(x_1, \dots, x_n) \in \Sigma$.
2. Exactly one of the following holds:
    - $\psi_m \in T_{m+1}$
    - $(\neg\psi_m) \in T_{m+1}$
3. If ψ_m is of the form $(\exists x_1, \dots, x_n)\chi(x_1, \dots, x_n)$ and $\psi_m \in T_{m+1}$, then $\chi(c_p^1, \dots, c_p^n) \in T_{m+1}$ for some $c_p^1, \dots, c_p^n \in C$.
4. Each T_m is a consistent theory of $\mathcal{L}'$ and a finite extension of T.

The construction of this sequence is shown by induction from the base case $T = T_0$. Assuming T_m satisfies the above properties, we construct T_{m+1} in the following manner:

T_m is a finite extension of T, so it can be written as $T_m = T \cup \{\mu_1, \dots, \mu_r\}$. Let $\mu = \mu_1 \wedge \dots \wedge \mu_r$. Replace all constants c_i^k in μ with the variable x_i^k, and then prefix the equation by $(\exists x_i^k)$ for all $i \not\equiv m$ to create the formula $\mu(x_m^1, \dots, x_m^n)$.

$\mu(x_m^1, \ldots, x_m^n)$ is consistent with the original theory T, so because T locally omits Σ, there must be some $\sigma \in \Sigma$ such that $\mu(x_m^1, \ldots, x_m^n) \wedge \neg\sigma(x_m^1, \ldots, x_m^n)$ is consistent with T (if not, $\mu(x_m^1, \ldots, x_m^n)$ would be the formula from which T would locally realize Σ). By adding $\neg\sigma(c_m^1, \ldots, c_m^n)$ to T_{m+1}, condition (1) of the sequence of theories holds.

Consider the theory $T_{k_m} = T_m \cup \{\neg\sigma(c_m^1, \ldots, c_m^n)\}$. If ψ_m is consistent with T_{k_m}, add ψ_m to T_{m+1}. Otherwise, add $\neg\psi_m$. From this, condition (2) holds.

If $\psi_m = (\exists x_1, \ldots, x_n)\chi(x_1, \ldots, x_n)$ is consistent with T_{k_m}, put $\chi(c_p^1, \ldots, c_p^n)$ into T_{m+1}. This satisfies condition (3).

All of the above steps preserve consistency and add only finitely many (from 1 to 3) formulas from T_m to T_{m+1}, so T_{m+1} is a consistent finite expansion of T and condition (4) holds. Also, by the completeness theorem, T_{m+1} is satisfiable.

Now consider the theory $T_\omega = \bigcup_{n<\omega} T_n$. From condition (2) in the formation of each T_n, T_ω is maximal, and from condition (4), T_ω is consistent, so T_ω is a maximal consistent theory in $\mathcal{L}'$. Because each T_n is satisfiable, T_ω is satisfiable by the compactness theorem. Let $\mathfrak{A}' = (\mathfrak{A}, a_0, a_1, \ldots)$ be a model of T_ω where the constants $a_0, a_1, \ldots$ are the interpretations of C in $\mathfrak{A}'$. Then from condition (3), we see that the universe A in $\mathfrak{A}$ is just $A = \{a_0, a_1, \ldots\}$, so $\mathfrak{A}$ is countable. The model $\mathfrak{A}$ is the reduct of $\mathfrak{A}'$ to $\mathcal{L}$, so $\mathfrak{A}$ is a model of T. Condition (1) shows that $\mathfrak{A}$ omits Σ; for each $a_1, \ldots, a_n \in A$, there is some $\sigma(x_1, \ldots, x_n) \in \Sigma$ that $a_1, \ldots, a_n$ does not satisfy.

(2) We prove the contrapositive: if T is complete and locally realizes Σ, then every model of T realizes Σ.

Suppose T locally realizes Σ. Then by definition there exists some formula $\psi = \psi(x_1, \ldots, x_n)$ such that $T \cup \{\psi\}$ is consistent and for each $\sigma(x_1, \ldots, x_n) \in \Sigma$, $T \models \psi \to \sigma$. By assumption, T is complete, so $T \models (\exists x_1, \ldots, x_n)\psi(x_1, \ldots, x_n)$ follows from $T \cup \{\psi\}$ being consistent. Therefore, for any model $\mathfrak{A}$ of T, there must be an n-tuple $a_1, \ldots, a_n \in A$ that satisfies $\psi(x_1, \ldots, x_n)$. Then $a_1, \ldots, a_n$ also satisfies $\Sigma(x_1, \ldots, x_n)$, so $\mathfrak{A}$ realizes Σ. $\square$

The proof of part 1 can be generalized to omit countably many sets of formulas.

