CASTELNUOVO'S CRITERION AND BIRATIONAL GEOMETRY OF SURFACES

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Abstract. We define and discuss the definition of a blowup, and then we study the birational geometry of projective surfaces, following Hartshorne chapter V and finishing with Castelnuovo's criterion and minimal models. Much of the material from Hartshorne chapter II is used without comment. More advanced material is occasionally used without proof but citations are given.

Contents

1. What is a blowup?

The first example anyone sees of a blowup is the blowup of $\mathbb{A}^2$ at the origin. Consider the subvariety $\widetilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{P}^1$, where $\widetilde{\mathbb{A}^2}$ is given by equation $\{(x,y) \times [t:u] : xy - yt = 0\}$. The first projection map induces a map $\widetilde{\mathbb{A}^2} \xrightarrow{\pi} \mathbb{A}^2$. Away from the origin, we can see that this map is an isomorphism, but $\pi^{-1}(0,0) \cong \mathbb{P}^1$, so the point $(0,0)$ now corresponds to a codimension-one subvariety of $\mathbb{A}^2$. Consider a line through the origin $L = V(ax - by) \subset \mathbb{A}^2$. the points of this line are paramentrized by $t \to (bt, at)$, so away from zero, the line $\pi^{-1}(L)$ is parametrized by $t \to (at, bt) \times [b : a]$. The closure of this line intersects $\pi^{-1}(0,0)$ at $[b : a]$. So, lines through the origin in $\mathbb{A}^2$ correspond to lines in $\widetilde{\mathbb{A}^2}$ intersecting $\pi^{-1}(0,0) = \mathbb{P}^1$ at the direction through which they pass through the origin in $\mathbb{A}^2$. We have taken the point at the origin and added in the first order information around it, letting us detect not only when subvarieties pass through the origin but at what direction they pass through. In general, given a noetherian scheme X and a closed subscheme Y, a blowup can be thought of as a way of adding in all of the first order information around Y. What this means is we want to create a new variety $\widetilde{X}$ such that subschemes of X will correspond in a natural way to subschemes of $\widetilde{X}$, and such that $\widetilde{X}$ contains information about both the intersection of a subscheme Z with Y and the directions the tangent space of Z has in common with the normal bundle of Y. From this heuristic picture, one can tell that over Y, $\widetilde{X}$ should look like the projectivization of the normal bundle of Y. Since the normal bundle corresponds to $\mathbf{Proj}_Y(\mathcal{J}_Y/\mathcal{J}_Y^2)$,[1] and this is just the ideal $\mathcal{J}_Y$ restricted to Y, we have some motivation for the following definition of Grothendieck.

Definition 1.1. Given a noetherian scheme X and a coherent sheaf of ideals $\mathcal{J}$ on X, we consider $\widetilde{X} = \mathbf{Proj}(\bigoplus \mathcal{J}^d)$, where $\mathcal{J}^0 = \mathcal{O}_X$. This is called the blowup of X along Y and is also written $Bl_Y(X)$, or $\widetilde{X}$ when there is no ambiguity about the sheaf of ideals in question.

Example 1.2. Let's look at the blowup of $\mathbb{A}^2$ at the sheaf $\mathcal{I} = (x, y)$. We have a surjection $\mathcal{O}_{\mathbb{A}^2}[t, u] \twoheadrightarrow \bigoplus \mathcal{I}^d$ given by $t \to x$, $u \to y$. This induces a closed embedding $\widetilde{\mathbb{A}^2} \hookrightarrow \mathbf{Proj}(\mathrm{Sym}(\mathcal{O}^2_{\mathbb{A}^2})) = \mathbb{A}^2 \times \mathbb{P}^1$. We can see that $\widetilde{\mathbb{A}^2}$ satisfies $xu - yt = 0$ in $\mathbb{A}^2 \times \mathbb{P}^1$, and since $V(xu - yt)$ is an irreducible closed subset, of the same dimension, it must be equal to $\widetilde{\mathbb{A}^2}$.

Remark 1.3. For "well-behaved" ideals $\mathcal{I} = (f_1, \dots f_r) \subset k[x_1, \dots x_n]$, the blowup of $\mathbb{A}^n$ at $\mathcal{I}$ will be given by $(x_1, \dots x_n) \times [y_1 : \dots : y_r] : f_i y_j - f_j y_i = 0$ for $i, j = 1, \dots r$. In particular, if there is a regular sequence generating $\mathcal{I}$ then our blowup will be given by these simple equations. Any radical ideal, which is an ideal corresponding to a reduced subscheme, will be well-behaved in this sense.[2] Since in this paper the only blowups we will be considering are of points on smooth projective varieties, the above definition of the blowup is sufficient, and much of the generality of the Proj construction is unnecessary. I included it both because of the beauty of the construction and because it makes the proof of resolution of singularities much more transparent; see theorem 5.4.

