NONLINEAR WAVE EQUATIONS

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Abstract. This paper explores the properties of nonlinear wave equations. The proof for the existence and uniqueness of solutions to the 1+1 dimensional linear wave equation with smooth data is given. The D'Alembert formula is then presented in its full generality for the nonlinear equation. Important properties like the domain of dependence and propagation of information are discussed and motivated. These are used to address subtle questions concerning local existence of the nonlinear wave equation. Finally, global and long-time properties are considered, showcasing the utility of energy estimates as well as geometric arguments in higher dimensions.

Contents

1. Introduction

The wave equation is a class of partial differential equation often presented as follows

$$\begin{cases} \Box u = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

where u is a function of (t,x) with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. The symbol $\Box u$ is defined

$$\Box u = -u_{tt} + \Delta u = -\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} \frac{\partial^2 u}{\partial (x^i)^2}$$

Some readers may be familiar with opposite signs in the box operator. We will adopt this convention here. When it is convenient to consider a system where the signs differ, the formula will be written explicitly.

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This setup has great utility, but is only one particular example of wave equations. We'll consider equations such as

$$u_{tt} - c^2(t,x)u_{xx} = 0$$

We will devote most of our energy to studying equations of the form

$$\Box u = F(u, \partial u)$$

These are non-linear equations and tend to have subtle answers to questions concerning existence of solutions and regularity of data.

Many questions posed in PDE are motivated by physical phenomena; wave equations are no exception. Much of the terminology used arises from their physical meaning and the properties of systems are often examined due to their of physical significance. One such property of the wave equation is its belonging to the family of *hyperbolic PDE.*

Hyperbolic differential equations are ones where the Cauchy problem has a unique local solution around a given point where it is *hyperbolic.* This definition is difficult to wield, so it is often given in conjunction with the wave equation. It turns out that given a 2^{nd} order PDE where the 2^{nd} order terms are of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0 \text{ where } B^2 - AC > 0$$

can be made via change of coordinates into a wave equation. This means that solutions of hyperbolic PDE have the properties of solutions to wave equations. For instance, finite speed of propagation of perturbations in data (i.e. the speed c a wave travels) along characteristics.

2. 1+1 Dimensional Linear Wave Equation

We shall to consider the following PDE:

$$\begin{cases} u_{tt} - c^2(t,x)u_{xx} = 0 \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = g \end{cases}$$

We will examine conditions where a unique, smooth, and global solution exists.

We are interested in the case where $c(t,x)$ is smooth, has limited propagation, and has bounded derivatives. The initial data is also smooth. These considerations are formalized as

$$\begin{cases} u_{tt} - c^2(t,x)u_{xx} = 0 \\ u|_{t=0} = f,\ \partial_t u|_{t=0} = g \\ f, g \in C^\infty(\mathbb{R}) \end{cases}$$

$$\text{Where } c(t,x) \in C^\infty(\mathbb{R}^2)$$

$$\text{And } 0 < m \le c \le M \text{ and } |\partial_t(t,x)| + |\partial_x(t,x)| \le M$$

Theorem 2.1. *Under the above conditions, the PDE has a unique, smooth, global solution.*

Proof. The first step is to convert this to a first order system.

$$\begin{bmatrix} 1 & 0 \\ 0 & c^2(t,x) \end{bmatrix} \begin{bmatrix} u_t \\ u_x \end{bmatrix}_t + \begin{bmatrix} 0 & -c^2(t,x) \\ -c^2(t,x) & 0 \end{bmatrix} \begin{bmatrix} u_t \\ u_x \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is a *symmetric hyperbolic system.*

Invert the first coefficient matrix $\begin{bmatrix} 1 & 0 \\ 0 & c^2(t,x) \end{bmatrix}$ and apply to the equation to obtain the following

$$\begin{bmatrix} u_t \\ u_x \end{bmatrix}_t + \begin{bmatrix} 0 & -c^2(t,x) \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_t \\ u_x \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For convenience, write $\mathcal{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathcal{A} = \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix}$ so that we can more compactly express the above as

$$\partial_t \mathcal{U} + A(t,x)\mathcal{U}_x = 0$$

We now compute the eigenvectors of A

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & c^2 \\ 1 & \lambda \end{bmatrix} = \lambda^2 - c^2$$

So we have the eigenvalues $\lambda = \pm c(t,x)$ and the eigenvectors

$$A \begin{bmatrix} c \\ 1 \end{bmatrix} = -c \begin{bmatrix} c \\ 1 \end{bmatrix} \text{ and } A \begin{bmatrix} c \\ -1 \end{bmatrix} = c \begin{bmatrix} c \\ -1 \end{bmatrix}$$

