ON THE GROUP-THEORETIC PROPERTIES OF THE AUTOMORPHISM GROUPS OF VARIOUS GRAPHS

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ABSTRACT. In this paper we provide an introduction to the properties of one important connection between the theories of groups and graphs, that of the group formed by the automorphisms of a given graph. We provide examples of important results in graph theory that can be understood through group theory and vice versa, and conclude with a treatment of Frucht's theorem.

CONTENTS

1. INTRODUCTION

Groups and graphs are two highly important kinds of structures studied in mathematics. Interestingly, the theory of groups and the theory of graphs are deeply connected. In this paper, we examine one particular such connection: that which emerges from the observation that the automorphisms of any given graph form a group under composition.

In section 2, we provide a framework for understanding the material discussed in the paper. In sections 3, 4, and 5, we demonstrate how important results in group theory illuminate some properties of automorphism groups, how the geometric properties of particular embeddings of graphs can be used to determine the structure of the automorphism groups of *all* embeddings of those graphs, and how the automorphism group can be used to determine fundamental truths about the structure of the graph. In section 6, we present and prove Frucht's theorem, a deep and elegant result that relates every finite group to a corresponding graph.

We will not assume knowledge of groups or graphs, will clearly define any important topics or results that appear in the paper but that a reader may not be familiar with, and will make an effort to present and explain our results in an intuitive manner.

2. Fundamental Definitions, Concepts, and Theorems

Definition 2.1. *A graph $X = (V, E)$ is a set of vertices V and a set of edges E connecting those vertices.*

An edge can be thought of as an (unordered, for our purposes) pair of vertices. Two vertices are said to be *adjacent* if and only if there is an edge connecting them.

Definition 2.2. *An automorphism ϕ of a graph $X = (V, E)$ is a mapping $\phi : V \mapsto V$ of vertices such that for all pairs of vertices $a, b \in V$, $\phi(a)$ is adjacent to $\phi(b)$ if and only if a is adjacent to b.*

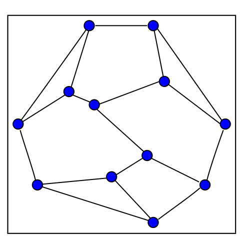
Definition 2.3. *The automorphism group of a graph X, $Aut(X)$, is the set of all its automorphisms.*

As the name suggests, the automorphism group forms a *group* under composition of automorphisms, the notion of which we shall formalize (see **Definition 2.6**).

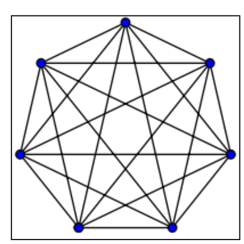
Proposition 2.4. *For an arbitrary graph X, $1 \leq |Aut(X)| \leq n!$.*

Proof. To show the lower bound, observe that the identity is always an automorphism. To show the upper bound, observe that all automorphisms are permutations of vertices, so the largest possible automorphism group is the symmetric group. $|S_n| = n!$. □

A graph is said to be an identity graph if its only automorphism is the identity. Additionally, there are many graphs with $|Aut(X)| = n!$. A graph is said to be complete if every pair of vertices is adjacent, and all complete graphs achieve the upper bound of $n!$ automorphisms. An identity graph and a complete graph are both shown.



(A) An identity graph. Image by Koko90, under CC-BY-SA-3.0 license.



(B) A complete graph. Image released into public domain by creator.

Definition 2.5. *A graph $X = (V, E)$ is said to be vertex-transitive if for all pairs of vertices $a, b \in V$, there exists an automorphism $\phi \in Aut(X)$ such that $\phi(a) = b$.*

Definition 2.6. *A group $G = (S, \cdot)$ is any set S with an operation $\cdot : S \times S \mapsto S$ (sometimes called "the group law") defined on it that satisfies the following axioms:*

Associativity: for any elements $a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

Existence of the identity: There is an element $e \in S$ such that for every element $a \in S$, it holds that $a \cdot e = e \cdot a = a$.

Existence of inverses: for every element $a \in S$, there is an "inverse element" a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$

In some cases where the group law is unambiguous, we omit its symbol: if $a, b \in G$, $a \cdot b = ab$.

Two elementary examples of groups are $\mathbb{R}$ under addition, and the set of permutations of a list of n elements under composition. When all permutations are allowed, this group is known as the *symmetric group* S_n.

It is easy to see that automorphisms of graphs satisfy these axioms if the group law is taken to be composition, which justifies our use of the term "automorphism group".

Note that since the uppercase G is often used to denote both graphs and groups, but we will often be working with both in this paper, we will generally write X for an arbitrary graph and G for an arbitrary group in order to avoid confusion.

