MODEL CATEGORIES: THEORY AND APPLICATIONS

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Abstract. This paper consists of two interlaced – but mathematically independent – parts. In the first, we briefly introduce the theory of model categories and its uses. This tour culminates in a proof of a general form of Whitehead's Theorem. We then focus our attention on a model category which plays a strong role in algebraic geometry: the category of small dg-categories. After presenting a method of Schwede and Shipley that helps transfer model structures, we place a model structure on a relative version of the category of small dg-categories.

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Introduction

Generally speaking, the theory of model categories provides an abstract framework for "doing homotopy theory." What might such a vague statement mean, and more to the point, who needs homotopy theory besides algebraic topologists? This paper will be an attempt to convince the reader that the theory of model categories provides a powerful common language for homotopy theory, homological algebra, and some aspects of algebraic geometry.

First, what do we mean by homotopy theory? In the strictly topological sense, homotopy theory is the study of homotopy classes of maps between topological spaces. But there are many categories in which one would like to identify maps

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that act in similar ways, or objects that look the same to these homotopy classes. For example, two projective resolutions of a chain complex give the same derived functors; if we accept that homological algebra is the study of homology and derived functors, then it makes sense to identify these objects in some way.

Model categories were developed by Daniel Quillen in [11] to turn these analogies into mathematics.

Although we provide many examples – better too many than too few – and attempt to explain each non-obvious diagram or construction that appears, we'll nevertheless assume familiarity with basic categorical language. For an introduction to the basic constructions of category theory we recommend the first chapter of [19]. The exposition there strikes a particularly nice balance between abstraction and calculations in **Set**, **Grp**, and other familiar categories.

This paper consists of two supporting but mathematically independent pieces. In the first half of the paper we give the basic definition of a model category and its associated homotopy category, providing a healthy flow of examples along the way. For our first reward, we show that the Whitehead Theorem from classical homotopy theory has a word-for-word translation in any model category. This first section also provides an opportunity for you the reader to become accustomed to the basic strategies of proof in model categories. We won't reprove the entire theory; this would take at least 30 pages to recreate and is already done well in [5] and [10]. Instead, our strategy is to highlight those proofs that show a clear relation between the theory of model categories and the model categories themselves.

In the second half of the paper we recall the basic constructions of differential graded algebra. We then use tools developed by Schwede and Shipley in [13] to place a model structure on $\mathbf{dgCat_A}$, the category of small categories enriched in modules over a fixed commutative DGA. The non-relative versions of these categories – in which the modules live only over a commutative ring – play a strong role in algebraic geometry. They are, as shown by Tabuada ([17]), the correct "home" for invariants like the Hochshild homology of an algebra. Tabuda's work uses a model structure on the category of dg-categories, and this structure is the starting point of our investigation.

Our expositions in Section 1 follow [5], which provides a leisurely and motivated introduction to the subject of model categories. A much cleaner approach appears via the use of the HELP Lemma in [10], but this treatment seems best for someone seeing model categories for the second time. The final proof assumes some knowledge about homotopy colimits and smallness, but a reader with categorical experience should be able to follow nonetheless.

It seems likely that the methods of [3] could have provided an alternative basis for the proof in the second half of the paper. Our current method has the advantage of bringing us explicitly through some of the important constructions of dg-categories such as path objects and composition of morphisms.

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1. First Definitions

When first reading the following axiomatization, the reader may do well to think of the words "weak equivalence," "fibration," and "cofibration" in terms of their meanings in the category of topological spaces. We emphasize, however, that while these words are meant to be evocative they refer only to abstract classes of maps.

Definition 1.1. A model category (or category with model structure) M consists of a category which we also denote by M, along with three distinguished classes of morphisms: weak equivalences, fibrations, and cofibrations. These classes must contain all identity morphisms and be closed under composition. They must also satisfy the following axioms, conveniently ordered in the same way in most current literature:

MC1: M is complete and cocomplete.
MC2: If f and g are two composable maps in M and two out of three of f, g, and gf are weak equivalences, then so is the third.

For this next axiom we need a quick definition.

Definition 1.2. We say a map $f : X \to Y$ is a retract of $g : A \to B$ if there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & B \arrow[r] & Y \end{tikzcd}$$

where the composition across each row is the identity; i.e. each row forms a retract in the normal sense.

MC3: If $g : X \to Y$ is a weak equivalence, a cofibration, or a fibration and f is a retract of g, then f is also a weak equivalence, cofibration, or fibration, respectively.

MC4: Suppose we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "f"] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. If either i or p is also a weak equivalence then there is a lift f from B to X as shown.

MC5: If $f : X \to Y$ is a map in M, it factors in two ways:

$$X \xrightarrow{i} W \xrightarrow{p} Y$$

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

In the first factorization i is a cofibration and p is a fibration that is also a weak eqivalence, while in the second j is a cofibration that is also a weak equivalence and q is just a fibration. We call a map that is both a cofibration and a weak equivalence *an acyclic cofibration.* Similarly, *an acyclic fibration* is a fibration that is also a weak equivalence. (In some literature this factorization is required to be functorial. In the case of a *cofibrantly generated* model category, any factorization can be made functorial. Other times this isn't the case, so we don't require functoriality here.)

Remark 1.3. At a first glance, the reader might feel cheated from the golden promise of "a way to axiomatize homotopy." We haven't even defined a homotopy in a model category yet! As we will see, however, the model structure picks out special classes of objects, called fibrant and cofibrant objects, which are the essential ingredients for a well-behaved homotopy category. Our first intuition might be to just formally add inverses to all of the weak equivalences, but these can behave notoriously badly: the morphisms between objects may well turn into proper classes!

Examples. Some standard and more exotic examples are in order. The objects in the table below may not all be familiar, but these examples are meant to illustrate, first, the astounding variety of categories on which one can put a model structure (compare **Top** and $\mathbf{Ch_R}$). Second, the examples should also indicate a large amount of similarity in the model structures themselves. This is good – it means the axiomatization indeed captures a related range of phenomena. We note here that model structures are determined by any two of the three classes of morphisms, so N/A in the table below means that an easy description of one of the classes is not available.

Category	**Weak equivalences**	**fibrations**	**Cofibrations**
Top	Classical weak equivalences	Serre fibrations	Retracts of relative cell complexes
Top	Homotopy equivalences	Hurewicz fibrations	Hurewicz cofibrations
SimplicialSet	Weak homotopy equivalences	Kan fibrations	Levelwise injections
$\mathbf{Ch_R}$ **(Chain Complexes Over a Commutative Ring R)**	Quasi-isomorphisms	Degreewise surjections	Degreewise injections with projective cokernel
Differential Graded Algebras Over a Commutative Ring R	Quasi-isomorphisms	Degreewise surjections	Degreewise injections with projective cokernel
Categories Enriched in $\mathbf{Ch_R}$ **(DG-Cats)**	Quasi-equivalences	See Theorem 2.21	N/A
Simplicial Presheaves on a Grothendieck Site (Global)	Object-wise weak equivalence	Object-wise Kan fibrations	N/A

Homotopy. Because our axioms are purely formal and didn't include anything like a unit interval, it takes some effort to define a homotopy between two maps. First of all, since we have assumed that M is complete and cocomplete, the coproduct $X \coprod X$ exists. By its universal property it also comes with a map $\nabla : X \coprod X \to X$, and two maps $X \to X \coprod X$. These maps can reasonably be denoted in_0 and in_1. This coproduct will form the "ends" of our cylinder.