Since we can pull back subschemes under a map, it makes sense that we should be able to pull back sheaves of ideals as well. However, some caution is required. Given a map $X \xrightarrow{f} Y$ and $\mathcal{J} \subset \mathcal{O}_Y$, the standard pullback $f^*\mathcal{J} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ may not actually be a subsheaf of $\mathcal{O}_X$. This is because tensoring with $\mathcal{O}_X$ is not necessarily left exact and relates to the failure of f to be a flat morphism. What we do is we take the image of the map $f^*\mathcal{J} \to f^*\mathcal{O}_Y = \mathcal{O}_X$ induced by the inclusion $\mathcal{J} \hookrightarrow \mathcal{O}_Y$. We call this the inverse image ideal sheaf and denote it by $f^{-1}\mathcal{J} \cdot \mathcal{O}_X$, or just $\mathcal{J} \cdot \mathcal{O}_X$. It is easy to check that the scheme on X cut out by $\mathcal{J} \cdot \mathcal{O}_X$ has underlying space equal to the preimage of the space cut out by $\mathcal{J}$, so this gives us a canonical method of putting a scheme structure on the preimage of a subscheme.

Theorem 1.4. *Let X, $\mathcal{J}$, $\pi : \widetilde{X} \to X$ be as above. Then:*

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[1]The relative Proj construction is discussed in Hartshorne II.7, but most of the details of the construction are left to the reader. For a thorough proof, see [4]

[2]See Matsumara.

(a) The inverse image ideal sheaf $\widetilde{\mathcal{J}} = \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$ *is invertible on* $\widetilde{X}$.
(b) If Y *is the closed subscheme corresponding to* $\mathcal{J}$, *then* π *is an isomorphism away from* Y.

Proof. (a) We can determine $\widetilde{J}$ locally. On an affine open set U, the coherent sheaf $\mathcal{J} \cdot \bigoplus_{d\geq 0} \mathcal{J}^d(U)$ is the module $\bigoplus_{d\geq 0} \mathcal{J}^{d+1}(U)$. But this is precisely $\mathcal{O}_{\widetilde{X}}(1)(U)$, so the sheaf $\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(1)$.

(b) Since $\mathcal{J}_{|U} = \mathcal{O}_U$, we have that $\pi^{-1}(U) = \mathbf{Proj}(\mathcal{O}_U[t]) = U$. □

Property (b) above tells us that π is birational. If X is a variety, then $\mathcal{J}(U)$ will be an integral domain for any $U \subset X$, so the sheaf $\bigoplus \mathcal{J}^d$ is a sheaf of integral domains, and the relative Proj construction tells us that $\widetilde{X}$ is also a variety. π is automatically proper from the relative Proj construction, and since by (b) above the image of π contains a dense open subset of X, π is surjective. In the case that X is a projective variety, we can see by the construction of $\widetilde{X}$ as a subset of $X \times \mathbb{P}^r$ that π is a projective morphism and that $\widetilde{X}$ is also a projective variety.

We have finally arrived at the theorem about blowups we are most interested in for the sake of this paper, the computation of the normal bundle of the exceptional divisor.

Theorem 1.5. *Let* X *be a smooth variety over* k, *and let* $Y \hookrightarrow X$ *be a smooth closed subvariety with ideal sheaf* $\mathcal{J}$. *Let* $\widetilde{X} \xrightarrow{\pi} X$ *be the blowing up of* $\mathcal{J}$, *and let* $Y' \hookrightarrow \widetilde{X}$ *be the subvariety defined by* $\mathcal{J}' = \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$. *Then:*

(a) Y', *with the induced map* $Y' \xrightarrow{\pi} Y$ *is isomorphic to* ***Proj***$(Sym^n(\mathcal{J}/\mathcal{J}^2)$.
(b) Under this isomorphism, $\mathcal{N}_{Y'/\widetilde{X}} \cong \mathcal{O}_{Y'}(-1)$.