Theorem 2.4. ***Extended Omitting Types Theorem:*** *Let T be a consistent theory in a countable language $\mathcal{L}$, and for each $s < \omega$, let $\Sigma_s(x_1, \ldots, x_{n_s})$ be a set of formulas with n_s free variables. If T locally omits each Σ_s, then T has a countable model which omits each Σ_s.*

Proof. The proof is almost identical to omitting types. The set of constants that extend $\mathcal{L}$ to $\mathcal{L}'$ are arranged in a different manner, with countably many constants added for each s:

$$c_s^s, c_{s+1}^s, c_{s+2}^s, \ldots$$

The construction of each T_n in the increasing sequence of theories follows the same pattern, except condition (1) is changed to ensure that each set Σ_s is omitted in T_ω. For each $s = 0, \ldots, m$, $(\neg\sigma(c_m^s)) \in T_{n+1}$ for some $\sigma \in \Sigma_s$. Instead of adding only one formula $\neg\sigma(x_n)$ to T_{n+1}, this construction adds countably many formulas, one for each set Σ_s. $\mathfrak{L}'$ is still countable, so the resulting model $\mathfrak{A}$ of T is countable and omits each Σ_s. $\square$

3. Atomic and Saturated Models

The rest of the paper will focus only on types of models for complete theories in countable languages.

3.1. Atomic Models.

Definition 3.1. Let T be a complete theory in a countable language $\mathcal{L}$. We say an $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ is **complete** in T iff for every $\mathcal{L}$-formula $\sigma(x_1, \ldots, x_n)$, exactly one of the following holds:

- $T \models \psi \to \sigma$
- $T \models \psi \to \neg\sigma$

We then say a formula σ is **completable** in T iff there exists a complete formula ψ in T such that

$$T \models \psi \to \sigma$$

σ is **incompletable** if it is not completable.

Definition 3.2. A theory T in a countable language $\mathcal{L}$ is **atomic** iff every $\mathcal{L}$-formula consistent with T is completable in T.

A model $\mathfrak{A}$ is an **atomic model** iff every n-tuple $a_1, \ldots, a_n \in A$ satisfies a complete formula of $Th(\mathfrak{A})$.

We now prove some theorems concerning the existence of and relation between atomic models of a theory.

Theorem 3.3. ***Existence Theorem for Atomic Models:*** *Let T be a complete theory. T has a countable atomic model iff T is atomic.*

Proof. First assume T has a countable atomic model $\mathfrak{A}$. Let $\psi = \psi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula consistent with T. Because T is complete, we know $T \models \psi(x_1, \ldots, x_n)$, and also $T \models (\exists x_1, \ldots, x_n)\psi(x_1, \ldots, x_n)$. Then there exists an n-tuple $a_1, \ldots, a_n \in A$ that satisfies $\psi(x_1, \ldots, x_n)$. $\mathfrak{A}$ is a countable atomic model, so by definition there exists a complete formula $\phi = \phi(x_1, \ldots, x_n)$ satisfied by $a_1, \ldots, a_n$. Because ϕ is a complete formula, we know $T \models \phi \to \psi$ or $T \models \phi \to \neg\psi$. But $a_1, \ldots, a_n$ satisfy ϕ and ψ, so it must be the case that $T \models \phi \to \psi$. Therefore, ψ is completable. Every formula consistent with T is completable, so T is atomic.

Now assume that T is atomic. For each $n < \omega$, let $\Sigma_n(x_1, \ldots, x_n)$ be the set consisting of the negation of all complete formulas in n free variables. Because T is atomic, every formula $\psi = \psi(x_1, \ldots, x_n)$ consistent with T is completable, so there exists some complete formula $\phi = \phi(x_1, \ldots, x_n)$ such that $\psi \wedge \phi$ is consistent with T. Because ϕ is complete, $\phi = (\neg\sigma)$ for some $\sigma \in \Sigma_n$, so $\psi \wedge (\neg\sigma)$ is consistent with T for some $\sigma \in \Sigma_n$. It follows that T locally omits each set Σ_n, as there is no formula $\psi(x_1, \ldots, x_n)$ consistent with T where $T \models \psi \to \sigma$ for all $\sigma \in \Sigma_n$. Then by the Extended Omitting Types Theorem, there exists a model $\mathfrak{A}$ of T that omits each Σ_n. Therefore, any n-tuple $a_1, \ldots, a_n$ does not satisfy Σ_n, so there exists a $\sigma \in \Sigma_n$ such that $a_1, \ldots, a_n$ satisfies $\neg\sigma$, i.e. $a_1, \ldots, a_n$ satisfies a complete formula. So $\mathfrak{A}$ is a countable atomic model. □

Theorem 3.4. ***Uniqueness Theorem for Atomic Models:*** *If $\mathfrak{A}$ and $\mathfrak{B}$ are countable atomic models such that $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. The universes A and B can both be well-ordered with order type ω because they are countable. Let a_0 be the first element of A. $\mathfrak{A}$ is an atomic model, so there exists some complete formula $\psi_0(x_0)$ that a_0 satisfies in $\mathfrak{A}$. Because $\mathfrak{A} \models \psi_0[a_0]$, we know $\mathfrak{A} \models (\exists x_0)\psi_0(x_0)$. Then because $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, $\mathfrak{B} \models (\exists x_0)\psi_0(x_0)$ because $(\exists x_0)\psi_0(x_0)$ is an $\mathcal{L}$-sentence. Choose $b_0 \in B$ that satisfies ψ_0. Let b_1 be the first element in the well-ordering of $B \setminus \{b_0\}$, and $\psi_1(x_0, x_1)$ a complete formula satisfied by b_0, b_1. Then because ψ_0 is a complete formula and $\mathfrak{B}$ an atomic model, we know $\mathfrak{B} \models (\forall x_0)(\psi_0(x_0) \to (\exists x_1)\psi_1(x_0, x_1))$. $\mathfrak{A}$ models the same sentence, so we can choose $a_1 \in A \setminus \{a_0\}$ such that a_0, a_1 satisfy ψ_1. Continuing this pattern using the back and forth method results in an ordering