Definition 1.4. A cylinder object for X, denoted $X \wedge I$, is an object that gives a factorization of $X \coprod X \to X$ as $X \coprod X \to X \wedge I \to X$, where the second map is a weak equivalence. We call $X \wedge I$ a *good* cylinder object if the first map is a cofibration, and *very good* if the second map is an acyclic fibration.

Remark 1.5. As before, the smash notation is meant to be evocative but does not refer to an actual product in M. In most of our examples it does make sense to define such an *interval object*. Interested readers should see Section 4 of [2], where one possible definition of such an object is given.

Proposition 1.6. *Every object in M has a very good cylinder object.*

Proof. By axiom **MC5**, $X \coprod X \to X$ factors as $X \coprod X \to Y \to X$, where the first map is a cofibration and the second is an acyclic fibration. □

Remark 1.7. It is tempting to forget about cylinder objects that are not very good, but many useful models of cylinder objects – most notably, the cylinder $X \times I$ in **Top** – are not very good. This example also illustrates the interplay between the general theory and specific model categories, in which a theoretically valuable concept may not be as useful in concrete circumstances.

We are ready to define the idea of a homotopy between two maps. As a first step, notice that any cylinder object $X \wedge I$ for X comes with two maps i_0 and i_1 from X, just by composing in_0 and in_1 with the map $X \coprod X \to X \wedge I$.

Definition 1.8. Let f and g be two maps from X to Y. Then by the universal property of the coproduct there is a map $f + g : X \coprod X \to Y$. We say f and g are *left homotopic* if there is a lift $H : X \wedge I \to Y$ in the following diagram:

$$\begin{tikzcd} X \wedge I \arrow[dr, dashed, "H"] & \\ X \coprod X \arrow[u] \arrow[r, "f+g"] & Y \end{tikzcd}$$

If f is left homotopic to g we write $f \overset{l}{\sim} g$. Finally, we call H a *good (or very good) left homotopy* if $X \wedge I$ is a good (or very good) cylinder object.

Any self-respecting notion of homotopy should give a equivalence relation on maps between objects. Unfortunately we can only guarantee that this holds for some objects in our category:

Definition 1.9. We call an object $A \in M$ *fibrant* if the map from A to the terminal object of M (denoted $\star$) is a fibration, and *cofibrant* if the map from the initial object of M (denoted $\emptyset$) to A is a cofibration. An object that is both fibrant and cofibrant is called bifibrant.

Remark 1.10. In many of the categories appearing in our table, all objects are cofibrant (**sSet**) or all objects are fibrant ($\mathbf{Ch_R}$). We will soon see that bifibrant objects are the key to model category theory.

Lemma 1.11. *Suppose $A \in M$ is cofibrant. Then for any $X \in M$, left homotopy is an equivalence relation on $M(A, X)$.*

We need a proposition first.

Proposition 1.12. *If $f \overset{l}{\sim} g$, there exists a good left homotopy from f to g. If in addition Y is fibrant, there exists a very good left homotopy from f to g.*

Proof. Say $H : X \wedge I \to Y$ is a left homotopy from f to g. Then by **MC5** we may factor the map $X \coprod X \to X \wedge I$ as $X \coprod X \xrightarrow{i} X \wedge I' \xrightarrow{j} X \wedge I$, with the first map a cofibration and the second map an acyclic fibration. Now define $H' : X \wedge I' \to Y$ as the composition of H and i. We need to check that $X \wedge I'$ is a cylinder object for X and that H' gives the desired homotopy. First, $X \wedge I'$ is a cylinder object because each map in the composition $X \wedge I' \xrightarrow{j} X \wedge I \to X$ is a weak equivalence, so the composition is too. By definition $X \wedge I'$ factors the required map. Since i is a cofibration, $X \wedge I'$ is a good cylinder object. Now H' is a left homotopy from f to g, since $X \wedge I'$ provides a factorization of the map $f + g : X \coprod X \to Y$.

Now suppose Y is fibrant. As above, factor $X \wedge I \to X$ as $X \wedge I \to X \wedge I'' \to X$, a cofibration followed by an acyclic fibration. By the two-out-of-three rule, the cofibration is also a weak equivalence. It's easy to see that $X \wedge I''$ is a very good cylinder object for X, and we need an extension of H to $X \wedge I''$. Since Y is fibrant we have a commutative diagram

$$\begin{tikzcd} X \wedge I \arrow[r, "H"] \arrow[d] & Y \arrow[d] \\ X \wedge I'' \arrow[r] \arrow[ur, dashed, "H''"] & \star \end{tikzcd}$$

and a lift H'' exists by **MC4**. We note that H'' is a left homotopy from f to g because $X \wedge I''$ also factors the map $f + g : X \coprod X \to Y$.

□

Now to a proof of the Lemma 1.11. The first two parts are simple, but the third highlights a key principle in model categories: when in doubt, abstract the construction from a model category you know.

Proof. **Reflexivity:** Suppose $f : A \to X$. To show $f \overset{l}{\sim} f$, we take A itself as a cylinder object: because identities are weak equivalences, the map

$$A \coprod A \xrightarrow{\nabla} A \to A$$

satisfies the conditions of a cylinder object. Then the obvious choice for a left homotopy from A to X is f itself, which is trivially a lift of f.

Symmetry: Suppose $f \overset{l}{\sim} g : X \to Y$. Then there is a map H such that the following diagram commutes:

$$\begin{tikzcd} X \wedge I \arrow[dr, "H"] & \\ X \coprod X \arrow[u] \arrow[r, "f+g"] & Y \end{tikzcd}$$

We would like such a homotopy in the case that the lower map is $g + f$ instead of $f + g$. We first construct a map *swap* : $X \coprod X \to X \coprod X$. We have the maps $in_0, in_1 : X \to X \coprod X$, and *swap* is defined by mapping them to $X \coprod X$ in the order in_1, in_0. Now H is a homotopy from g to f since we can take $X \wedge I$ as a path object for the composite:

$$X \coprod X \overset{swap}{\to} X \coprod X \to X.$$

Transitivity: Here is where we will use our lemma. Suppose that $f : X \to Y$ is left homotopic to g and that g is left homotopic to h. In the setting of topological spaces we would glue two copies of the interval together, i.e. take a coproduct. Here we do the same. By Proposition 1.12 we may choose good left homotopies $H : X \wedge I \to Y$ from f to g and $H' : X \wedge I' \to Y$ from g to h. Let $X \wedge I''$ be the pushout of the following diagram:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \\ X \wedge I' & \end{tikzcd}$$

The reader may easily check that $X \wedge I''$ is a cylinder object for X. Furthermore, since we are given H and H', the universal property of the coproduct gives a map

$$H'' : X \wedge I'' \to Y,$$

and extensions

$$in_0 : X \to X \wedge I'', in_1' : X \wedge I''.$$

A run through the diagrams shows that the composite

$$X \coprod X \to X \wedge I'' \to Y$$

agrees with $H \circ in_0 + H' \circ in_1'$. In other words, it is $f + h$.