Proof. We know that since Y is smooth, its ideal $\mathcal{J}$ is locally generated by a regular sequence, and therefore $\mathcal{J}/\mathcal{J}^2$ is locally free $\mathcal{J}^d/\mathcal{J}^{d+1} \cong \mathrm{Sym}^d(\mathcal{J}/\mathcal{J}^2)$.[3] So, we have

$$\begin{aligned} Y' &= \mathbf{Proj} \bigoplus (\mathcal{J}^d \otimes \mathcal{O}_X/\mathcal{J}) \\ &= \mathbf{Proj} \bigoplus \mathcal{J}^d/\mathcal{J}^{d+1} \\ &= \mathbf{Proj} \bigoplus \mathrm{Sym}^d(\mathcal{J}/\mathcal{J}^2) \end{aligned}$$

By the proof of theorem 1.4 (a), $\mathcal{J}' = \mathcal{O}_X(1)$, so by the functorial properties of the relative proj construction, $\mathcal{J}' \otimes \mathcal{O}_{Y'} = \mathcal{J}'/\mathcal{J}'^2 = \mathcal{O}_{Y'}(1)$, and therefore $\mathcal{N}_{Y'/\widetilde{X}} = \mathcal{O}_{Y'}(-1)$.

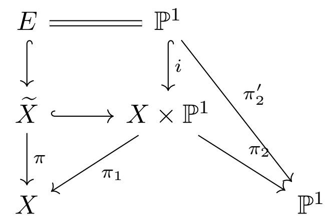
□

Remark 1.6. The above proof also tells us that $\widetilde{Y'}$ is smooth, since it is locally isomorphic to $Y \times \mathbb{P}^{r-1}$, where $r = \mathrm{Codim}(Y, X)$. Since $\widetilde{X}$ is smooth away from Y', and Y' is smooth, we know

[3]For details, see Hartshorne II.8.22, or for a complete proof see Matsumara p110

that $\widetilde{X}$ is smooth.[4] Also, it tells us that $\mathcal{J}^n/\mathcal{J}^{n+1} = \mathcal{O}_{Y'}(n)$, a fact which we will use repeatedly in later computations.

For those who are happy to only work with blowups of points on surfaces, here's a proof that the normal bundle of the exceptional divisor is $\mathcal{O}_E(-1)$. We can do the computation of the blowup locally, so without loss of generality assume X is an affine variety. Then the exceptional set Y is cut out by local parameters $I = (x, y)$, and as in example 1.2, $\widetilde{X} = Z(ux - ty) \subset X \times \mathbb{P}^1$.



The normal bundle of $E = Y \times \mathbb{P}^1$ in $X \times \mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}$ when restricted to any fibre over a point $y \in Y$, because it is cut out by elements of zero degree in $\mathcal{O}_X[t, u]$. Over Y it looks like $I/I^2 = (x, y)/(x^2, xy, y^2)$ which is the free module $\mathcal{O}(Y) \oplus \mathcal{O}(Y)$. Thus $\mathcal{N}_{E/X\times\mathbb{P}^1} = \mathcal{O}_E \oplus \mathcal{O}_E$.
To calculate the normal bundle of $\widetilde{X}$ inside $X \times \mathbb{P}^1$, we have

$$(N_{\widetilde{X}/X\times\mathbb{P}^1})|_E = i^* N_{\widetilde{X}/X\times\mathbb{P}^1} \tag{1.7}$$

$$= i^* \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1) \tag{1.8}$$

$$= \pi_2'^* \mathcal{O}_{\mathbb{P}^1}(1) \tag{1.9}$$

$$= \mathcal{O}_E(1) \tag{1.10}$$

(1.8) follows from the fact that $ux - ty$ is a section of of $\mathcal{O}_{X\times\mathbb{P}^1}(1) = \pi_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. So, the normal exact sequence

$$0 \longrightarrow N_{E/\widetilde{X}} \longrightarrow N_{E/X\times\mathbb{P}^1} \longrightarrow N_{\widetilde{X}/X\times\mathbb{P}^1}|_E \longrightarrow 0$$

becomes

$$0 \longrightarrow N_{E/\widetilde{X}} \longrightarrow \mathcal{O}_E \oplus \mathcal{O}_E \longrightarrow \mathcal{O}_E(1) \longrightarrow 0$$

So taking determinants we see that $N_{E/\widetilde{X}} = \mathcal{O}_E(-1)$.