Definition 2.7. *A subgroup A of a group G is a subset of G that is also a group under the same operation $(\cdot)$.*

Definition 2.8. *If G is a group, a coset is a set gA where $g \in G$ and A is a subgroup of G, defined as $gA = \{ga | a \in A\}$.*

Proposition 2.9. *The union of all cosets of a group is the group itself. Furthermore, any two cosets have the same cardinality.*

Proof. To show that any two nonequal cosets of a group G are disjoint, take $g_1, g_2 \in G$, and suppose the intersection of the cosets g_1A and g_2A is not the empty set. It must be true that there exists some pair of elements $a_1, a_2 \in A$ such that $g_1a_1 = g_2a_2$, but then $g_1 = g_2a_2a_1^{-1}$, and $g_2 = g_1a_1a_2^{-1}$, so for all elements $g_1a \in g_1A$, it holds that $g_1a = g_2a_2a_1^{-1}a \in g_2A$, and for all elements $g_2a \in g_2A$, it holds that $g_2a = g_1a_1a_2^{-1}a \in g_1A$. Thus $g_1A = g_2A$.

To show any two cosets have equal cardinality, observe that if $a \in A$, then $a \mapsto ga$ is a bijection. Finally we show that

$$G = \bigcup_{g \in G} gA$$

Since $g \in G$, $g = ge$ and $e \in A$ always, we have that each element g of the group is found within at least "its own coset", gA. □

Theorem 2.10 (Lagrange's Theorem)**.** *For a subgroup A of a finite group G, $|G| = |A| \cdot |G : A|$ where $|G : A|$ is the number of distinct cosets of A.*

Proof. Since all cosets are disjoint and have equal size, it is clear that the size of the group is the number of its equally-sized components multiplied by the number of its components, so we use **Proposition 2.9** to obtain $|G| = |A| \cdot |G : A|$. □

Definition 2.11. *A group G is said to act on a set S if we have a mapping $\phi : G \times S \mapsto S$ where for every element $s \in S$, the following are satisfied: $\phi(e, s) = s$, and for all pairs of elements $g_1, g_2 \in G$, it holds that $\phi(g_1, \phi(g_2, s)) = \phi(g_1 \cdot g_2, s)$.*

In a looser sense, we can say that if G acts on S, G is a group of mappings which are defined from S to itself such that the identity element of G is also the identity transformation and the group law is equivalent to composition of mappings.

Definition 2.12. *If a group G acts on a set S, then the orbit of an element $s \in S$ is the set of elements in S that s is mapped to by members of G. In notation,*

$$Orb(s) = \bigcup_{g \in G} g(s)$$

Definition 2.13. *If a group G acts on a set S, then the stabilizer of an element $s \in S$ is the set of all elements G that map s to itself. In notation,*

$$Stab(s) = \{g \in G | g(s) = s\}$$

3. Example 1: The Orbit-Stabilizer Theorem and its Application to Graph Automorphisms

Theorem 3.1 (The Orbit-Stabilizer Theorem)**.** *If G is finite group acting on a finite set S, and s is an arbitrary element of S, $|G| = |Orbit(s)||Stab(s)|$.*

Proof. Take an arbitrary $s \in S$. Clearly, for every element $g \in Stab(s)$, $g \in G$. Thus $Stab(s)$ inherits associativity from G. By the definition of the stabilizer, the identity $e \in G$ is in the $Stab(s)$ as well.

Finally we must show that for every element $g \in Stab(s)$, $g^{-1} \in Stab(s)$ as well. Now observe that

$$\begin{aligned} s &= es \\ &= g^{-1}gs \\ &= g^{-1}s \end{aligned}$$

and it becomes clear that $Stab(s)$ is a subgroup of G.

Now we need to show that there are exactly as many elements in $Orb(s)$ as there are distinct cosets of $Stab(s)$, which we can do by establishing a bijection between them. An appropriate mapping is $\phi : G \mapsto Orbit(s)$ given by $\phi(gs) = gStab(s)$. Since this is both injective and surjective, it is a bijection. □

The orbit-stabilizer theorem provides an extremely simple way to compute the order of any graph's automorphism group. We shall make use of it in the next section. Suppose a graph $G = (V, E)$. Pick an arbitrary vertex $v \in V$. Now, each automorphism ϕ that maps v to itself is an element of its stabilizer. The orbit of v is every vertex that v can be mapped to by automorphisms. Thus, $|Aut(X)| = |Orb(v)||Stab(v)|$.