$$A = \{a_0, a_1, \dots\} \qquad B = \{b_0, b_1, \dots\}$$

for the universes of $\mathfrak{A}$ and $\mathfrak{B}$. We now show that the mapping $f : a_m \to b_m$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Let $\phi = \phi(x_1, \dots, x_n)$ be an $\mathcal{L}$-sentence such that that the n-tuple $a_1, \dots a_n$ satisfies ϕ in $\mathfrak{A}$:

$$\mathfrak{A} \models \phi[a_1, \dots, a_n].$$

Then because $\mathfrak{A}$ is an atomic model, $a_1, \dots, a_n$ satisfy some complete formula $\psi = \psi(x_1, \dots, x_n)$. We know that $\mathfrak{A} \models \psi(a_1, \dots, a_n) \to \phi(a_1, \dots, a_n)$, so $\mathfrak{A} \models (\exists x_1, \dots, x_n)(\psi(x_1, \dots, x_n) \to \phi)$. Then $\mathfrak{B}$ also models this sentence. From the first part of this proof, we know the mapping $f : a_m \to b_m$ holds for complete formulas. Then $\mathfrak{B} \models \psi[b_1, \dots, b_n]$, and from the above sentence $\mathfrak{B} \models \phi[b_1, \dots, b_n]$.

Let $\mathcal{L}$ be the language that $\mathfrak{A}$ and $\mathfrak{B}$ model, and r_i an n-placed relation symbol in $\mathcal{L}$, with $r_i^{\mathfrak{A}}$ its interpretation in $\mathfrak{A}$ and $r_i^{\mathfrak{B}}$ its interpretation in $\mathfrak{B}$. Consider the formula $\phi(x_1, \dots, x_n) = r_i(x_1, \dots, x_n)$. Let $a_1, \dots, a_n \in A$ be an n-tuple satisfying $r_i^{\mathfrak{A}}$, so $\mathfrak{A} \models \phi[a_1, \dots, a_n]$. Then $\mathfrak{B} \models \phi[b_1, \dots, b_n]$, so

$$r_i^{\mathfrak{A}}(a_1, \dots, a_n) \iff r_i^{\mathfrak{B}}(b_1, \dots, b_n)$$

which is the first equivalence for isomorphism.

Now let g_j be an n-placed function symbol of $\mathcal{L}$, with interpretations $g_j^{\mathfrak{A}}$ and $g_j^{\mathfrak{B}}$. Consider the formula $\phi(x_1, \dots, x_{n+1}) = (g_j(x_1, \dots, x_n) = x_{n+1})$. Let $a_1, \dots, a_n \in A$ be an n-tuple with $g_j^{\mathfrak{A}}(a_1, \dots, a_n) = a_k$ for some $a_k \in A$. Then $\mathfrak{A} \models \phi[a_1, \dots, a_n, a_k]$. So $\mathfrak{B} \models \phi[b_1, \dots, b_n, b_k]$, which means $b_k = g_j^{\mathfrak{B}}(b_1, \dots, b_n)$. But $b_k = f(a_k) = f(g_j^{\mathfrak{A}}(a_1, \dots, a_n))$, so

$$f(g_j^{\mathfrak{A}}(a_1, \dots, a_n)) = g_j^{\mathfrak{B}}(b_1, \dots, b_n)$$

which is the second equivalence for isomorphism.

Let c_k be a constant symbol of $\mathcal{L}$, with interpretations $c_k^{\mathfrak{A}}$ and $c_k^{\mathfrak{B}}$. Consider the formula $\phi(x) = (x = c_k)$. Let $a_k \in A$ be the element represented by $c_k^{\mathfrak{A}}$, so $\mathfrak{A} \models \phi[a_k]$. Then $\mathfrak{B} \models \phi[b_k]$. So $b_k = c_k^{\mathfrak{B}}$, but $b_k = f(a_k) = f(c_k^{\mathfrak{A}})$, so

$$f(c_k^{\mathfrak{A}}) = c_k^{\mathfrak{B}}$$

which is the final equivalence for isomorphism.

Therefore, the mapping $f : a_m \to b_m$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, so $\mathfrak{A} \cong \mathfrak{B}$. □

The next theorem demonstrates that an atomic model of a theory is "small" compared to other models of the theory; first, we introduce a new definition for what "small" means in this context.

Definition 3.5. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of $\mathcal{L}$. The function $f : A \to B$ is an **elementary embedding** of $\mathfrak{A}$ into $\mathfrak{B}$ iff for all $\mathcal{L}$-formulas $\phi(x_1, \ldots, x_n)$ and n-tuples $a_1, \ldots, a_n \in A$

$$\mathfrak{A} \models \phi[a_1, \ldots, a_n] \iff \mathfrak{B} \models \phi[f(b_1), \ldots, f(b_n)]$$

We say $\mathfrak{A}$ is **elementarily embedded** in $\mathfrak{B}$, with the notation $\mathfrak{A} \prec \mathfrak{B}$.