□

We now let $\pi^l(x, y)$ denote the set of maps $M(X, Y)$ modulo the equivalence relation generated by left homotopy.

The proofs of the following statements are not difficult, but they do no more to illustrate the proof methods of model categories than the proofs we have already given; we therefore cite them without proof.

Lemma 1.13 ([5], Lemma 4.9)**.** *If A is cofibrant and $p : X \to Y$ is an acyclic fibration, then composition with p induces a bijection between $\pi^l(A, Y)$ and $\pi^l(A, X)$.*

Lemma 1.14 ([5], Lemma 4.10)**.** *Suppose X is fibrant, and that we have two left homotopic maps $f, g : A \to X$, and that $h : A' \to A$ is any map. Then fh is left homotopic to gh.*

Lemma 1.15 ([5], Lemma 4.11)**.** *If X is fibrant, then composition in M induces a map $\pi^l(A, A') \times \pi^l(A, X) \to \pi^l(A', X)$.*

The term "left homotopy" may have raised some suspicions that right homotopy also exists. Indeed it does. We recall that, under the standing assumption that M is complete, the product $X \times X$ exists for every object $X \in M$, and by its universal property it comes with a map $\Delta : X \to X \times X$.

Definition 1.16. A *path object* for X is an object X^I which factors the diagonal map $\delta : X \to X \times X$ into a weak equivalence followed by an arbitrary map, like so:

$$X \to X^I \to X \times X$$

We say that X^I is *good* if the second map is a fibration, and *very good* if the first map is an acyclic cofibration.

Right homotopy uses these path objects instead of cylinder objects.

Definition 1.17. We say two maps $f, g : X \to Y$ are right homotopic if there exists a map $X \to Y^I$ that lifts the map $X \to Y \times Y$ given by the product of f and g.

The basic facts surrounding right homotopy are precisely dual to those of left homotopy, so we omit them here.

Obviously we would be out of luck if these two notions of homotopy weren't related. Fortunately they are.

Lemma 1.18. *Suppose X is cofibrant and $f, g : X \to Y$ are left homotopic. Then f and g are right homotopic. Similarly, if Y is fibrant then right homotopy implies left homotopy.*

Proof. Choose a good cylinder object for X (with structure map $j : X \wedge I \to X$) and a good path object for Y, and suppose f and g are left homotopic via a map H. Now consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "in_0"] & Y \arrow[r] & Y^I \arrow[d] \\ X \wedge I \arrow[rr, "(j \circ f) \times H"] & & Y \times Y \end{tikzcd}$$

A lift in this diagram, a map $X \wedge I \to Y^I$, would give us a right homotopy via the composition $X \to X \wedge I \to Y^I$. And we're close: the map $Y^I \to Y$ is a fibration by the choice of a good path object, so we need $i_0 : X \to X \wedge I$ to be an acyclic cofibration (as we would obviously expect in the topological case.) This is indeed the case, and it follows from the fact that pushouts of cofibrations are cofibrations; see [5], Lemma 4.4 for the full details. $\square$

Whitehead's Theorem. From now on we use the phrase "homotopy of maps between bifibrant objects" to refer to both right and left homotopies, which are equivalent by Lemma 1.18. A map is called a homotopy equivalence if it has a homotopy inverse (i.e. compositions are homotopic to the identity), and objects are called homotopy equivalent if there is a homotopy equivalence between them. We are ready for Whitehead's theorem, which in the topological category says that a weak equivalence between CW complexes is a homotopy equivalence.

Theorem 1.19. *Suppose $X, Y \in M$ are bifibrant. Then X and Y are weakly equivalent if and only if they are homotopy equivalent.*

Proof. Suppose first that there is a weak equivalence $f : X \to Y$. Just like when we were trying to get very good left homotopies, we use the fibrancy of X to get a "lift" of f from Y to X. As we will see, we need a cofibrancy condition, so we factor f as

$$X \xrightarrow{q} C \xrightarrow{p} Y,$$

where q is an acyclic cofibration and p is a fibration. We will provide a homotopy inverse for q, denoted r below. Then, noting that by the two-out-of-three axiom p is a weak equivalence, the dual argument will provide a homotopy inverse for p.

We thus begin with the observation that we have a lift $r : C \to X$ in the following diagram:

$$\begin{tikzcd} X \arrow[r, "Id"] \arrow[d, "q"] & X \arrow[d] \\ C \arrow[r] \arrow[ur, dashed, "r"] & * \end{tikzcd}$$

We conclude that $rq = Id$. What can we say about the composite qr? We have that X is fibrant, and the map from C to a point is a retract of the map from X to a point. By **MC3**, then, C is fibrant. Then by the dual to our Lemma 1.13, there is a bijection between $\pi^r(C, C)$ and $\pi^r(X, C)$ induced by composition with q. Under this composition qr gets mapped to q, but so does the identity on C. Thus qr is homotopic to the identity.

We apply the dual argument to get a homotopy inverse for p; the composition of the two homotopy inverses is a homotopy inverse for f.

For the converse, suppose X and Y are homotopy equivalent, so there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity of Y via a left homotopy H. Again, we factor f as $X \xrightarrow{q} C \xrightarrow{p} Y$, with q an acyclic cofibration and p a fibration. We show that p is a weak equivalence, which will imply that f is too.

Again, we work just with C. We have that p is a fibration and i_0 is an acyclic cofibration since Y is fibrant, so there is a lift H':

$$\begin{tikzcd} Y \arrow[r, "Id"] \arrow[d, "i_0"] & C \arrow[d, "p"] \\ Y \wedge I \arrow[r, "H"] \arrow[ur, dashed, "H'"] & Y \end{tikzcd}$$

Now set $s = H' \circ i_1 : Y \to C$, so that H' is by definition a homotopy from s to qg. By the commutativity of the above diagram, ps is the identity on Y. If we can show sp is homotopic to the identity on C we'll be nearly done, since then it will be a weak equivalence – an easy argument shows that any map homotopic to a weak equivalence is a weak equivalence.

Since q is a weak equivalence, it follows from the first half of this proof that it has a homotopy inverse, $r : C \to X$. Now $f = pq$, so fr is homotopic to pqr by our Lemma 1.14, which is homotopic to f. Substituting and using Lemma 1.14 repeatedly, we have

$$sp \sim qgp \sim qgfr \sim qr \sim Id_C.$$

Thus sp is a weak equivalence.