Define a matrix P of the eigenvectors $P = \begin{bmatrix} c & c \\ 1 & -1 \end{bmatrix}$ and define the corresponding $\Lambda = P^{-1}AP$. Note that $\begin{bmatrix} u_t \\ u_x \end{bmatrix} = P\mathcal{W}$

$$A\mathcal{U} = AP\mathcal{U} = P\Lambda\mathcal{W}$$

Then from the equation above we have that

$$\partial_t u + A\partial_x u = P\partial_t W + (\partial_t P)\mathcal{W} + A(\partial_x P\mathcal{W} + P\partial_x \mathcal{W}) = 0$$

We deduce

$$\partial_t \mathcal{W} + P^{-1}AP\partial_x \mathcal{W} = (-P^{-1}\partial_t P - P^{-1}A\partial_x P)\mathcal{W}$$

Define

$$(-P^{-1}\partial_t P - P^{-1}A\partial_x P) = B$$

It is known from the definition that $B \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+$, real 2×2 matrices), and that B satisfies the bound

$$||B||_{L^\infty_{t,x}} \leq C(m, M)$$

where the norm is the sum of the absolute values of entries and C is a function of the bounds on c.

We are now ready to set up coupled equations, for $\mathcal{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$:

$$\begin{cases} \partial_t W_1 - c(t,x)\partial_x W_1 = (BW)_1 \\ \partial_t W_2 - c(t,x)\partial_x W_2 = (BW)_2 \end{cases}$$

We now assume that there is a smooth solution u and we now have scalar equations we can work with.

In the particular case $c \equiv 1$, we have that $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. This is rotation of 45 degrees and we have the corresponding characteristics. In addition, $B \equiv 0$, so the equations decouple.

In general, we have coupled equations, which we can work with

$$\mathcal{W}|_{t=0}(x) = P^{-1}(0,x)\begin{bmatrix} g(x) \\ f^{'}(x) \end{bmatrix}$$

Given $u_t - c(t,x)u_x = 0$ characteristic $dx + c(t,x)dt = 0$ so $\dot{x}(t) = -c(t,x)$ Define $x(0) = a$. We can deduce the following about the solution on the characteristic

$$\frac{d}{dt}u(t,x(t,\cdot)) = \partial_t u + \dot{x}u = \partial_t u - c(t,x)\partial_x u = 0$$

This implies that the solution is constant on the characteristic.

Now take the equations above and we have produced characteristic equations giving a system of ODEs to solve

$$\begin{cases} \dot{x}_{\pm}(t) = \pm c(t, x_{\pm}(t)) \\ x_{-}(0) = a \ , \ x_{+}(0) = b \end{cases}$$

This satisfies the Lipschitz condition

$$|c(t,x) - c(t,y)| \leq L|x-y|$$

because of the condition on the derivatives.

If $(B\mathcal{W})_1 = 0$, then $W_1(t,x))$ is constant on $x_{-}(t,a)$, and in general we have

$$W_1(t,x(t,\cdot)) = W_1(0,a) + \int_0^t (B\mathcal{W})_1(s, x_{-}(s,a))ds$$

When we restrict to $t = 0$, we can use the initial data

$$\mathcal{W}|_{t=0}(x) = P^{-1}(0,x)\begin{bmatrix} g(x) \\ f^{'}(x) \end{bmatrix} = \frac{1}{2c}\begin{bmatrix} 1 & c \\ 1 & -c \end{bmatrix}\begin{bmatrix} g(x) \\ f^{'}(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{2c}g & \frac{1}{2}f^{'} \\ \frac{1}{2c}g & -\frac{1}{2}f^{'} \end{bmatrix}$$

Which gives us

$$W_1(0,a) = \frac{1}{2c(0,a)}(g(a) + cf^{'}(a))$$

and similarly

$$W_2(t,x(t,b)) = \frac{1}{2c(0,b)}(g(b) - cf^{'}(b)) + \int_0^t (B\mathcal{W})_2(s, x_{+}(s,b))ds$$

Now we can take a sequence of Picard iterates $\{W^n(t,x)\}$ where we first define

$$W^0(t,x) = \frac{1}{2c(0,x)}\begin{bmatrix} g(x) & c(0,x)f^{'}(x) \\ g(x) & -c(0,x)f^{'}(x) \end{bmatrix}$$

and then build the sequence with $W^n = \begin{bmatrix} W_1^n \\ W_2^n \end{bmatrix}$ where

$$W_1^{n+1}(t, x_{-}(t,a)) + W_1^0(a) + \int_0^t (BW^n)_1(s, x_{-}(s,a))ds$$

$$W_2^{n+1}(t, x_{+}(t,b)) + W_2^0(b) + \int_0^t (BW^n)_2(s, x_{+}(s,b))ds$$

Then look for the suitable Picard bound

$$||W^{n+1} - W^n||_{L^\infty_{t,x}([0,T]\times\mathbb{R})} \leq C \int_0^T ||(W^n - W^{n-1})(s,\cdot)||_{L^\infty_x} dx \text{ for n} \leq 1$$

and

$$||W^1 - W^0||_{L^\infty([0,T]\times\mathbb{R})} \leq T||W^0||_{L^\infty_x}$$

We can then conclude by induction that

$$||W^n - W^{n-1}||_{L^\infty([0,T]\times\mathbb{R})} \leq \frac{cT^n}{n!}||W^0||_{L^\infty_x} \text{ (the Picard bound)}$$