Theorem 3.6. ***Smallness of Atomic Models:*** *Let $\mathfrak{A}$ be a countable atomic model. $\mathfrak{A}$ is elementarily embedded in every model of $Th(\mathfrak{A})$.*

Proof. This proof is similar to the Uniqueness Theorem for atomic models, only using half of the back and forth construction. Let $T = Th(\mathfrak{A})$, $\mathfrak{A}$ a countable atomic model, and let $\mathfrak{B}$ be any model of T. The universe A of $\mathfrak{A}$ is countable, so it can be well-ordered as $A = \{a_0, a_1, \ldots\}$. Let $\psi_0(x_0)$ be a complete formula satisfied by a_0. Then $T \models (\exists x_0)(\psi_0(x_0))$, so we can choose some $b_0 \in B$ that satisfies $\psi_0(x_0)$. Let $\psi_1(x_0, x_1)$ be a complete formula satisfied by a_1. In a similar manner to the Uniqueness Theorem, we know $T \models (\forall x_0)(\psi_0(x_0) \to (\exists x_1)\psi_1(x_0, x_1))$. So we can choose $b_1 \in B$ such that b_0, b_1 satisfy ψ_1. Continuing in this pattern, we obtain an elementary embedding $f : a_m \to b_m$. The proof that f is an elementary embedding follows the same logic as the second half of the Uniqueness Theorem, except for the limitation here that f is not necessarily surjective. $\square$

3.2. **Saturated Models.** Given a model $\mathfrak{A}$, a language $\mathcal{L}$, and a finite set of constants $C = \{c_0, \ldots, c_n\}$, we write $\mathcal{L}_C = \mathcal{L} \cup C$ and $\mathfrak{A}_C = (\mathfrak{A}, c_0, \ldots, c_n)$.

Definition 3.7. We say that a model $\mathfrak{A}$ is **ω-saturated** iff for every finite subset $C \subset A$, every set of formulas $\Sigma(x)$ in $\mathcal{L}_C$ consistent with $Th(\mathfrak{A}_C)$ is realized in $\mathfrak{A}_C$. $\mathfrak{A}$ is **countably saturated** iff $\mathfrak{A}$ is ω-saturated and countable.

Definition 3.8. A **type** in n variables $x_1, \ldots, x_n$ is a maximal consistent set of formulas $\Sigma = \Sigma(x_1, \ldots, x_n)$. For any theory $T \subseteq \Sigma$, Σ is a **type of** T.

An example of a type is an atomic type, a type axiomatized by a single formula in the set. In fact, this single formula is a complete formula, while all other formulas in the set are the corresponding completable formulas.

Example 3.9. The theory of dense linear orders without endpoints is over the language consisting of a binary relation and constant symbols named for all rational numbers $\mathcal{L} = \{<, q : q \in \mathbb{Q}\}$. The axioms of the theory are

- $(\forall x \forall y)(x < y) \to (\exists z)((x < z) \wedge (z < y))$
- $(\forall x)(\exists y)x < y$
- $(\forall x)(\exists y)y < x$
- $p < q$ iff the rational named by p is less than the rational named by q

A model of the theory $\mathfrak{A} = (\mathbb{Q}, <^{\mathfrak{A}}, q_0^{\mathfrak{A}}, \ldots)$ contains the universe $\mathbb{Q}$, which has a defined linear order that satisfies these axioms. The binary relation consists of all pairs of strictly increasing elements following this order, and each constant represents the rational number in $\mathbb{Q}$ it names. Then a type in one variable $\Sigma(x)$ defines a Dedekind cut. Partition $\mathbb{Q}$ such that $\mathbb{Q} = K_1 \sqcup K_2$, where for all elements $k_1 \in K_1$ and $k_2 \in K_2$, $k_1 < k_2$. Then the maximum set of all consistent formulas in one variable is the deductive closure of the set

$$\Sigma(x) = \{x < p : p \in K_2\} \cup \{x > p : p \in K_1\}$$

A type in n variables would consist of all formulas defining a certain n elements, as well as formulas concerning the relation between variables.

The following theorems for saturated models are very similar to the corresponding theorems for atomic models, just expanded to all types instead of only atomic types. Because of this, they are presented in a more concise way. The Existence Theorem for saturated models is saved for the next section - it is much stronger than the one for atomic models.

Theorem 3.10. ***Uniqueness Theorem for Countably Saturated Models:*** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be elementarily equivalent countably saturated models. Then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. This proof follows the same back and forth construction as the theorem for atomic models. Given that the universes A and B are countable, they can be well-ordered with order type ω. Let a_0 be the first element of A, with $C_0 = \{a_0\}$. Then all sets of formulas $\Sigma(x_0)$ in $\mathcal{L}_{C_0}$ consistent with $Th(\mathfrak{A}_{C_0})$ are realized in $\mathfrak{A}_{C_0}$. By replacing a_0 with x_0 and prefixing with $(\exists x_0)$ in each formula, each formula turns into a sentence, so they are also realized in $\mathfrak{B}$. So we can choose $b_0 \in B$ that satisfies $\Sigma(x_0)$, and $(\mathfrak{A}, a_0) \equiv (\mathfrak{B}, b_0)$. Following the same pattern, we can use all elements in the well-ordering of A and B to arrive at

$$(\mathfrak{A}, a_0, a_1, \ldots) \qquad \equiv \qquad (\mathfrak{B}, b_0, b_1, \ldots)$$