Last but not least, p is a retract of sp via the diagram

$$\begin{tikzcd} C \arrow[r, "Id"] \arrow[d, "p"] & C \arrow[r, "Id"] \arrow[d, "sp"] & C \arrow[d, "p"] \\ X \arrow[r, "s"] & C \arrow[r, "p"] & X \end{tikzcd}$$

We conclude that p is a weak equivalence, so f is too, and we are done.

□

Whitehead's Theorem tells us that the theory of homotopy classes of maps between objects can be completely recovered by only considering the fibrant-cofibrant objects. It actually does us a bigger favor, by allowing us to keep track of maps between *arbitrary* objects using fibrant-cofibrant objects. This advantage allows us extra flexibility, since some natural constructions might not preserve fibrancy or cofibrancy – think about taking colimits in **Top**, for example.

Any object $X \in M$ has a "fibrant replacement" FX gotten by factoring the unique map $\emptyset \to X$ as a fibration followed by an acyclic cofibration. By definition the map $FX \to X$ is a weak equivalence and FX is fibrant. We can similarly define the cofibrant replacement QX of X. It is a fact that QFX is both fibrant and cofibrant, and that it is weakly equivalent to X. In fact, for a map $g : X \to Y$, Q and F induce maps Qg and Fg. Because we didn't assume functoriality of factorization in **MC5**, Q and F needn't be functors. They nonetheless suffice to define the homotopy category.

Definition 1.20. Given a model category M, the homotopy category $Ho(M)$ has the same objects as M and morphisms

$$Ho(M)(X, Y) = M(QFX, QFY).$$

Since weak equivalences between fibrant-cofibrant objects have honest-to-goodness homotopy inverses by Whitehead's Theorem, we have succeeded in inverting the weak equivalences. Of course there are a few checks to do here, but we leave this to [5], Theorem 6.2.

Theorem 1.21. *The functor $L : M \to Ho(M)$ given by the identity on objects and QF on morphisms is a localization of M with respect to its weak equivalences. In other words, weak equivalences in M turn into isomorphisms in $Ho(M)$, and given any other functor $K : M \to C$ in which weak equivalences go to isomorphisms, there is a natural factorization of K though L.*

Cofibrant Generation. Suppose we are given the cofibrations and weak equivalences in a model category M, and we'd like to check whether a map f in M is a fibration. In general, we must check lifting properties against a proper class of acyclic cofibrations, the members of which might be extremely complicated. In a cofibrantly generated model category we require that we can check lifting properties against *sets* of maps which often take a simple form.

Definition 1.22. Let I be a class of maps in a category C. We say $p : E \to B$ has the right lifting property with respect to $i : A \to X$ if there is a lift in every square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \arrow[r] & B \end{tikzcd}$$

We call p I-injective if it has the right lifting property with respect to every map in I.

Unfortunately, because cofibrant generation is used to dodge set-theoretic difficulties, its definition requires some set-theoretic background and a technical "smallness" condition. To avoid being overly technical here, we leave the details to [10], Chapter 15.

Definition 1.23. Suppose C is a model category. We say that C is cofibrantly generated if there are sets I and J of maps such that:

1: The domains of the maps of I are small relative to I-cell;
2: The domains of the maps of J are small relative to J-cell;
3: The class of fibrations consists precisely of the J-injective maps.
4: The class of trivial fibrations consists precisely of the I-injective maps.

Since the fibrations and cofibrations – along with knowledge of which maps lift – determine a model structure, cofibrant generation says that two sets of maps determine the entire structure.

Homotopy Limits and Colimits. The last model-category theoretic tool that we'll need in the second half is the notion of a homotopy limit. We'll refer the reader to a rigorous treatment in [5] or [10], but the notion of a homotopy limit is itself quite simple. As we mentioned during the discussion of Whitehead's Theorem, category-theoretic constructions often fail to be homotopy invariant. One canonical example comes from the following two diagrams of inclusions of topological spaces, where D^n is a disc and S^n is a sphere:

$$pt \longleftarrow S^{n-1} \longrightarrow D^n$$

$$pt \longleftarrow S^{n-1} \longrightarrow pt$$

Even though the two diagrams are homotopy equivalent, in the sense that there is a commutative diagram that connects the two and contracts D^n, the colimits of the two diagrams are not homotopy equivalent. Indeed, the top diagram has S^n as a colimit and the bottom colimit is just a point.

Homotopy limits and colimits – written *holim* and *hocolim*, respectively – fix this problem. For our purposes all we will need is the universal property of the homotopy pullback: Suppose we have a diagram D in a model category, with objects D_i. Then giving a map $X \to holimD$ is the same thing as giving a map from X to each object of D that is homotopy coherent; i.e. every composition of maps that would normally have to agree on the nose is now required to be related by a homotopy. Those homotopies must be connected by homotopies, and so on. In particular, if the maps $X \to D_i$ are coherent in the normal sense, then a map $X \to holimD$ is induced as usual. Homotopy colimits are defined dually.

Finally, any "nice" functor F between model categories has homotopy-invariant versions, like homotopy limit and colimit. These functors are called the (left and right) derived functors of F, and are calculated in general on an object X by applying F to a cofibrant or fibrant approximation of X.

2. Transferring Model Structures: The Main Characters

Though ten pages can't do justice to the power of model categories and their associated homotopy categories, even such a small taste reveals that a model structure on a category provides a wealth of tools to a working mathematician. But a question lingers: How does one *obtain* a model structure on an arbitrary category? This request seems a bit greedy, since categories come in all shapes and sizes, so it might be more reasonable to demand: If there are two categories related by an adjunction and one has a model structure, there ought to be a way to transfer the model structure to the other.

In this section we formulate an example of such a pair of categories, and provide a number of examples of why differential graded algebra is essential as both an algebraic and topological tool. In the next section we detail an affirmative answer to the demand, and use this answer to place a model structure on a category of interest. We recall the following definitions:

Definition 2.1. A cochain complex M over a commutative ring R is a sequence of R -modules $\{M_i | i \in \mathbb{Z}\}$ with maps $d_i : M_i \to M_{i+1}$, satisfying $d_{i+1} \circ d_i = 0$.

Remark 2.2. The maps d_i are called differentials. It is standard to abuse notation and simply write d for every differential, forgetting the subscript. Since we have required that $d \circ d = 0$, we may take the cohomology of the complex, defined by

$$H^i(M) = ker(d_i)/im(d_{i-1}).$$

It is an easy exercise to check that a map of complexes induces maps on each level of cohomology.

From here on we fix a commutative ring R and assume every cochain complex we mention lives over R. Although they might not look it at first, cochain (and chain) complexes are natural homotopical objects. In fact, an essential step in the development of cohomology is the passage from simplicial sets to simplicial abelian groups and finally to cochain complexes of abelian groups. The Dold-Kan Theorem makes this connection into a theorem (see [9] for more details):

Theorem 2.3. *The category of simplicial abelian groups is equivalent to the category of non-negatively graded chain complexes of abelian groups.*

This theorem provides strong evidence that homotopical tools such as model categories should apply in an algebraic context.