Another way to compute $\widetilde{J}_E/\widetilde{J}_E^2$ is to compute locally, taking some affine open set $V \ni P$ and local coodinates x, y around P such that $\pi^{-1}(V) = Z(yt - xu) \subset V \times \mathbb{P}^1$. Then on the affine set $A_t^2 = \{t \neq 0\}$, $\mathcal{J}^n = (x^n)$, and $\mathcal{O}_{A_t^2}/\mathcal{I} = k[u/t]$, and on the affine set $A_u^2 = \{u \neq 0\}$, $\mathcal{J}^n = (y^n)$, and $\mathcal{O}_{A_t^2}/\mathcal{J} = k[t/u]$. Thus on the overlap of the two sets, since $x^n = (t/u)y^n$, the transition map of $\mathcal{J}^n/\mathcal{J}^{n+1}$ is given by multiplication by $(t/u)^n$, so $\mathcal{J}^n/\mathcal{J}^{n+1} = \mathcal{O}_E(n)$, which implies the result about the normal bundle and will also be used in later computations.

[4]If a quotient of a noetherian local ring by an element which is not a zero divisor is regular, then the local ring itself is regular. See Hartshorne II.8.24

2. Cohomological premininaries

Definition 2.1. Given a map $X \xrightarrow{f} Y$ of topological spaces and a sheaf $\mathcal{F}$ of abelian groups on X, for any $i \geq 0$ we define the higher direct image functor $R^i f_*(\mathcal{F})$ to be the sheaf associated to the presheaf

$$V \to H^i(f^{-1}(V), \mathcal{F}_{|f^{-1}(V)}).$$

If we consider X a scheme with its canonical map to Spec $\mathbb{Z}$, then the global sections of the higher direct image functor are just the cohomology groups of $\mathcal{F}$, or if $Y =$ Spec A is any affine scheme the global sections are the cohomology modules with coefficients in A. Thus the higher direct image functors can be thought of as a globalized version of cohomology.

Theorem 2.2. *If $X \xrightarrow{f} Y$ is a map of spaces, and $\mathcal{F}$ a sheaf of abelian groups on X, and $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$, then for all $i \geq 0$,*

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

Proof. Consider some injective resolution $\mathcal{F} \to \mathcal{J}^*$ of $\mathcal{F}$, and then consider the pushforward $f_*\mathcal{F} \to f_*\mathcal{J}^*$. The fact that $R^i f_*\mathcal{F} = 0$ is equivalent to f_* being exact, so the pushforward is also a resolution. Then the cohomology of $\mathcal{F}$ is the cohomology of the complex $\Gamma(X, \mathcal{J}^*)$ and the cohomology of $f_*\mathcal{F}$ is the cohomology of the complex $\Gamma(Y, f_*\mathcal{J}^*)$, so the two are equal.

□

Fitting into the heuristic of "globalized cohomology", this theorem tells us that if $\mathcal{F}$ is cohomologically trivial relative to Y, then one can pass between $\mathcal{F}$ and $f_*\mathcal{F}$ without affecting the cohomology groups. In particular, this will always happen whenever $\mathcal{F}$ is an affine morphism. It will turn out that in fact a blowup of a surface at a point wiill have trivial higher direct images, which tells us that a blowup preserves the cohomology of a surface. We will use higher direct images to establish geometric invariants between birational varieties. The most important technical tool we will need is the Theorem on Formal Functions. The setup of this theorem is as follows: Given a map $X \xrightarrow{f} Y$, we take a point $y \hookrightarrow Y$ and thicken it by looking instead at Spec $\mathcal{O}_y/\mathfrak{m}_y^n$. This is the point y but its structure sheaf remembers information of up to degree n around y. Then we can consider $X_n = X \times_Y$ Spec $\mathcal{O}_y/\mathfrak{m}_y^n$ to be the thickened fiber $f^{-1}(y)$, and its structure sheaf $\mathcal{F}_n$ the pullback $v^*\mathcal{F}$ where v is the map induced by the fibre product. Then the Thereom of Formal Functions is as follows:

Theorem 2.3 (The Theorem of Formal Functions)**.** *Let $X \xrightarrow{f} Y$ be a projective morphism of noetherian schemes, let $\mathcal{F}$ be a coherent sheaf on X, and let $y \in Y$. then the natural map*

$$\widehat{R^i f_*(\mathcal{F})}_y \to \varprojlim H^i(X_n, \mathcal{F}_n)$$

is an isomorhism for all $i \geq 0$.