We know that $W^n \in C^\infty_{t,x}(\mathbb{R}\times\mathbb{R})$ and therefore on every compact interval $[0,T]$

$$W^n(t,x) \to W(t,x) \in C([0,T]\times\mathbb{R}) \text{ uniformly in } (t,x)$$

We can also observe from the integral equation that W is C^1 in t.

Taking ∂_x above the Picard estimate for $\partial_x W^{n-1} - \partial_x W^n \to W \in C^1$ in x is obtained, meaning that the process can be repeated to produce as many continuous partial derivatives as desired, giving the smoothness desired. □

This proof illustrates the utility of ODE methods like Picard iteration to prove existence and uniqueness in PDE settings. In this way, using the method of characteristics and reducing the problem to an ODE paid significant dividends.

3. D'Alembert's Formula For the Inhomogeneous Wave Equation

We begin to consider nonlinear forms of the wave equation with the analysis of the following PDE

$$\begin{cases} u_{tt} - u_{xx} = F(t,x) \\ u(t_0,x) = f(x) \\ u_t(t_0,x) = g(x) \end{cases}$$

This is a basic inhomogeneous setup. Given certain regularity, we can find a unique and explicit solution to this equation. This solution is given by a general form of *d'Alembert's formula*. Often d'Alembert's formula is given for the linear form where $F \equiv 0$, the more general solution can be easily seen to reduce to the more familiar version.

Theorem 3.1. *Consider the inhomogeneous wave equation above. Fix $k \geq 1$ and take $t_0 \in (T_-,T_+)$. If $f \in C^{k+1}(\mathbb{R}), g \in C^k(\mathbb{R}), F \in C^k[(T_-,T_+)\times\mathbb{R}]$, then there exists a unique solution $u \in C^{k+1}[(T_-,T_+)\times\mathbb{R}]$ given by*

$$u(t,x) = \frac{1}{2}f(x+t-t_0) + \frac{1}{2}f(x-t+t_0) + \frac{1}{2}\int_{x-t+t_0}^{x+t-t_0} g(s)ds + \frac{1}{2}\int_{t_0}^{t}[\int_{x+s-t}^{x+t-s} F(s,v)dv]ds$$

Proof. First, assume there is a solution to the PDE.

Define the following function

$$h_-(s) = (u_t - u_x)(s, x_0+s) \text{ for parameter } x_0$$

This function comes to us naturally as

$$\frac{dh_-}{ds} = (u_{tt} + u_{tx} - u_{xt} - u_{xx})(s,x_0+s) = (u_{tt} - u_{xx})(s,x_0+s) = F(s,x_0+s)$$

The mixed partials are equal as $u \in C^2$, so we have that $\frac{dh_-}{ds}$ is composed of known quantities (in particular F).

The curve $(s, x_0 + s)$ comes about as a characteristic. This would be a line on which light or a wave travels in 2-dimensional Minkowski space.

Integrate $\frac{dh_-}{ds}$ to obtain

$$(u_t - u_x)(t, x_0 + t) = h_-(t) = (u_t - u_x)(t_0, x_0 + t_0) + \int_{t_0}^{t} F(s, x_0 + s)ds$$

Now use the initial conditions

$$h_-(t) = (g - \partial_x f)(x_0 + t_0) + \int_{t_0}^{t} F(s, x_0 + s)ds$$

Let $x_0 = x - t$ so that $(t, x_0 + t) = (t, x)$

$$(u_t - u_x)(t, x) = (g - \partial_x f)(x - t + t_0) + \int_{t_0}^{t} F(s, x + s - t)ds$$

We note that $(s, x_0 - s)$ is another characteristic with the properties as discussed above, so define

$$h_+(s) = (u_t - u_x)(s, x_0 - s)$$

and we will obtain through a similar derivation (put $x_0 = x + t$)

$$(u_t + u_x)(t, x) = (g + \partial_x f)(x + t - t_0) + \int_{t_0}^{t} F(s, x + t - s)ds$$

Add $(u_t + u_x)(t, x)$ and $(u_t - u_x)(t, x)$

$$u_t(t, x) = \frac{1}{2}[(g - \partial_x f)(x - t + t_0) + (g + \partial_x f)(x + t - t_0)$$

$$+ \int_{t_0}^{t} F(s, x + t - s)ds + F(s, x + s - t)ds]$$

To obtain $u(t, x)$, integrate

$$u(t, x) = f(x) + \frac{1}{2}\int_{t_0}^{t} [(g - \partial_x f)(x - s + t_0) + (g + \partial_x f)(x + s - t_0)ds$$

$$+\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{v} F(s, x + v - s)ds + F(s, x + s - v)ds$$