Recall that the tensor product of two cochain complexes M and N is defined by the rule

$$(M \otimes N)_k = \bigoplus_{i+j=k} M_i \otimes N_j.$$

From here on, tensor product will always mean tensor product over the ground ring R. Differentials on the tensor product are given in each summand by

$$d(m \otimes n) = d(m) \otimes n + (-1)^{deg(m)} m \otimes d(n).$$

A map of cochain complexes from $M = \{M_i\}$ to $N = \{N_i\}$ consists of maps from M_i to N_i for each i that commute with the differentials. Actually, $Hom_R(M, N)$ can be given the structure of a cochain complex itself!

Definition 2.4. Let $M[k]$ denote the kth forward shift of M, whose nth degree is given by M_{n-k}, with differentials $(-1)^k d$. Then for $M, N \in \mathbf{Ch_R}$, we define a chain complex $Hom_R(M, N)$ by

$$Hom_R(M, N)_k = Hom_R(M[k], N),$$

with differentials

$$d(f) = d_N \circ f - (-1)^k f \circ d_M.$$

Remark 2.5. Note that elements of $Z^i(Hom_R(M,N))$, the kernel of the differential at level i, are exactly the "true" maps of complexes from $M[k]$ to N as defined above. However, this richer hom-complex makes $\mathbf{Ch_R}$ into a closed symmetric monoidal category.

Two characters play a key role in the theory of differential graded algebra. First, there are those maps that induce *isomorphisms* on cohomology.

Definition 2.6. A map $f : M \to N$ of R-cochain complexes is called a *quasi-isomorphism* if $f : H^i(M) \to H^i(N)$ is an isomorphism for each i.

Definition 2.7. An R-module P is called *projective* if, for all surjections $q : M \to N$, if there is a map $f : P \to N$ then there is a lift of f, say f, from P to M.

These characters play essential roles in homological algebra. There, one would like derived functors of modules to exist abstractly. In practice, the way to construct them is by first taking a sequence of projective modules that is quasi-isomorphic to the module one would like to work with, and only then applying the desired functor.

The category of cochain complexes is rich, but there are times when the problem one is studying has even more structure.

Definition 2.8. A differential graded algebra (or DGA) A over a commutative ring R is a cochain complex (which we also denote by A) together with maps $\mu : A \otimes A \to A$ and $\eta : R \to A$ (R being also the complex with only R in degree zero) such that μ is associative and η is unital, so the following diagram commutes:

$$\begin{tikzcd} R \otimes A \arrow[r, "\eta\otimes Id"] \arrow[dr, "\simeq"'] & A \otimes A \arrow[d, "\mu"] & A \otimes R \arrow[l, "Id\otimes\eta"'] \arrow[dl, "\simeq"] \\ & A & \end{tikzcd}$$

Example 2.9. One source of DGAs is algebraic geometry. If R is a commutative ring, the Koszul complex of elements $x_1, \ldots, x_n \in R$ is a DGA denoted $K(x_1, \cdots, x_n)$. Then $K(x_1, \cdots, x_n)$ can be used to understand when $x_1, \ldots, x_n$ is a regular sequence, a property which is related to smoothness and regularity properties of the scheme $Spec(R)$ See [20, Section 4.5] for more details.

Example 2.10. Another source for DGAs lies in homotopy theory. One can model *rational* homotopy types – in other words, spaces whose homotopy groups are all rational vector spaces – using special DGAs called minimal Sullivan algebras:

Theorem 2.11 ([6], Cor. 1.26)**.** *Rational homotopy types of simply connected spaces of finite rational type are in bijective correspondence with isomorphism classes of minimal Sullivan algebras.*

These algebraic tools then provide a powerful tool for calculating homotopy type modulo torsion in homotopy groups.

Given the connections between homotopy theory and differential graded algebra, it is not surprising that a number of model category structures exist on the categories of chain complexes and DGAs. The first such theorem, proved by Quillen, regards nonnegatively graded chain complexes.

Theorem 2.12 ([11])**.** *There is a cofibrantly generated model structure, called the q-model structure, on the category of nonnegatively graded chain complexes. It has weak equivalences given by quasi-isomorphisms, fibrations given by degreewise surjections in positive degree, and cofibrations given by degreewise injections with degreewise projective cokernel.*

There is a model structure on the category of unbounded cochain (or chain) complexes that has quasi-isomorphisms as weak equivalences and degree-wise surjections as fibrations, but cofibrations are a bit harder to describe. This structure is called the q-model structure. Furthermore, the homotopy theory of the q-model structure entirely recovers much of classical homological algebra. This is easy to see in the case of Quillen's category, since the cofibrant-fibrant objects are precisely the projective complexes. The machinery of the first part of this paper thus implies that derived functors can be calculated using projective resolutions.

Remark 2.13. There are at least two more model structures on $\mathbf{Ch_R}$. One is similar, except it emphasizes injective objects instead of projective objects. The other has weak equivalences given by homotopy equivalences. A homotopy equivalence of complexes is stronger than a weak equivalence, so the corresponding theory is less rich.

Such theorems are just the beginning for the contemporary mathematician, who seeks above all "the relative case," in which algebraic objects ought to be replaced by maps of algebraic objects: schemes lying over schemes, vector bundles lying over manifolds. So indeed we can examine cochain complexes not only over a ring R, but over a fixed commutative DGA A. These investigations are not just intellectual hedonism. In the case of modules, the relative case is applied in [1] to clarify classical constructions in homological algebra.

Definition 2.14. Fix once and for all A, a graded-commutative DGA over R; by this we mean $ab = (-1)^{deg(a)deg(b)}ba$ for $a, b, \in A$. An A-module is an R-cochain complex M together with a map of R-cochain complexes

$$\phi : A \otimes M \to M$$

such that the action of A satisfies associativity and unit conditions.

A map between A-modules is a map of underlying R-modules that is equivariant with respect to the action of A; we may of course tensor two A-modules to get another A-module, and these operations make the category of A-modules into a closed symmetric monoidal category; see [15]. We will denote it $\mathcal{M}_A$.

Because elements of $\mathcal{M}_A$ now have an action of both R and A, more interesting algebra arises. In fact, [1] places not one, not two, but six model structures on $\mathcal{M}_\mathcal{A}$.

Example 2.15. Every R-module M gives rise to an A-module through the functor $F(M) = A \otimes M$. This functor will be very important in the next section.

DG-Categories. Our ultimate objects of study are upgraded differential-algebraic objects called dg-categories.