In the case that $i = 0$, this tells us that the completion of the global sections $\Gamma(X_y, \mathcal{F}_{|X})$ is equal to the inverse system which comes from the global sections of the thickenings. The proof of this theorem is long and of a very different flavor than the rest of this paper, so I won't include it. It is Theorem III.11.1 in Hartshorne.

A useful application of this is that if a fibre has multiple connected components, its global sections will split into a direct sum of rings. Thus the Theorem of Formal Functions gives us control over the number of connected components of the fibres of a morphism. The specific application of this we need is called Zariski's main theorem.

Theorem 2.4 (Zariski's Main Theorem)**.** *Let $X \xrightarrow{f} Y$ be a birational projective morphism of noetherian integral schemes, with Y normal. Then for every $y \in Y$, $f^{-1}(y)$ is connected.*

Proof. Suppose that $f^{-1}(y) = X' \cup X''$, where X' and X'' are disjoint closed subsets. Then for each n, we would have

$$H^0(X_n, \mathcal{O}_{X_n}) = H^0(X'_n, \mathcal{O}_{X_n}) \oplus H^0(X''_n, \mathcal{O}_{X_n}),$$

so by the theorem,

$$(\widehat{f_*\mathcal{O}_X})_y = \varprojlim H^0(X_n, \mathcal{O}_{X_n}) = \varprojlim H^0(X'_n, \mathcal{O}_{X_n}) \oplus \varprojlim H^0(X''_n, \mathcal{O}_{X_n}).$$

Now we want to show that $f_*\mathcal{O}_X = \mathcal{O}_Y$ for a projective birational morphism. Choose an affine open set Spec $A \hookrightarrow Y$ containing y. By an elementary fact about projective morphisms[5] we have that $(f_*\mathcal{O}_X)_{|\text{Spec } A}$ is a finite module over A. The two share the same fraction field because the map is birational. Since Y is normal and therefore A is an integrally closed domain, a finite ring over A with the same fraction field must be equal to A. Thus $f_*\mathcal{O}_X = \mathcal{O}_Y$, so $(\widehat{f_*\mathcal{O}_X})_y = \widehat{\mathcal{O}_y}$, which is a local ring. A local ring cannot be the direct product of two rings, so we have a contradiction.[6] □

3. Intersection theory on a surface

An intersection theory on a smooth projective surface X is a map Pic $X \times$ Pic $X \to \mathbb{Z}$ such that:

(1) If C and D are nonsingular and meet transversally, then $C \cdot D = \#(C \cap D)$.
(2) $C \cdot D = D \cdot C$.
(3) $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$.

[5] Hartshorne II.5.19

[6] Take the spectrum of each side. A local ring is connected because it only has one closed point, and Spec $R_1 \oplus R_2 =$ Spec $R_1 \coprod$ Spec R_2.

We will show that for a given X, there is exactly one intersection theory. We use the fact that any divisor is the difference of two very ample divisors.[7] Now, let's investigate the generic representative of a very ample divisor. By taking the embedding induced by the divisor, we can consider X as a surface in some $\mathbb{P}^n$ and consider the generic hyperplane section.

Lemma 3.1. *Given a very ample divisor H on X and $C_1, \dots C_r$ irreducible curves on X, a nonempty open subset of $|H|$ will be a smooth irreducible curve on X which intersects $C_1, \dots, C_r$ transversally.*

Proof. We take the embedding of X into $\mathbb{P}^n$ and view $|D|$ as the set of hyperplane sections. Consider all of the singular points of $C_1, \dots C_r$. Since each of them is a curve, there are only finitely many singular points, so the set of hyperplanes passing through any of them is a closed subset Z of $|D|$. Now a hyperplane H' meeting C_i transversally is equivalent to the intersection $C_i \cap H$ being a reduced (and therefore smooth) 0-dimensional scheme, so by Bertini's theorem applied to each C_i the set of hyperplanes which intersect C_i non-transversally is a proper closed subset Z_i of $|H|$. Finally, by Bertini's theorem, the set of hyperplanes whose intersection with X is singular or reducible is also a closed subset Z' of $|D|$, so $U = |D| \setminus (Z \cup Z' \cup Z_1, \cup, \dots, \cup Z_r)$ is a an open subset of hyperplanes whose intersection with X is a smooth irreducible curve meeting a given finite set of curves transversally.

□

Now we want to show that for a curve C and an effective divisor D, for any representative of $|D|$ meeting C transversally the intersection multiplicity will be well-defined.