For the first part we have

$$\int_{t_0}^{t} [\partial_x f(x + s - t_0) - \partial_x f(x - s + t_0)]ds = f(x + t - t_0) + f(x - t + t_0) - 2f(x)$$

and by change of variables

$$\int_{t_0}^{t} [g(x-s+t_0)+g(x+s-t_0)]ds = \int_{x+t-t_0}^{x-t+t_0} g(s)ds$$

For the second part, we can apply Fubini's theorem and change of variables

$$\int_{t_0}^{t} [\int_{t_0}^{v} [F(s,x+v-s)+F(s,x+s-v)]ds]dv = \int_{t_0}^{t} [\int_{x+s-t}^{x+t-s} F(s,v)dv]ds$$

Adding these expressions we have the desired formula. Uniqueness follows from the formula's use on the data given. Existence of a solution u, which was assumed at the start, follows when the formula is substituted into the PDE.

The fact that $u \in C^{k+1}[(T_-, T_+) \times \mathbb{R}]$ follows from the regularity of the data in a similar manner to the discussion that u is at least C^2 □

The crucial idea above was the use of $(u_t \pm u_x)$ along the characteristics. This technique is only usable in 1 + 1 dimensions as it yields $u_{xx} - u_{tt}$.

3.1. **Propagation of Information.** The d'Alembert formula has consequences for the propagation of information. Take solution u and (t, x) where $t > t_0$. We can then see from the formula for u that $u(t, x)$ is a function of f and g in the interval $[x-t+t_0, x+t-t_0]$ and F in a triangle with base $\{0\} \times [x-t+t_0, x+t-t_0]$ and vertex (t, x). This means that if t_0 is taken as the starting point, then information only in this region can effect $u(t, x)$. This makes this wave equation a good model for waves where the speed of light is 1, as information cannot move more quickly than light as modeled by special relativity.

This has consequences for the relationship between the uniqueness of solutions and the form of the data. Take two PDE

$$\begin{cases} u^{(1)}_{tt} - u^{(1)}_{xx} = F(t,x) \\ u^{(1)}(t_0,x) = f^{(1)}(x) \\ u^{(1)}_t(t_0,x) = g^{(1)}(x) \end{cases} \qquad \begin{cases} u^{(2)}_{tt} - u^{(2)}_{xx} = F(t,x) \\ u^{(2)}(t_0,x) = f^{(2)}(x) \\ u^{(2)}_t(t_0,x) = g^{(2)}(x) \end{cases}$$

We then have solutions $u^{(1)}$ and $u^{(2)}$ corresponding to initial data $f^{(1)}$,$g^{(1)}$ and $g^{(2)}$,$f^{(2)}$. Then if we have that $f^{(1)} = f^{(2)}$ and $g^{(1)} = g^{(2)}$ in interval $[x-|t|, x+|t|]$. It follows that $u^{(1)}(t, x) = u^{(2)}(t, x)$.

This effect is intuitive in the following example. When we have solution u to

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

and $f(x) = g(x) = 0$ when $|x| \geq C$, then $u(t, x) = 0$ for $|x| \geq C + |t|$. Informally, this means that, for the wave equation, information only moves at finite time. Therefore a wave which begins life at 0 outside of some bounded set is 0 at any point outside that set at least as long as it would take light to reach the point.

This property differs from other systems. The heat equation is such an example.

$$\begin{cases} u_t - \Delta u = 0 \\ u(0,x) = f(x) \end{cases}$$

$$x \in \mathbb{R}^n,\ t \in \mathbb{R}\ , f \in C_0^\infty(\mathbb{R}^n) \text{ where } f \geq 0$$

$$\text{and } f(x) = 1 \text{ for } |x| \leq 1 \text{ and } f(x) = 0 \text{ for } |x| \geq 2$$

There is a well-known explicit solution

$$u(t,x) = \frac{1}{(\sqrt{4\pi t})^n} \int e^{-\frac{|x-y|^2}{4t} f(y) dy}$$

Now at the initial time $t = 0$, $u(0,x) = 0$ when $|x| \geq 2$. When we have any $t > 0$ this becomes $u(t,x) > 0\ \forall x$. This implies that the information that $u(0,x) \neq 0$ when $|x| \leq 1$ was instantaneously known at all $x \in \mathbb{R}^n$, so information traveled with infinite speed.

This useful and important property is formalized by the *Domain of Dependence.*

Take interval $I = [a,b]$ and $|t - t_0| < \frac{b-a}{2}$, then define $I_{t_0,t} = [a + |t - t_0|, b - |t - t_0|]$.