Definition 2.16. A dg-category $\mathcal{A}$ is a category enriched in $\mathbf{Ch_R}$. In other words, it consists of a collection of objects $Ob(\mathcal{A})$, and for every two objects x and y in $\mathcal{A}$ an R-cochain complex $\mathcal{A}(x, y)$. We also require a unit map, $\eta : R \to \mathcal{A}(x, x)$. Furthermore, given any other object z, we have maps of complexes

$$\mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \to \mathcal{A}(x, z).$$

These maps are required to be associative and unital in the obvious sense.

A map G between dg-categories $\mathcal{A}$ and $\mathcal{B}$ consists of a enriched functor: a map

$$G : Ob(\mathcal{A}) \to Ob(\mathcal{B}),$$

and for all $x, y \in Ob(\mathcal{A})$ a map of chain complexes

$$G : \mathcal{A}(x, y) \to \mathcal{B}(Gx, Gy).$$

This map is required to respect composition and units, meaning that the following two diagrams diagram must commute for all $x, y, z \in Ob(\mathcal{A})$:

$$\begin{tikzcd}
\mathcal{A}(y,z) \otimes \mathcal{A}(x,y) \arrow[r, "G\otimes G"] \arrow[d] & \mathcal{B}(Gy,Gz) \otimes \mathcal{B}(Gx,Gy) \arrow[d] \\
\mathcal{A}(x,z) \arrow[r, "G"] & \mathcal{B}(Gx,Gz)
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & \mathcal{A}(x,x) \arrow[d, "G"] \\
& \mathcal{B}(Gx,Gx)
\end{tikzcd}$$

In the first diagram the vertical arrows denote composition. We write **dgCat** for the category of small dg-categories. Tabuada actually considers pointed dg-categories, and we shall follow his lead; every unpointed dg-category determines a pointed dg-category by adding a distinguished object which has only zero maps to all other objects. Since there is little risk of confusion, we'll write **dgCat** for the category of pointed dg-categories and we draw all categories without their distinguished object.

Example 2.17 (Going Back)**.** Given a dg-category $\mathcal{A}$, there are a number of ways to produce an unenriched category. The first construction we'll need here is the category $H^0(\mathcal{A})$, also called the homotopy category of $\mathcal{A}$. It has the same objects as $\mathcal{A}$, but we set

$$H^0(\mathcal{A})(x, y) = H^0(\mathcal{A}(x, y)).$$

The motivation for this definition is that, if A is a DGA, the category of A-modules and homotopy classes of maps between them is equivalent to $H^0(\mathcal{M}_A)$; see [8] for more details.

The second construction is $Z^0(\mathcal{A})$, which also has the same objects as $\mathcal{A}$, but its morphisms are the 0-cocycles of the morphism chain cocomplex:

$$Z^0(\mathcal{A})(x, y) = Z^0(\mathcal{A}(x, y)).$$

Before we proceed, we give some important examples of dg-categories.

Example 2.18. Associated to any DGA A is a dg-category. It has one object, and the endomorphism complex of this object is A itself. Composition comes from multiplication in A, and the unit comes from the unit map from R into A. The one-object category is a standard example, but we can encode more exotic differential-graded objects as dg-categories. For example, a dg-module M over a fixed commutative DGA A is typically defined using a map of modules $A \otimes M \to M$ which satisfies the usual associativity conditions. But this data is also encapsulated by the following dg-category, in which hom-complexes are written as one arrow:

$$\begin{tikzcd} \bullet \arrow[loop left, "R"] \arrow[r, bend left, "M"] & \bullet \arrow[l, bend left, "0"] \arrow[loop right, "A"] \end{tikzcd}$$

Example 2.19. To an affine scheme $X = Spec(R)$ we may associate the category of complexes of quasicoherent sheaves on X, which is equivalent to the category $\mathbf{Ch_R}$. Of course the same construction works in the non-affine case. This category turns out to have a dg-structure, and provides a concrete motivation for being able to compare different dg-categories via a model category structure.

Such a model category structure was constructed by Tabuada in [18]. As expected, its weak equivalences are a combination of quasi-isomorphisms and equivalences of categories.

Definition 2.20. A quasi-equivalence of dg-categories is a functor of dg-categories that induces quasi-isomorphisms on all hom-algebras, and restricts to an equivalence on the underlying homotopy categories.

Theorem 2.21 ([18], Theorem 0.1)**.** *There is a cofibrantly generated model structure on* **dgCat**. *The weak equivalences in this structure are the quasi-equivalences, and the fibrations are the enriched functors $F : \mathcal{A} \to \mathcal{B}$ with the following properties:*

- **1:** *F induces componentwise surjections on all hom-complexes.*
- **2:** *Given an isomorphism $g : F(X) \to Y$ in $H^0(\mathcal{B})$, there is an isomorphism f living in $H^0(\mathcal{A})$ that maps to g under F. (This is called the isofibration property.)*

Remark 2.22. The definition of a fibration in this structure may seem mysterious, but it arises from the so-called *canonical* model structure on the category of small categories. Amazingly, there is a *unique* model structure on the category of small categories in which the weak equivalences are equivalences of categories. In this unique model structure the fibrations are precisely the isofibrations. See [12] for a full proof of these statements.

We will in fact need the full description of the generating cofibrations and acyclic cofibrations for **dgCat**. They consist of the following functors, where the maps on complexes are either identity or inclusion, and K is a 2-object category we will not need to worry about. We have listed the source of each functor on the left and its target on the right.

R(n): $\begin{tikzcd} \bullet \arrow[loop left, "R"] \arrow[r, bend left, "0"] & \bullet \arrow[l, bend left, "0"] \arrow[loop right, "R"] \end{tikzcd}$ $\qquad$ $\begin{tikzcd} \bullet \arrow[loop left, "R"] \arrow[r, bend left, "D^n"] & \bullet \arrow[l, bend left, "0"] \arrow[loop right, "R"] \end{tikzcd}$

F: $\begin{tikzcd} \bullet \arrow[loop left, "R"] \end{tikzcd}$ $\qquad$ K

S(n): $\begin{tikzcd} \bullet \arrow[loop left, "R"] \arrow[r, bend left, "S^{n-1}"] & \bullet \arrow[l, bend left, "0"] \arrow[loop right, "R"] \end{tikzcd}$ $\qquad$ $\begin{tikzcd} \bullet \arrow[loop left, "R"] \arrow[r, bend left, "D^n"] & \bullet \arrow[l, bend left, "0"] \arrow[loop right, "R"] \end{tikzcd}$

Q: $0 \circlearrowright \bullet$ $\qquad\qquad$ $\bullet \circlearrowleft R$

The following standard conventions appear above: S^n denotes the cochain complex with R in the n-th position, zero modules everywhere else, and all zero differentials; D^n denotes the cochain complex with R in the n-th and $(n-1)$-st positions and the identity differential between them; R denotes the complex with R in the zeroth position only and zero differentials; "0" denotes the complex with all zero modules.