We also see an immediate corollary:

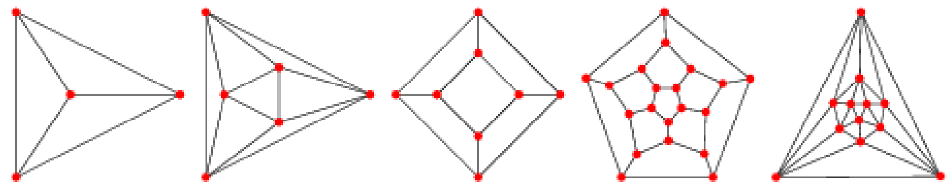
Corollary 3.2. *For any vertex-transitive graph X on n vertices, $|Aut(X)| \geq n$.*

This is clear since the orbit of any vertex in X is the whole set of vertices, with cardinality n. This bound is achieved on any graph with $n = 1$ or $n = 2$.

4. Example 2: On the Automorphism Groups of the Platonic Solid Skeleton Graphs

Definition 4.1. *A Platonic solid is a regular polyhedron such that all faces are regular polygons and the same number of faces meet at each vertex.*

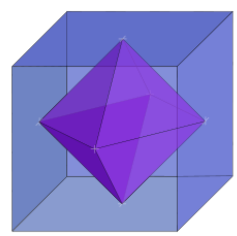
There are five Platonic solids: the tetrahedron, the cube, the octahedron, the icosahedron, and the dodecahedron. Image: [4]



Definition 4.2. *Two solids are said to be dual to each other if one can be constructed by taking the center of each face and letting it become a new vertex which is adjacent to the vertices replacing those faces that the original face was adjacent to.*

Theorem 4.3. [6] *If two solids are dual, they have identical automorphism groups.*

Although a Platonic graph is a Platonic graph no matter whether or not it is drawn in the form of the corresponding Platonic solid, we shall assume they are all drawn in solid form in order to exploit the geometric symmetries of each graph to generate the automorphism group.



The cube and octahedron are dual to each other. Image from Wikimedia Commons, 4c commonswiki, distributed under CC-BY-SA-3.0.

Proposition 4.4. *If T is the tetrahedron graph, $|Aut(T)| = 24$.*

Proof. We have that it is simply an embedding of the complete graph K_4 in 3-space, so any permutation of vertices is an automorphism. □

Proposition 4.5. *If C is the cube graph, $|Aut(C)| = 48$.*

Proof. We first obtain 24 automorphisms through rotations of the cube. Lastly we can invert each corner of the cube with the other, so we multiply again by a factor of 2, giving us $|Aut(C)| = 24 \cdot 2 = 48$. □

Proposition 4.6. *If O is the octahedron graph, $|Aut(O)| = 48$.*

Proof. For the octahedron, we can use **Theorem 4.3** to obtain $|Aut(O)| = |Aut(C)| = 48$. □

Proposition 4.7. *If I is the icosahedron graph, $|Aut(I)| = 120$.*

Proof. For the icosahedron, we can use the orbit-stabilizer theorem. For each element $v \in V$, $|Aut(I)| = |Orbit(v)||Stab(v)|$. Pick an arbitrary vertex as v and observe that $|Orbit(v)| = 12$ by vertex-transitivity. Furthermore, $|Stab(v)| = 10$ since we can rotate around vertex v in 5 ways while always mapping v to itself, and we can also reflect the solid over a plane parallel to the axis through v and the opposite vertex (which again will map v to itself), obtaining another factor of 2. $|Aut(I)| = (12)(10) = 120$. □

Proposition 4.8. *If D is the dodecahedron graph, $|Aut(D)| = 120$.*

Proof. For the dodecahedron, we again use **Theorem 4.3** to obtain $|Aut(I)| = |Aut(D)| = 120$. □

5. Example 3: A Tight Bound on the Product of the Chromatic Number and Independence Number of Vertex-Transitive Graphs

Definition 5.1. *For a graph X, an independent set is a set of vertices chosen such that no two chosen vertices are adjacent.*

Definition 5.2. *For a graph X, the independence number $\alpha(X)$ is the cardinality of the largest possible independent set.*

Definition 5.3. *For a graph X, the chromatic number $\chi(X)$ is the minimum number of distinct colors needed to color each vertex such that no two adjacent vertices share the same color. Equivalently, $\chi(X)$ is the minimum number of independent sets needed to cover X.*

Proposition 5.4. *For any graph X,*

$$\alpha(X)\chi(X) \geq n$$

Proof. Assume we have some χ-coloring of G. Now suppose the number of vertices colored by the kth color is C_k. We have that $n = \sum_{k=1}^{\chi} C_k$ and since $C_k \leq \alpha(X)$ for every k,

$$n \leq \sum_{k=1}^{\chi} \alpha(X) = \alpha(X)\chi(X)$$

as desired. □

Theorem 5.5 (Babai)**.** *For a vertex-transitive graph X,*

$$\alpha(X)\chi(X) \leq n(1 + ln(n))$$

This establishes that the lower bound is, while not exact, quite tight for all vertex-transitive graphs.