Definition 3.2. The *Domain of Dependence* is defined as follows

$$D_{I,t_0,t_1} = \{(s,x) | s \in [t_0,t_1], x \in I_{t_0,s}\}$$

This means that for initial data in the interval I for $t = t_0$ and F in D_{I,t_0,t_1} determine the solution in $\{t_1\} \times I_{t_0,t_1}$.

This is useful when writing solutions. A function u is in $C^k(D_{I,t_0,t_1})$ if u is in C^k in the interior of D_{I,t_0,t_1} and the derivatives up to order k can be continuously extended to the boundary of D_{I,t_0,t_1}.

4. Local Existence for 1+1 Dimension Nonlinear Wave Equations

In order to formulate general theorems concerning the local existence of solutions for 1+1 dimensional nonlinear wave equations, a suitable function space for data must be found.

Definitions 4.1. A *multi-index* α is a coordinate $\alpha = (\alpha_1, ..., \alpha_n)$ where $\alpha_i \in \mathbb{N}$.

Define $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

If $f \in C^{|\alpha|}(\Omega)$ for $\Omega \in \mathbb{R}^n$, open, then define

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} f \cdots \partial_{x_n}^{\alpha_n} f}$$

We do not want to use $C^\infty(\Omega)$ as $\frac{1}{x} \in C^\infty([0,1])$ and is not bounded. $C_0^\infty(\Omega)$ is usually not complete. We define function spaces best suited for our purposes.

Definition 4.2. For $\Omega \subset \mathbb{R}^n$, open, $m, k \geq \alpha$, then

$$C_b^m(\Omega, \mathbb{R}^k) = \{f \in C^m(\Omega, \mathbb{R}^k) \text{ where } \forall \alpha \text{ multi-index, } |\alpha| \leq m, \exists C_\alpha < \infty \text{ s.t. } |\partial^\alpha f(x)| \leq C^\alpha\ \forall x \in \Omega\}$$

and for $f \in C_b^m(\Omega, \mathbb{R}^k)$

$$||f||_{C_b^m(\Omega, \mathbb{R}^k)} = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha f(x)|$$

This is a Banach space when equipped with the above norm. This space is closer to what we want, however, the following fact can be proved which disqualifies it.

Proposition 4.3. *There is an $f \in C_b(\mathbb{R})$ such that $\psi(t,x) = f(t+x)$ such that $\psi \notin ([-1,1], C_b(\mathbb{R}))$*

Note that if $f \in C^2$, then ψ is a solution of $\psi_{tt} - \psi_{xx} = 0$. Simply take $f(x) = \sin(x^2)$. This means that we can't use an iteration in this space as the iterates may not be in the right space. However, we can now define the space which will be used.

Definition 4.4. Define

$$C_d^m(\mathbb{R}^n, \mathbb{R}^k) = \{f \in C_b^m(\mathbb{R}^n, \mathbb{R}^k) | \forall \epsilon > 0, \exists M \text{ s.t. } |x| \geq M \text{ implies } \sum_{|\alpha < m} \sup_{|x| \geq M} |\partial^\alpha f(x)| \leq \epsilon\}$$

The definition above forms the space C_d^m as a subspace of C_b^m with functions vanishing at infinity. It is also a Banach space with the norm inherited by C_b^m. The proposition above which disqualified C_b^m does not hold for C_d^m. This is because the function $f(x)$ is small when $|x|$ is large, so consider the compact subset of $\mathbb{R}$ where $|x| \leq M$ for some large M and the part when $f(x)$ is small separately.

We now want to define estimates in order to control iterates in C_d^m. First define a quantity for a solution.

$$\mathcal{E}[u](t) = ||(u_t - u_x)(t, \cdot)||_{C_b(\mathbb{R})} + ||(u_t + u_x)(t, \cdot)||_{C_b(\mathbb{R})}$$

In the proof of the d'Alembert formula h_+, h_- were constant.

$$(u_t \pm u_x)(t, x_0 \mp t) = (g \pm f_x)(x_0 \mp t_0)$$

therefore

$$||(u_t - u_x)(t, \cdot)||_{C_b(\mathbb{R})} = ||g - f_x||_{C_b(\mathbb{R})}$$
$$||(u_t + u_x)(t, \cdot)||_{C_b(\mathbb{R})} = ||g + f_x||_{C_b(\mathbb{R})}$$

This quantity also bounds the sup norm of $|u_t| + |u_x|$

It is also useful to bound the derivatives with a quantity which is also conserved in the solution.

$$\mathcal{E}_j = ||(\partial x^j \partial_t u - \partial x^{j+1} u)(t, \cdot)||_{C_b(\mathbb{R})} + ||(\partial x^j \partial_t u + \partial x^{j+1} u)(t, \cdot)||_{C_b(\mathbb{R})}$$

In order to control u, we define

$$E_k[u](t) = \sum_{j=0}^{k} \mathcal{E}_j + ||u(t, \cdot)||_{C_b(\mathbb{R})}$$

These are all of use for proving the following theorem.