Then the mapping $f : a_m \to b_m$ is an isomorphism, $\mathfrak{A} \cong \mathfrak{B}$. □

Theorem 3.11. ***Largeness of Saturated Models:*** *Let $\mathfrak{A}$ be a countably saturated model. Then for any countable model $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$, $\mathfrak{B}$ is elementarily embedded in $\mathfrak{A}$.*

Proof. This follows the same back and forth method relating to types as shown in the previous theorem, but only done in one direction, as in theorem 3.6. After well-ordering $\mathfrak{B}$ with order type ω, we can find a sequence $a_0, a_1, \ldots \in A$ such that

$$(\mathfrak{B}, b_0, b_1, \ldots) \qquad \equiv \qquad (\mathfrak{A}, a_0, a_1, \ldots)$$

so the map $f : b_m \to a_m$ is an elementary embedding. □

4. Vaught's Theorem

Definition 4.1. Given a theory T and a cardinal α, the **spectrum of T in α** $I(T, \alpha)$ is defined as the number of non-isomorphic models of T of cardinal α. This section proves the peculiar result that for any complete theory T, $I(T, \aleph_0) \neq 2$, i.e. no complete theory has exactly two non-isomorphic countable models.

4.1. Existence of Atomic and Saturated Models.

The theorems in this section work to prove that if $n = I(T, \aleph_0)$ is finite, then one model is saturated, and one is atomic. Additionally, if $n > 1$, the saturated and atomic models are non-isomorphic.

Lemma 4.2. *Let $C = c_1, \ldots, c_m$ be a set of m constants, and the language $\mathcal{L}' = \mathcal{L} \cup C$ an extension of $\mathcal{L}$. For $n > m$, there is a 1-to-1 correspondence between types in n variables in $\mathcal{L}$ and types in $n - m$ variables in $\mathcal{L}'$.*

Proof. Let $\Sigma = \Sigma(x_1, \ldots, x_n)$ be a type in $\mathcal{L}$. We claim that the corresponding set $\Sigma' = \Sigma'(c_1, \ldots, c_m, x_{m+1}, \ldots, x_n)$ is a type of $\mathcal{L}'$ in $n - m$ variables. To prove this, we show that Σ' is maximally consistent in $\mathcal{L}'$ using contradiction. Assume there is an $\mathcal{L}'$-formula $\phi = \phi(x_{m+1}, \ldots, x_n)$ such that $\Sigma' \cup \{\phi\}$ and $\Sigma' \cup \{\neg\phi\}$ are consistent.

By the completeness theorem, there exist models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{L}'$ such that $\mathfrak{A} \models (\Sigma' \cup \{\phi\})[a_{m+1}, \ldots, a_n]$ and $\mathfrak{B} \models (\Sigma' \cup \{\neg\phi\})[b_{m+1}, \ldots, b_n]$ for some $a_{m+1}, \ldots, a_n \in A, b_{m+1}, \ldots, b_n \in B$. Construct the formula $\psi = \psi(x_1, \ldots, x_n)$ from ϕ by replacing c_i with x_i for $i \leq m$. Let $a_1, \ldots, a_m$ be the elements of A signified by $c_1, \ldots, c_m$ in $\mathfrak{A}$, and $b_1, \ldots, b_m$ the elements of B signified by $c_1, \ldots, c_m$ in $\mathfrak{B}$. Then we have two pairs of equivalent sentences: $\phi[a_{m+1}, \ldots, a_n] = \psi[a_1, \ldots, a_n]$ and $\phi[b_{m+1}, \ldots, b_n] = \psi[b_1, \ldots, b_n]$. So $\mathfrak{A} \models (\Sigma' \cup \{\psi\})[a_1, \ldots, a_n]$ and $\mathfrak{B} \models (\Sigma' \cup \{\neg\psi\})[b_1, \ldots, b_n]$. Therefore, the sets $\Sigma' \cup \{\psi\}$ and $\Sigma' \cup \{\neg\psi\}$ are still both consistent in $\mathcal{L}'$.

Construct the set of $\mathcal{L}'$-formulas $\Sigma''(x_1, \ldots, x_n)$ by applying the process to construct ψ from ϕ for each $\sigma \in \Sigma$. $\Sigma'' \cup \{\psi\}$ and $\Sigma'' \cup \{\neg\psi\}$ are still consistent in $\mathcal{L}'$ by the same logic. But now they contain no elements in C, so Σ'' and ψ are actually just $\mathcal{L}$-formulas, and $\Sigma'' = \Sigma$. Then $\Sigma \cup \{\psi\}$ and $\Sigma \cup \{\neg\psi\}$ are both consistent in $\mathcal{L}$. This contradicts the assumption that Σ is maximally consistent. Therefore Σ' is a type of $\mathcal{L}'$. $\square$

Theorem 4.3. ***Existence Theorem of Saturated Models:*** *Let T be a complete theory. Then for each $n < \omega$, T has only countably many types in n variables iff T has a countably saturated model.*