Proof. [3] Let $m = \frac{n}{\alpha}\lceil ln(n) \rceil$. Take some maximum independent set A. A, by **Definition 5.2**, has cardinality α. If we let ϕ be a member of $Aut(X)$, then by vertex-transitivity, the probability that some vertex $v \in V$ is not in $\phi(A)$ is less than or equal to $(1 - \frac{\alpha}{n})$. The probability that v is not covered by the union of m such translates is therefore less than or equal to $(1 - \frac{\alpha}{n})^m$, so the probability that the *entire* graph is not covered by the union of m such randomly chosen automorphisms is therefore less than or equal to $n(1 - \frac{\alpha}{n})^m$.

Thus, if the statement $n(1 - \frac{\alpha}{n})^m < 1$ holds (see **Claim 5.6**), there is some m-coloring of X. So, then, $\frac{n}{\alpha}\lceil ln(n) \rceil \geq \chi(X)$ so

$$\alpha(X)\chi(X) \leq n\lceil ln(n) \rceil \leq n(1 + ln(n))$$

as desired. □

Claim 5.6. $n(1 - \frac{\alpha}{n})^m < 1$.

Proof. First observe that $(1 - \frac{\alpha}{n}) < e^{\frac{\alpha}{n}}$. Then it will be sufficient to show that

$$n(e^{-\frac{\alpha}{n}})^{\frac{n}{\alpha}\lceil ln(n) \rceil} < 1, \text{ or equivalently, } n(e^{-\lceil ln(n) \rceil}) < 1$$

which holds whenever $e^{-\lceil ln(n) \rceil} > n$. Now use that

$$e^{\lceil ln(n) \rceil} \geq e^{ln(n)} = n$$

and our result is clear. □

6. Frucht's Theorem

Definition 6.1. *An isomorphism between two groups* $G_1 = (S_1, \cdot)$ *and* $G_2 = (S_2, \oplus)$ *is a bijection* $\phi : G_1 \mapsto G_2$ *such that for any pair of elements* $x, y \in G_1$*, it holds that*

$$\phi(x \cdot y) = \phi(x) \oplus \phi(y)$$

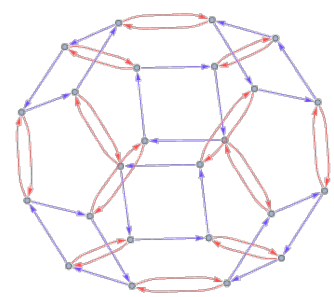
Two groups are said to be *isomorphic* if there exists an isomorphism between them, sometimes written $G_1 \simeq G_2$.

Definition 6.2. *A directed graph is a graph* $X = (V, E)$ *where* V *is the set of vertices and* E *is a set of ordered pairs representing adjacencies.*

Recall that in an undirected graph E is a set of unordered pairs. Thus the edges of a directed graph can be informally visualized as arrows pointing from one vertex to another, instead of only linking them.

Definition 6.3. *A Cayley graph is a directed graph* X *associated with a group* G *and a subset* A *of* G *(with the stipulation that* $e \notin A$*) defined by every vertex being associated with some* $g \in G$ *and* $(g_1, g_2) \in E$ *if and only if* $g_2 = g_1 a$*, for some* $a \in A$*. Such a graph is usually called the Cayley graph of* G *with respect to* A*.*

An example of a Cayley graph is shown. Often, the subset A is a set of *generators* for G, and that is the case with which we are concerned.



A Cayley graph for a finite group. Image: [5]

Definition 6.4. *A subset* S *of a group* G *may be called a set of generators for* G *if for every element* $g \in G$*,* $g = s_1 s_2 ... s_n$ *where* $\{s_1, s_2, ..., s_n\} \subseteq S$*.*

Sets of generators can be found for every group, indeed, the set of all members of the group is always a set of generators.

Theorem 6.5 (Frucht's Theorem)**.** *For any finite group* G*, there is an associated connected, undirected graph* X *such that* $Aut(X)$ *is isomorphic with* G*.*

It can be shown that there are infinitely many graphs with any desired finite automorphism group, although we are unconcerned with that fact.