Theorem 4.5. *Take $k \geq 1$ and*

$$\begin{cases} u_{tt} - u_{xx} = F(u, \partial u) \text{ for } F \in C^\infty(\mathbb{R}) \text{ and } F(0,0) = 0 \\ u(0, x) = f(x) \text{ for } f \in C_d^{k+1}(\mathbb{R}) \\ u_t(0, x) = g(x) \text{ for } g \in C_d^k(\mathbb{R}) \end{cases}$$

Then there is an $\epsilon_k > 0$ which is a function of $||f||_{C_b^{k+1}}$, $||g||_{C_b^k}$, and F so that there is a unique solution in $C^{k+1}[(-\epsilon_k, \epsilon_k) \times \mathbb{R}]$ where $u \in C[(-\epsilon_k, \epsilon_k), C_d^{k+1}(\mathbb{R})]$ and $\partial_t u \in C[(-\epsilon_k, \epsilon_k), C_d^k(\mathbb{R})]$

Proof. The idea of the proof is as follows. Take a sequence $\{u_n\}$ where

$$\begin{cases} \partial_t^2 u_0 - \partial_x^2 u_0 = 0 \\ u_0(0, x) = f(x) \\ \partial_t u_0(0, x) = g(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t^2 u_n - \partial_x^2 u_n = F(u_{n-1}, \partial u_{n-1}) \\ u_n(0, x) = f(x) \\ \partial_t u_n(0, x) = g(x) \end{cases}$$

It can be proven that for $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ (for some $k \geq 1$) above, then for $\{u_n\}$, $u_n \in C^{k+1}(\mathbb{R}^2)$ and $u_n \in C[\mathbb{R}, C_d^{k+1}(\mathbb{R})]$ and $\partial_t u_n \in C[\mathbb{R}, C_d^k(\mathbb{R})]$.

Now we can do rough control. Use $E_k(u_n)$. Estimation lemmas tell us that we have the bound

$$E_k[u_n](t) \leq 2E_k[u_0] + 1$$

The next step is to analyze convergence. Consider $\hat{u}_n = u_{n+1} - u_n$ and $E_k[\hat{u}_n]$. The quantity $E_k[\hat{u}_n]$ dominates the C_b^{k+1}-norm of $\hat{u}_n$ and the C_b^k-norm of $\partial\hat{u}_n$. Standard ODE arguments can be used to analyze the convergence of $\{\hat{u}_n\}$ by considering $F(u_n, \partial u_n) = F(u_{n-1}, \partial u_{n-1})$ and using that F is smooth to bound it. It can be proved that this is a Cauchy sequence by considering the $\mathcal{E}_j$ values of each u_n. The usual fixed point argument for ODEs completes the proof of existence. Uniqueness follows from the technical estimate above and a Gronwall's inequality applied to a difference between two solutions. □

This proof does not give any information concerning the limit of ϵ_k as $k \to \infty$. The next example gives a counterexample to existence for smooth data.

The local existence of solutions to the 1+1 dimension nonlinear wave equation is a subtle matter. Consider the system

$$\begin{cases} u_{tt} - u_{xx} = F(u, \partial u) \\ u(0, x) = f(x) \\ u_t(0, x) = g(x) \end{cases}$$

In order to illustrate the nuances in local existence, below is an explicit example where the local existence of a smooth solution given smooth initial data fails. This is precisely stated as follows.

Theorem 4.6. *Given system*

$$\begin{cases} u_{tt} - u_{xx} = u_t^2 \\ u(0, x) = f(x) \\ u_t(0, x) = g(x) \end{cases}$$

There exists initial data $f, g \in C^\infty(\mathbb{R})$ where for any $\epsilon > 0$ there is no $u \in C^\infty[(-\epsilon, \epsilon) \times \mathbb{R}]$ which is a solution.

Proof. Consider $u_{tt} = u_t^2$, and initial data for $u_t = g(x) = k > 0$. Then we have

$$u_t = \frac{k}{1 - kt} \text{ and } u = u(0) - \ln(1 - kt)$$

This implies that u blows up at $t = \frac{1}{k}$. We now want to use this fact to build initial data to fit our theorem.

Take $g(x) = k$ and $f = 0$ when $x \in [a - \frac{1}{k}, a + \frac{1}{k}]$. Then we have that the solution is $u = -\ln(1 - kt)$. This is the unique solution in the domain of dependence, a consequence of propagation of information discussed above. Therefore this u is a solution in the triangle with base $\{0\} \times [a - \frac{1}{k}, a + \frac{1}{k}]$ and vertex $(\frac{1}{k})$

Now take $\phi \in C^\infty(\mathbb{R})$ where $\phi(x) = 1$ when $|x| \leq 1$ and $\phi(x)$ vanishes when $|x| \geq 2$. Then for integer $k \geq 1$, construct $g_k(x) = k\phi(x - 4k), f(x) = 0$. This implies that $g_k(x) = k$ when $x \in [4k - \frac{1}{k}, 4k + \frac{1}{k}]$. Just as above, solution u_k will blow up at $t = \frac{1}{k}$. Observe that for distinct k, the corresponding g_k are non-zero on distinct intervals.