Just as in the non-categorical setting, we can look at categories enriched in DGAs over a fixed commutative DGA A:

Definition 2.23. A *dg-category over* A is a category $\mathcal{B}$ enriched in $\mathcal{M}_A$. That is, it consists of a collection of objects $Ob(\mathcal{B})$, and for every two objects x and y in $\mathcal{B}$ an A-module $\mathcal{B}(x,y)$. Furthermore, given any other object z, we have maps of A-complexes

$$\mathcal{B}(y,z) \otimes_A \mathcal{B}(x,y) \to \mathcal{B}(x,z),$$

and a unit map

$$\eta : A \to \mathcal{B}(x,x).$$

These maps are required to be associative and unital in the same sense as with dg-categories.

Maps between dg-categories over A are exactly analogous to those over R. We denote the category of small dga-categories by $\mathbf{dgCat_A}$.

Remark 2.24. Now we may encode any DGA over A as a one-object category in the same way as before. We emphasize that in the above construction the hom-sets are not themselves DGAs, but rather modules over a fixed DGA. It is possible to consider categories enriched over algebras, but this world becomes extremely strange. For example, the classical Eckmann-Hilton argument implies that the endomorphism algebra of every object in such a category is commutative. Even stranger things happen in general, where multiplication of algebras is required to satisfy a sort of interchange law. This law is satisfied if every algebra in sight is commutative, but there do exist small categories enriched over DGAs which are not commutative.

Monoidal Monsters. Here is another important construction in the theory of enriched categories that we will soon need. Suppose we have a functor F between symmetric monoidal closed categories $\mathscr{V}$ and $\mathscr{W}$, both with product denoted $\otimes$. It would be tempting to say that F induces a functor from the category of small categories enriched in $\mathscr{V}$ to those enriched in $\mathscr{W}$ by simply applying F to each hom-object. This is almost true.

Let C be a small category enriched in $\mathscr{V}$, and suppose that x, y, and z are objects of C. Since C is $\mathscr{V}$-enriched, it comes with $\mathscr{V}$-maps

$$c_{x,y,z} : C(y,z) \otimes C(x,y) \to C(x,z).$$

By applying F we get composition maps

$$Fc_{x,y,z} : F(C(y,z) \otimes C(x,y)) \to FC(x,z).$$

But in order to have a $\mathscr{W}$-enriched structure, we actually require maps

$$F(C(y,z)) \otimes F(C(x,y)) \to FC(x,z)$$

.

We thus deduce that, in order for F to induce the desired map, it must come with natural transformations

$$\tau_{\alpha,\beta} : F(\alpha) \otimes F(\beta) \to F(\alpha \otimes \beta),$$

for all α and β in $\mathscr{V}$. A functor between two monoidal categories that comes with natural transformations of this form, along with a map from the unit of $\mathscr{V}$ to F applied to the unit of $\mathscr{W}$, is called a *lax monoidal* functor. These maps must of course be associative and unital in the appropriate sense. A lax monoidal functor then induces a functor from the small $\mathscr{V}$-categories to small $\mathscr{W}$-categories.

Having thus introduced our main characters, we may turn to the rising action.

3. If You Give a Mouse a Category...

In their article [13], Schwede and Shipley provide a general method for transferring model structures along monads. To detail the entire theory of monads would take us too far afield. Instead, we define a monad here and immediately specialize the theory for our needs.

Definition 3.1. A monad on a category C is a triple (T, η, μ), where $T : C \to C$ is a functor, η is a natural transformation from the identity on C to T, and μ is a natural transformation from $T \circ T$ to T. We also require that the following diagrams of natural transformations commute, where horizontal composition of natural transformations is denoted $\circ$:

Associativity:

$$\begin{tikzcd} T \circ T \circ T \arrow[r, "\mu \circ Id_T"] \arrow[d, "Id_T \circ \mu"'] & T \circ T \arrow[d, "\mu"] \\ T \circ T \arrow[r, "\mu"] & T \end{tikzcd}$$

Unit:

$$\begin{tikzcd} T \arrow[r, "\eta \circ Id_T"] \arrow[d, "Id_T \circ \eta"'] \arrow[dr, "Id"] & T \circ T \arrow[d, "\mu"] \\ T \circ T \arrow[r, "\mu"] & T \end{tikzcd}$$

An algebra for a monad T is an object $x \in C$, along with a map $\theta : Tx \to x$ such that the following diagrams commute:

$$\begin{tikzcd} TTx \arrow[r, "\mu(x)"] \arrow[d, "T(\theta)"'] & Tx \arrow[d, "\theta"] \\ Tx \arrow[r, "\theta"] & x \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, "\eta"] \arrow[dr, "Id"'] & Tx \arrow[d, "\theta"] \\ & x \end{tikzcd}$$

Remark 3.2. It can be convenient to think of monads as theories, and algebras over monads as concrete instances of that theory. For example, there is a monad corresponding to the group axioms; an algebra over that monad is simply a group.

We saw earlier that lax monoidal functors allow movement between enriching categories. Monads, too, have their favored counterparts in the enriched world: a monoidal monad is one that induces a monad on an enriched category. (We use both T and F to denote monads below. This notation arises because F will be a monoidal monad and T will be the monad induced by F on the enriched level. Though related, they are not the same.)

Definition 3.3. A monoidal monad on a monoidal category is a monad (F, μ, η) for which F is lax monoidal and μ and η commute with the monoidal functor structure maps given with F. (All of the relevant diagrams are written out in [14]; we omit them here for the sake of space.)

We can now introduce our main tool. We say that a T-algebra y has a path object in a model category C if y has a good path object y^I, and that the structure maps $y \to y^I$ and $y^I \to y \wedge y$ are maps of T-algebras.

Theorem 3.4 ([13], Lemma 2.3)**.** *Suppose C is a cofibrantly generated model category with generating cofibrations I and acyclic cofibrations J, and that T is a monad on C. Let I_T and J_T be the images of the sets I and J under the functor T.*

Now if S is a set of maps in C, we let $S - cof_{reg}$ denote the set of (possibly transfinite, again see [7] for set-theoretic background) compositions of pushouts of maps in I. Assume that the domains of I_T and J_T are small relative to $I_T - cof_{reg}$ and $J_T - cof_{reg}$ respectively. Suppose that

1: *Every regular J_T-cofibration is a weak equivalence, or*

2: *Every object of C is fibrant and every T-algebra has a path object.*

Then the category of T-algebras is a cofibrantly generated model category with I_T a generating set of cofibrations and J_T a generating set of acyclic cofibrations.

In our application of the above theorem, T will be a "change of base" monad on dg-categories.