Proof. By assumption, for every finite n, T has at most countably many types in n variables. Let $C = \{c_0, c_1, \ldots\}$ be a set of countably infinite many constants, and $\mathcal{L}' = \mathcal{L} \cup C$ the expansion of $\mathcal{L}$, the language of T. Let $Y = \{y_1, \ldots, y_n\}$ be a subset of C with n constants. From the above lemma, there is a 1-to-1 correspondence between the types in one variable $\Sigma(x)$ of T in $\mathcal{L}_Y$ and the types in $n+1$ variables $\Sigma(x_0, \ldots, x_n)$ of T in $\mathcal{L}$. So there are only countably many types $\Sigma(x)$ of T in $\mathcal{L}_Y$ for any $Y \subset C$, and because C is countable, there are only countably many finite subsets Y. Therefore, we can enumerate all types of T in all $\mathcal{L}_Y$ where Y is a finite subset of C:

$$\Sigma_1(x), \Sigma_2(x), \ldots$$

$\mathcal{L}$ and C are both countable, so $\mathcal{L}'$ is also countable. Then we can also enumerate the sentences of $\mathcal{L}'$:

$$\psi_1, \psi_2, \ldots$$

We now construct an increasing sequence of theories, similar to the proof of the Omitting Types Theorem

$$T = T_0 \subset T_1 \subset T_2 \subset \ldots$$

with the following properties:

(1) Exactly one of the following holds:
   - $\psi_n \in T_{n+1}$
   - $(\neg\psi_n) \in T_{n+1}$

(2) If ψ_n is in T_{n+1} and of the form $\psi_n = (\exists x)\phi(x)$, then $\phi(c_m) \in T_{n+1}$ for some $c_m \in C$.

(3) If $\Sigma_n(x)$ is consistent with T_{n+1}, then $\Sigma_n(c_m) \subset T_{n+1}$ for some $c_m \in C$.

(4) T_n is consistent and contains only finitely many constants from C.

The process of this construction by induction is straightforward. Assuming the theory T_n has the properties listed above, we construct T_{n+1} in the following manner.

If $T_n \cup \{\psi_n\}$ is consistent, then add ψ_n to T_{n+1}. Otherwise, add $\neg\psi_n$. From this, condition (1) holds.

If ψ_n was added to T_{n+1} in the previous step and is of the form $\psi_n = (\exists x)\phi(x)$, add $\phi(c_m)$ to T_{n+1} for some $c_m \in C$. Then condition (2) holds.

If ψ_n is consistent with T_n, then if $\Sigma_n(x)$ is consistent with $T_n \cup \{\psi_n\}$, there is some $c_m \in C$ such that $\Sigma_n(c_m) \subset T_n \cup \{\psi_n\}$. For every $\sigma(x) \in \Sigma_n(x)$, add $\sigma(c_m)$ to T_{n+1}. So condition (3) holds. If ψ_n is not consistent with T_n, use $\neg\psi_n$ instead of ψ_n.

Every step above maintained consistency in T_{n+1} and added only finitely many constants to the set of constants used in T_n. Then condition (4) holds. By the completeness theorem, T_n is satisfiable.

Now consider the model $T_\omega = \bigcup_{n<\omega} T_n$. From conditions (1) and (4) in the construction of each T_n, we know T_ω is a maximally consistent theory of $\mathcal{L}'$. By the compactness theorem, T_ω is satisfiable. By condition (2), let $\mathfrak{A}' = (\mathfrak{A}, a_0, a_1, \ldots)$ be a model of T_ω where the constants $a_0, a_1, \ldots$ are the interpretations of C in $\mathfrak{A}'$. Then from condition (3), we see that the universe A in $\mathfrak{A}$ is just $A = \{a_0, a_1, \ldots\}$, so $\mathfrak{A}$ is countable. $\mathfrak{A}$ is the reduct of $\mathfrak{A}'$ to $\mathcal{L}$, so $\mathfrak{A}$ is a model of T.

We show that $\mathfrak{A}$ is ω-saturated, and therefore countably saturated. Take a finite subset $Y \subset A$. Any set of formulas $\Gamma(x)$ consistent with $Th(\mathfrak{A}_Y)$ can be extended to a maximally consistent set, which is a type $\Sigma_n(x)$ for some n. We know all types consistent with $Th(\mathfrak{A}_Y)$ are consistent with T_ω, so it follows from the construction of T_ω that $\Sigma_n(x)$ is consistent with T_{n+1}. Then some $c_m \in C$ satisfies $\Sigma_n(x)$ in T_ω, so the corresponding $y_m \in Y$ satisfies $\Sigma_n(x)$ in $\mathfrak{A}_Y$. It follows that $\mathfrak{A}_Y$ realizes $\Gamma(x)$, so A is ω-saturated.