Proof of Frucht's Theorem. Let G be a finite group. Take a set of generators $S = \{s_1, s_2, ..., s_m\}$ for g. Now we construct the Cayley graph $X_0 = (V, E)$ of G with respect to S, and we also impose an edge coloring on E. For each edge $e \in E$, Let $C(e)$ be the color of edge e. Edge $e = (g_1, g_2) \in E$, exists when there also exists some element $s_c \in S$ such that $g_2 = g_1 s_c$, so we color it as $C(e) = c$.

Claim 6.6. *The group* $Aut_C(X_0)$ *of automorphisms that preserve edge-coloring of* X_0 *as well as adjacency is isomorphic to* G*.*

Proof. First we show that an automorphism ϕ preserves color if and only if for all elements $s \in S, g \in G$, it holds that $\phi(gs) = \phi(g)s$. To demonstrate the "if" direction, assume that $\phi(gs) = \phi(g)s$. If an arbitrary edge (g_1, g_2) exists, then $g_2 = g_1 s_c$ where c is the color of the edge, and

$$\phi(g_2) = \phi(g_1 s_c) = \phi(g_1)s_c$$

so the color (c) of the corresponding edge is preserved. For "only if", assume $\phi(g_2) = \phi(g_1)s_c$ when $g_2 = g_1 s_c$. Now,

$$\phi(g_2) = \phi(g_1 s_c) = \phi(g_1)s_c$$

and since our assumption is for arbitrary $s_c \in S$ and $g_2 \in G$, $\phi(gs) = \phi(g)s$ always.

Now we define $\phi_i : V \mapsto V$ as $\phi_i(g) = g_i g$. It is clear that ϕ_i is always color preserving by our previous observation, since

$$\begin{aligned} \phi_i(gs) &= g_i(gs) \\ &= (g_i g)s \\ &= \phi_i(g)s \end{aligned}$$

Every one of the ϕ_i is a color-preserving automorphism, so if it also holds that if color-preserving automorphism is one of the ϕ_i, then the set of ϕ_i and the set of color-preserving automorphisms are the same.

To show that they are, let ϕ be a color-preserving automorphism. We know that for all elements $s \in S, g \in G$, it holds that $\phi(gs) = \phi(g)s$. Now we recognize that (since S is a set of generators), an arbitrary g can be written as a product of generators $s_1 s_2 s_3 ... s_m$.

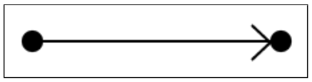
$$\begin{aligned} \phi(g) &= \phi(s_1 s_2 ... s_{m-1} s_m) \\ &= \phi(e s_1 s_2 ... s_{m-1}) s_m \\ &= \phi(e s_1 s_2 ...) s_{m-1} s_m \\ &= \phi(e)(s_1 s_2 ... s_{m-1} s_m) \\ &= \phi(e) g \end{aligned}$$

but since $\phi(e)$ is a member of G, one of the ϕ_i satisfies $\phi_i(g) = \phi(e)g$, so $\phi = \phi_i$ for some i as desired.

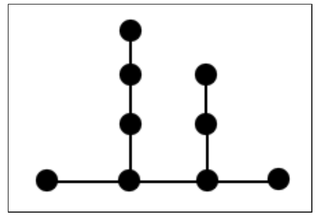
Now, we establish that the mapping $M : G \mapsto Aut_C(X_0)$ given by $g_i \mapsto \phi_i$ is an isomorphism (see **Definition 6.1**). It is injective, since if $\phi_i(g_1) = \phi_i(g_2)$ then $(g_i)g_1 = (g_i)g_2$, and we can operate on the left by g_i^{-1} to obtain $g_1 = g_2$. Because every color-preserving automorphism is one of the ϕ_i, it is surjective. Finally we must show that the mapping preserves the group law. We define $\phi_{jk} = \phi_i$ when $g_i = g_j g_k$.

$$\begin{aligned} \phi_{(jk)}(g) &= g_j g_k g \\ &= \phi_j(g_k g) \\ &= \phi_j(\phi_k(g)) \end{aligned}$$

so $Aut_C(X_0) \simeq G$. □



(A) We replace this edge of color 2



(B) with this subgraph, which can be considered an "artificial directed edge". For other colors c, let the height of the "tower" on the left be $c+1$ and on the right be c.

If we create a new graph X such that each edge of X_0 is replaced with an identity graph unique to *the edge's color*, as illustrated, it is clear that $Aut_C(X_0) \simeq Aut(X)$.

But since $Aut_C(X_0) \simeq G$, it is also true that $Aut(X) \simeq G$. We have then shown that every finite group is isomorphic to the automorphism group of some undirected, uncolored graph, and our proof is complete. □

7. Acknowledgements

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