Define a smooth function

$$g(x) = \sum_{k=1}^{\infty} g_k(x)$$

Then we know $g(x) = k$ when $x \in [4k - \frac{1}{k}, 4k + \frac{1}{k}]$, so take this g and $f \equiv 0$ as initial data. Then any solution u blows up at time $\frac{1}{k}$ for any $k \in \mathbb{N}$. □

This example illustrates that smooth initial data is not enough to guarantee the local existence of a smooth solution. In particular it suggests that additional conditions must be imposed for large values of x in any theorem addressing local existence of smooth solutions given smooth initial data

5. Energy and Global Properties of the Wave Equation

Consider the wave equation

$$\Box u = F(u, \partial u)$$

and define the energy quantity

Definition 5.1. The *energy* in one dimension of a function u is $E[u]$ where

$$E[u] = \frac{1}{2} \int_{\mathbb{R}} [u_t^2 + u_x^2 + \int F du](t, x) dx$$

In general this takes the form

$$E[u] = \frac{1}{2} \int_{\mathbb{R}^n} [u_t^2 + |\nabla u|^2 + \int F du](t, x) dx$$

Energy is conserved over time in the linear case. In one dimension, this is easy to see.

$$\frac{dE}{dt} = \frac{1}{2} \int_{\mathbb{R}} [u_t u_{tt} + u_t u_{xx}] dx = \frac{1}{2} \int_{\mathbb{R}} [u_{tt} - u_{xx}] u_t dx = 0$$

This follows from similar arguments in higher dimensions.

In general for the non-linear form, when F has compact support in x for all finite time we have the following non-zero result

$$\frac{dE}{dt} = \frac{1}{2} \int_{\mathbb{R}} [u_t F] dx$$

Energy which is not conserved in the non-linear form has great utility when moving from local to global existence. This is demonstrated in the following theorem.

Theorem 5.2. *Consider*

$$\begin{cases} \Box u = -u^k \text{ where } k \text{ is odd} \\ u(0, x) = f(x) \;\; f \in C_0^\infty(\mathbb{R}) \\ u_t(0, x) = g(x) \;\; g \in C_0^\infty(\mathbb{R}) \end{cases}$$

There is a unique $u \in C^\infty(\mathbb{R}^2)$ solving the above system in all finite time

Proof. First note that $u(t,x) = 0$ for $|x| \geq C + |t|$ as $f(x), g(x) = 0$ for $|x| \geq C$. This is a consequence of the uniqueness and finite propagation discussed earlier.

We already have a local solution. We want to examine the energy with a sign modification

$$\hat{E}(t) = \frac{1}{2}\int_{-\infty}^{\infty}[u_t^2 + u_x^2 + \frac{2}{k+1}u^{k+1}](t,x)dx$$

The restriction that k be odd is critical here as it implies that all terms in the integrand are non-negative. We can then obtain

$$\frac{d\hat{E}}{dt} = \int_{-\infty}^{\infty}[u_{tt} - u_{xx} + u^k]u_t dx = 0$$

We have that $\hat{E}$ is conserved. It can be shown that for the non-linear wave equation with smooth F, f, g where the data is also compactly supported that either it exists in all time or $||u(t,\cdot)||_{C_b(\mathbb{R})}$ is unbounded on the maximum existence interval.

To prove the theorem, we simply need to show that u does not blow up in finite time. To do this, invoke the following. It is a fact that for $\phi \in C_0^1(\mathbb{R})$

$$|\phi(x)| \leq (\int_{-\infty}^{\infty}[\phi^2 + (\phi^{'})^2](s)ds)^{\frac{1}{2}} \ \forall x \in \mathbb{R}$$

Applying this to the problem reduces it to showing that

$$\int_{-\infty}^{\infty}[u^2 + u_x^2](t,x)dx$$

does not blow up in finite time. By the above, $\hat{E}$ is conserved, so there is a constant C such that

$$\int_{-\infty}^{\infty}[u_t^2 + u_x^2](t,x)dx \leq C$$

Now define

$$H(t) = \int_{-\infty}^{\infty}u^2(t,x)dx$$

and compute

$$|\frac{dH}{dt}| = 2|\int_{-\infty}^{\infty}uu_t dx| \leq \int_{-\infty}^{\infty}[u^2 + u_t^2]dx \leq C + H$$

We can then conclude that $H + C > 0$ as long as u is not identically zero, so $|\frac{d(H+C)}{dt}| \leq H + C$. Integrating this shows that F does not blow up in finite time. □