For the backwards direction, let $\mathfrak{A}$ be a countably saturated model of T. Each type of T in n variables is realized in $\mathfrak{A}$. No n-tuple of A can realize more than one type, and there are only countably many n-tuples of A, so T has only countably many types in n variables. $\square$

Corollary 4.4. *Let T be a complete theory with $I(T, \aleph_0)$ finite or ω. Then T has a countably saturated model.*

Proof. Each type of T is realized in some countable model of T. There are only countably many non-isomorphic countable models of T, and each model can only realize countably many types. Then T has only countably many types, and from the above theorem, T has a countably saturated model. $\square$

Lemma 4.5. *Let T be a non-atomic complete theory, and $\psi = \psi(x_1, \ldots, x_n)$ a consistent, incompletable formula. Then there exist two formulas $\phi_0 = \phi_0(x_1, \ldots, x_n)$ and $\phi_1 = \phi_1(x_1, \ldots, x_n)$ consistent with T such that*

- $T \models \phi_0 \to \psi$
- $T \models \phi_1 \to \psi$
- $T \models \neg(\phi_0 \wedge \phi_1)$

Proof. Let $\sigma = \sigma(x_1, \ldots, x_n)$ be a formula consistent with T such that $T \models \sigma \to \psi$. We know there exists some formula σ because of the trivial example where $\sigma = \psi$. Then because ψ is incompletable, σ cannot be a complete formula. This means there exists some formula $\gamma = \gamma(x_1, \ldots, x_n)$ such that not exactly one of the following holds:

- $T \models \sigma \to \gamma$

- $T \models \sigma \to \neg\gamma$

It also cannot be such that both hold; then $T \cup \{\sigma\} \models (\gamma \wedge (\neg\gamma))$, but σ was assumed to be consistent with T. So $T \not\models \sigma \to \gamma$ and $T \not\models \sigma \to \neg\gamma$. Because T is complete, the negations of these formulas are modeled by T:

- $T \models \neg(\sigma \to \gamma)$
- $T \models \neg(\sigma \to \neg\gamma)$

These are logically equivalent to

- $T \models (\sigma \wedge \neg\gamma)$
- $T \models (\sigma \wedge \gamma)$

These statements can be weakened to $T \models \gamma \to \sigma$ and $T \models \neg\gamma \to \sigma$. The lemma follows from the example $\sigma = \psi$, with $\phi_0 = \gamma, \phi_1 = \neg\gamma$. □

Theorem 4.6. ***Atomic Model from Saturated:*** *Let T be a complete theory. If T has a countably saturated model, then it also has an atomic model.*

Proof. We prove the contrapositive: if T does not have an atomic model, then it does not have a countably saturated model. If T does not have an atomic model, T is not atomic by the Existence Theorem for atomic models. Then there exists some formula $\psi = \psi(x_1, \ldots, x_n)$ consistent with T that is incompletable. By the above lemma, we can take two formulas ϕ_0, ϕ_1 consistent with T but not with each other. Note that these two formulas are also incompletable - if they were completable, the complete formula satisfying them would also satisfy ψ. Then for ϕ_0, we can again find two formulas ϕ_{00}, ϕ_{01} consistent with T but not with each other (and for ϕ_1, there are ϕ_{10}, ϕ_{11}). Continuing in this pattern, we see that there are 2^ω sets of formulas in n variables all inconsistent with each other in T. Each set of formulas can be extended to a type of T, so T has uncountably many types. Then from the backwards direction of the Existence Theorem for saturated models, T does not have a countably saturated model. □

4.2. **Categoricity and Proof of Vaught's Theorem.** We know that if $I(T, \aleph_0)$ is finite, T has a countable atomic model and a countably saturated model. We must show that if $I(T, \aleph_0) \geq 2$, these models are non-isomorphic.

Lemma 4.7. *Let T be a complete theory in a countable language $\mathcal{L}$. Given a countable atomic model $\mathfrak{A}$ and a countably saturated model $\mathfrak{B}$ of T, if $\mathfrak{A} \cong \mathfrak{B}$, then all models of T are isomorphic.*

Proof. Let $\mathfrak{A}$ be a countable atomic model and $\mathfrak{B}$ a countably saturated model of a complete theory T, and suppose $\mathfrak{A} \cong \mathfrak{B}$. Let $\Sigma = \Sigma(x_1, \ldots, x_n)$ be a type in n variables of T. Then because $\mathfrak{A}$ is isomorphic to a countably saturated model, there exists an n-tuple $a_1, \ldots, a_n \in A$ that satisfy Σ. Because $\mathfrak{A}$ is atomic, this n-tuple must also satisfy some complete formula $\gamma = \gamma(x_1, \ldots, x_n)$. Then $\gamma \in \Sigma$, so every type contains a complete formula.

Now let $\Gamma = \Gamma(x_1, \ldots, x_n)$ be the set of the negation of all complete formulas ψ_i in n variables in T. Γ then cannot be extended to a type of T, because each type contains a complete formula. So Γ is inconsistent with T. Then because inconsistency is defined by finitely many deductions, there exists a subset $\{\neg\psi_1, \ldots, \neg\psi_k\} \subset \Gamma$ that is inconsistent with T. T is complete, so $T \models \neg(\neg\psi_1 \wedge \ldots \wedge \neg\psi_k)$, which is equivalent to $T \models \psi_1 \vee \ldots \vee \psi_k$. The set of consequences of each $T \cup \{\psi_i\}$ is a type of T. Every n-tuple of a model of T satisfies some ψ_i for $i \leq k$, so there are only k many types. So for each n, T has only finitely many types in n variables.