Theorem 3.5. *There is a "change-of-base" monoidal monad T on* **dgCat** *for which* $\mathbf{dgCat_A}$ *is precisely the category of T-algebras. The underlying functor of T is just tensoring each hom-complex with A (i.e. applying F to each hom-complex), and μ and η are induced by multiplication in A and the unit map for A, respectively.*

Proof. We must first show that F is a monoidal monad, i.e. that there are maps $F(M) \otimes F(N) \to F(M \otimes N)$ for $M, N \in \mathbf{Ch_R}$. This map is just the one induced by the multiplication of A. Now suppose $\mathcal{B}$ is a T-algebra via an enriched functor $\theta : T(\mathcal{B}) \to \mathcal{B}$. For all $x, y \in \mathcal{B}$, θ induces a map

$$\theta_{x,y} : A \otimes \mathcal{B}(x, y) \to \mathcal{B}(x, y).$$

The associativity and unit diagrams for T-algebras then imply that every hom-complex is an A-module. Finally, the fact that T is an enriched functor implies that the composition maps are maps of A-modules.

The proof that every dg-category over A is a T-algebra is similar, so we omit it here.

□

We spend the remainder of the paper checking the second condition in the theorem of Schwede and Shipley.

4. Applying the Theorem

Theorem 4.1. *There exists a cofibrantly generated model category structure on* $\mathbf{dgCat_A}$ *induced by the change-of-base monad* T.

Proof. Our first order of business is checking smallness conditions on the domains of J and I. This check is unenlightening and would take us too far afield into set theory, so we omit it here.

Second, the terminal object of **dgCat** is a category with one object and the zero R-module as its endomorphisms. It is thus obvious that every element of **dgCat** is fibrant.

Finally, we must check that every T-algebra has a path object. Fortunately, path objects in **dgCat** are constructed in [16]. We need only check that the construction given there provides maps of dg-categories over A. Suppose $\mathcal{B}$ is a dg-category over A, with generic composition map $c_{x,y,z}$. Then we define the objects of $\mathcal{B}^I$ to be the morphisms $f : x \to y$ of $Z^0(\mathcal{B})$ that have an inverse in $H^0(\mathcal{B})$. Now suppose we have another map $f' : x' \to y'$. We define $Hom_{\mathcal{B}^I}(f, f')$ as the homotopy pullback (in the q-model structure on $\mathbf{Ch_R}$) of the following diagram:

$$\begin{tikzcd} & Hom_{\mathcal{B}}(y,y') \arrow[d, "c_{x,y,y'}(f,-)"] \\ Hom_{\mathcal{B}}(x,x') \arrow[r, "c_{x,x',y'}(-,f')"] & Hom_{\mathcal{B}}(x,y') \end{tikzcd}$$

This is the homotopy analogue of maps between two morphisms being represented by commutative squares whose sides are those two morphisms. If we denote the homotopy pullback of this diagram by Y, the center of the above cospan by X_3, and the legs by X_1 and X_2, then a model of Y can also be given explicitly by the formula

$$Y = X_1 \times_{X_3} X_3^I \times_{X_3} X_2.$$

Here X_3^I denotes the complex

$$Hom_{\mathcal{M}_A}(I_R \otimes A, X_3),$$

and

$$I_R = \cdots 0 \to R \to R \oplus R \to 0 \cdots$$

with the nontrivial differential in degree zero being $d(x) = (x, 0) - (0, x)$ for x a generator. One way to prove that this explicit construction is indeed a homotopy pullback of the diagram (they are not unique, but are unique up to weak equivalence) is by showing it has the universal property mentioned in the introduction.

Suppose Z is a cochain complex, and that there exist maps $f : Z \to X_1$ and $g : Z \to X_2$ such that the resulting diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_3 \end{tikzcd}$$

commutes up to a right homotopy $H : Z \to X_3^I$ between f and g. Then there exists a map $k : Z \to Y$ given by

$$k(z) = (f(z), H(z), g(z)),$$

and it is clear that this map gives the required universal property.

Now $\mathcal{B}^I$ is first of all a dg-category over A, so long as $\mathcal{B}$ is: the hom-complexes are certainly A-modules, since A acts on each coordinate of Y. Second, the composition maps are maps of A-modules. Indeed, suppose we have maps

$$f : x \to y, g : x' \to y', h : x'' \to y''.$$

It is most helpful to visualize this situation with the following diagram, though the fact that we are working with a homotopy pullback makes it just a visual aid:

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"] & x' \arrow[r] \arrow[d, "g"] & x'' \arrow[d, "h"] \\ y \arrow[r] & y' \arrow[r] & y'' \end{tikzcd}$$

Then composition of morphisms from f to g with morphisms from g to h in $\mathcal{B}^I$ is given by component-wise composition in right and left factors of the product. In the middle factors,

$$Hom(x, y')^I \text{ and } Hom(x', y'')^I,$$

one first post- and pre-composes respectively with the maps

$$y' \to y'' \text{ and } x \to x'$$

that are given as elements of the first and third factors of each hom-complex. This process gives two elements of $Hom(x, y'')^I$, and one then composes these two "paths" in the same way one composes chain homotopies; see [10], p. 377 for the required formulas, which just involve projection and addition and are thus maps of A-modules. We conclude that compositions of maps in $\mathcal{B}^I$ are maps of A-modules.

A functor from $\mathcal{B}$ to $\mathcal{B}^I$, denoted κ, is given by sending an object x to the map $x \xrightarrow{1} x$, where 1 is the image of R in the unit map for the hom-complex; on maps, this functor sends $f : x \to y$ to the diagram in the homotopy colimit with f horizontally on top and bottom, identity maps in between, and a trivial homotopy.

We note here that $\mathcal{B} \times \mathcal{B}$ is concretely a category whose objects are $Ob\mathcal{B} \times Ob\mathcal{B}$, and whose hom-complexes are just the product of the hom-complexes of $\mathcal{B}$, so that

$$Hom((x, y), (x', y')) = Hom(x, x') \times Hom(y, y').$$

The functor π from $\mathcal{B}^I$ to $\mathcal{B} \times \mathcal{B}$ sends an object $f : x \to y$ to the object (x, y). On maps, this "endpoint functor" forgets the homotopy coordinate of the homotopy pullback. The description of κ and π as inclusions and projections makes it clear that they are maps of dg-categories over A.

This concludes our proof of the main theorem.

□

The Road Ahead. Because I wanted to highlight the uses of Schwede and Shipley's transfer-of-structures method, I began with Tabuada's model structure. But his structure is, in turn, clearly a product of the q-model structure on $\mathbf{Ch_R}$, and so the resulting structure on $\mathbf{dgCat_A}$ resembles most the q-model structure on $\mathcal{M}_A$. (In the language of [3], Tabuada's structure is the canonical model structure induced by the q-model structure on the enriching category.) Our method thus lacks some interesting features that appear in other model structures on $\mathcal{M}_A$. For example, the so-called r-model structure investigated in [1] in some sense remembers both A and the ground ring R in its model structure.

An obvious next step, then, would be to place a model structure on $\mathbf{dgCat_A}$ that reflects any of the other five model structures on $\mathcal{M}_A$ presented by Barthel, May, and Riehl. It would also be nice to understand the applications of this model structure to concrete problems in homological algebra. After all, the model structure created by Tabuada was designed for concrete applications to Morita theory.

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