6. Decay of Waves

In order to study the asymptotic behavior of waves, we want to consider the following system which has 3 space dimensions.

$$\begin{cases} u_{tt} - \Delta u = F \text{ where } F \in C^2(\mathbb{R}) \\ u(0,x) = f(x) \; f \in C_0^3(\mathbb{R}^3) \\ u_t(0,x) = g(x) \; g \in C_0^2(\mathbb{R}^3) \end{cases}$$

Theorem 6.1. [1] *The solution to u above is given by the formula*

$$u(t,x) = \frac{1}{4\pi t^2} \int\limits_{|x-y|=t} [tg(y) + f(y) - (\partial_j f)(y)(x^j - y^j)] d\sigma(y)$$

$$+ \frac{1}{4\pi} \int\limits_0^t \int\limits_{S^2} (t-s) F[s, x + (t-s)y] d\sigma(y) ds$$

The measure $d\sigma$ is the standard measure on the surface of the 2-sphere.

Immediate consequences are that determining the solution at (t,x) only necessitates knowing f and g on the sphere $\{y : |x-y| = t\}$ and F on the cone with the base of this sphere and vertex (t,x).

We can also derive the following decay condition for the linear equation.

Theorem 6.2. *Consider the PDE*

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(0,x) = f(x) \; f \in C_0^\infty(\mathbb{R}^3) \\ u_t(0,x) = g(x) \; g \in C_0^\infty(\mathbb{R}^3) \end{cases}$$

where f and g vanish for $|x| \geq R$. Then there is C_R a constant depending only on R such that, for all time

$$|u(t,x)| \leq \frac{C_R}{\sqrt{1+t^2}} [||f||_{C_b(\mathbb{R}^3)} + \sum_j ||\partial_j f||_{C_b(\mathbb{R}^3)} + ||g||_{C_b(\mathbb{R}^3)}]$$

Proof. First take $0 < t \leq 2(R+1)$, then we have

$$|u(t,x)| \leq \frac{1}{4\pi t^2} \int\limits_{|x-y|=t} [tg(y) + f(y) - (\partial_j f)(y)(x^j - y^j)] d\sigma(y)$$

$$\leq [||f||_{C_b(\mathbb{R}^3)} + t \sum_j ||\partial_j f||_{C_b(\mathbb{R}^3)} + t||g||_{C_b(\mathbb{R}^3)}$$

Because $t \leq 2(R+1)$, use

$$C_R = 2(R+1)\sqrt{1 + 4(R+1)^2}$$

We can then work on the integral terms

$$\int\limits_{|x-y|=t} |f(y)| d\sigma(y) \leq ||f||_{C_b(\mathbb{R}^3)} \int\limits_{|x-y|=t, |y| \leq R} d\sigma(y)$$

We know $t \geq 2(R+1)$, so maximize the area of the intersection of a sphere of radius t and ball of radius R. Assume that the center of the sphere is not inside the ball as otherwise the intersection is empty. Center the sphere at the origin. Set θ to be the angle between any point and the z axis where the z axis has been placed to go through the center of the sphere (the origin) and the center of the ball. Then take any point p in the intersection of the sphere and ball. We then know $\sin(\theta) \leq \frac{R}{t}$. And for maximum angle θ_{max} we have the bound

$$t^2 \int_0^{2\pi} d\phi \int_0^{\theta_{max}} \sin\theta d\theta = 2\pi t^2(1-\cos\theta_{max}) = 2\pi t^2[1-\sqrt{1-\frac{R^2}{t^2}}] = 2\pi R^2[1+\sqrt{1-\frac{R^2}{t^2}}]^{-1} \leq 2\pi R^2$$

This is a bound in terms of R as desired. The other terms can be done similarly for a bound in terms of R. □

This means that the solution to such a system has a decay of $\frac{1}{t}$ in 3 dimensions where the 1 dimensional form had no decay.

In general, it can be shown that in n+1 dimensions there is decay of order $t^{-\frac{(n-1)}{2}}$[1]. This suggests something is special about dimensions $n = 3, 5, 7....$ These are where *Huygen's Principle* applies. This says that when the initial data (the *initial disturbance*) are compactly supported, the solution is supported on a small volume (i.e. it has a *sharp trailing edge*). This is why sound enters our ears but then quickly dies off, in contrast to waves on the surface of a pond which reverberate many times.

There is contemporary research concerning the behavior of systems like the one above. For instance, it can be shown that for the one-dimensional system, the *defocusing nonlinear wave equation* $\Box u = |u|^{p-1}u$ $(p > 1)$, there are global, smooth solutions given smooth, compactly supported data. In a 2011 paper, Lindblad and Tao show that the solution decays in the L^∞ norm as $t \to \infty$[3]. This is in contrast to the linear case where the solutions to the linear form in 1 dimension do not decay.

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