Then there are only finitely many formulas in T in n variables up to equivalence. Now let $\mathfrak{C}$ be any model of T. Let $\phi_1, \dots, \phi_k$ be the list of all formulas satisfied by an n-tuple $c_1, \dots, c_n \in C$. Then the conjunction of these formulas $\phi_1 \wedge \dots \wedge \phi_k$ is a complete formula satisfied by $c_1, \dots, c_n$. Every n-tuple of C satisfies a complete formula, so $\mathfrak{C}$ is atomic. Every model of T is then atomic, so all models of T are isomorphic. □

Definition 4.8. A theory T is **α-categorical** iff $I(T, \alpha) = 1$, i.e. all models of T of cardinal α are isomorphic. In the example above, if a countable atomic model and a countably saturated model of T are isomorphic, then T is ω-categorical.

So if $I(T, \aleph_0) \geq 2$, T has an atomic model and a saturated model that are not isomorphic.

Theorem 4.9. *Let T be a complete theory. Then $I(T, \aleph_0) \neq 2$.*

Proof. Assume that for some complete theory T, $I(T, \aleph_0) = 2$. As shown above, the two non-isomorphic countable models of T are an atomic model $\mathfrak{A}$ and a countably saturated model $\mathfrak{B}$. We know $\mathfrak{B}$ is not atomic, so there exists some n-tuple $b_1, \dots, b_n \in B$ that does not satisfy a complete formula. Consider the theory of $\mathfrak{B}$ expanded by this n-tuple $T' = Th(\mathfrak{B}, b_1, \dots, b_n)$. $(\mathfrak{B}, b_1, \dots, b_n)$ is just a finite expansion of $\mathfrak{B}$ by elements in B, so $(\mathfrak{B}, b_1, \dots, b_n)$ is also countably saturated. Then T' has a countably saturated model, so it must also have an atomic model $(\mathfrak{C}, c_1, \dots, c_n)$.

The reduct $\mathfrak{C}$ is a model of T. We must show that $\mathfrak{C}$ is not an atomic or countably saturated model. The n-tuple $c_1, \dots, c_n \in C$ does not satisfy a complete formula in T, so $\mathfrak{C}$ is not atomic. By assumption, T is not ω-categorical. Then the expanded theory T' is not ω-categorical, so $(\mathfrak{C}, c_1, \dots, c_n)$ is not isomorphic to $(\mathfrak{B}, b_1, \dots, b_n)$. From the Uniqueness Theorem for saturated models, $(\mathfrak{C}, c_1, \dots, c_n)$ is not countably saturated. Then its reduct $\mathfrak{C}$ is not countably saturated. $\mathfrak{C}$ is not atomic or countably saturated, so it is not isomorphic to $\mathfrak{A}$ or $\mathfrak{B}$. Therefore, if a complete theory has two non-isomorphic countable models, it has at least one more as well. □

Example 4.10. While $I(T, \aleph_0) \neq 2$ for any complete theory T, we can have $I(T, \aleph_0) = n$ for any $n \geq 3$.

Consider the language consisting of a binary relation and countably many constants $\mathcal{L} = \{<, c_0, c_1, \dots\}$ with a model $\mathfrak{A}$. Let T be a complete theory axiomatized by $c_i < c_{i+1}$ that $\mathfrak{A}$ models. There are 3 non-isomorphic models, depending on the values in A that the constants represent.

(1) The sequence $c_0, c_1, \dots$ is unbounded.
(2) The sequence $c_0, c_1, \dots$ is bounded but does not converge to a limit
(3) The sequence $c_0, c_1, \dots$ converges.

Here, $I(T, \aleph_0) = 3$. For $n > 3$, expand $\mathcal{L}$ by $n-2$ unary relations $R_1, \dots, R_{n-2}$. Then expand T by including axioms such that for each unary relation R_k, the values c_m such that R_k holds are dense, and R_1 is true for all c_i. For this theory, the first two non-isomorphic models are the first two in the above list where $I(T, \aleph_0) = 3$. There are an additional $n-2$ non-isomorphic models where the sequence $c_0, c_1, \dots$

converges to a limit, as the limit can belong to $n-2$ different unary relations. So $I(T, \aleph_0) = n$.

We omit in this example the proofs for the completeness of these theories; this is done in Vaught's paper [2] through use of the Löwenheim-Skolem Theorem.

From this example, we know that there exist complete theories T such that $I(T, \aleph_0) = n$ for any $n \neq 2$.

Example 4.11. We now show that Vaught's Theorem is limited to countable models, as there exists a complete theory T such that $I(T, \aleph_1) = 2$.

Consider the language of one binary relation $\mathcal{L} = \{\sim\}$. Let T be a complete theory, and $\mathfrak{A}$ be a model of T of power $\aleph_1$ such that there are two equivalence classes of $\sim$, each being of infinite cardinality.

Remark 4.12. Given two infinite cardinals α_1 and α_2, $\alpha_1 + \alpha_2 = \max(\alpha_1, \alpha_2)$.

Because $\mathfrak{A}$ is of power $\aleph_1$, at least one equivalence class must be of cardinality $|\aleph_1|$. Then the second equivalence class must be infinite, so its cardinality could be $|\aleph_0|$ or $|\aleph_1|$. The two models realizing these possibilities are not isomorphic, as there cannot exist a bijection between the equivalence classes of different cardinalities. Therefore $I(T, \aleph_1) = 2$.

Again, the proof of completeness for this theory is omitted above. This can be done using Robinson's test [4].

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