Theorem 3.2. *Let C be an irreducible smooth curve on X, and let D be any curve meeting C transversally. Then $\#(C \cap D) = deg_C(\mathcal{L}(D) \otimes \mathcal{O}_C)$.*

Proof. Since $\mathcal{L}(-D)$ is the ideal sheaf of D, we have the exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and tensoring with $\mathcal{O}_C$ we get

$$0 \to \mathcal{L}(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{D \cap C} \to 0$$

So we know that as sheaves on C, $(\mathcal{L}(-D) \otimes \mathcal{O}_C)^\vee = \mathcal{L}(D) \otimes \mathcal{O}_C = \mathcal{O}_{D \cap C}$, and so the degree of $C \cap D$ is just the length of the scheme-theoretic intersection of C and D, which is the same as the set theoretic intersection since we assumed the interesections were transversal. □

[7] If D is a divisor, pick an ample divisor H. Then by definition of ample there is some n_0 such that $D + n_0 H$ is generated by global sections. But there is also some n_1 such that for all $n > n_1$ nH is very ample. The sum of a divisor generated by global sections and a very ample divisor is very ample, (See Hartshorne II Ex 7.5d) so for all $n > n_0 + n_1$, $D + nH$ is very ample. But also nH is very ample, so $D = D + nH - nH$ is the difference of two very ample divisors.

Theorem 3.2 tells us that the intersection depends only on $\mathcal{L}(D)$, so it doesn't matter which representative we choose.

Theorem 3.3. *There exists a unique intersection theory on any smooth projective variety X.*

Proof. We start by defining the intersection of two elements of Pic X, C_1' and C_2', by setting $C_1' = C_1 - D_1$ and $C_2' = C_2 - D_2$, where the C_i and D_i are smooth irreducible curves which all meet transversally, and then defining

$$C_1' \cdot C_2' = C_1 \cdot C_2 - C_1 \cdot D_2 - C_2 \cdot D_1 + D_1 \cdot D_2.$$

Thus, the intersection pairing of any two divisors is determined by the pairing on the very ample divisors, and by lemma 3.1 and condition (1) the intersection of very ample divisors is determined, because we can choose representatives that meet transversally. This proves uniqueness. Now, all we have to do is show that our intersection pairing doesn't depend on our choice of very ample divisors. Suppose $C_1' = C_1 - D_1 = E_1 - F_1$ are two different reprentations of C_1' as differences of irreducible curves. Given another divisor $C_2' = C_2 - D_2$ where C_2 and D_2 are transverse to all the curves representing C_1', we have that since $C_1 + F_1 = D_1 + E_1$, $(C_1 + F_1) \cdot C_2 = (D_1 + E_1) \cdot C_2$, so by linearity $(C_1 - D_1) \cdot C_2 = (E_1 - F_1) \cdot C_2$, and similarly for D_2. Thus, the map is well-defined. □

Now that we have an intersection theory on a surface, there are some things to note. Firstly, the intersection of any effective divisor with a very ample divisor must be non-negative. This is because we can pick a representative of the effective divisor, then pick a representative of the very ample divisor meeting it transversally. This freedom to move around very ample divisors guarantees positive intersection. An interesting case of this is self-intersection. By theorem 3.2 we know that for a divisor C, $C \cdot C = \deg_C(\mathcal{L}(C) \otimes \mathcal{O}_C)$. In the case that C is a curve, we have that the ideal sheaf $\mathcal{I}_C$ is equal to $\mathcal{L}(-C)$, so

$$\begin{aligned}
\mathcal{L}(C) \otimes \mathcal{O}_C &= (\mathcal{L}(-C) \otimes \mathcal{O}_C)^\vee \\
&= (\mathcal{I}_C \otimes \mathcal{O}_X/\mathcal{I}_C)^\vee \\
&(\mathcal{I}_C/\mathcal{I}_C^2)^\vee \\
&= \mathcal{N}_{C/X}
\end{aligned}$$

So the self intersection of a curve is the degree of its normal bundle. If we view the normal bundle as the set of "infinitesimal deformations" of the curve, this makes sense, because if we could pick a section s of the normal bundle and move C infinitesimally along it, it makes intuitive sense that the intersections of C with $C + \Delta s$ would be the zero set of s. Thus, having negative self-intersection tells us that the curve can't "infinitesimally move", so is in some sense rigid in X. We can make this heuristic rigorous if we look at C as a divisor. We know that if our curve has another effective reprentative C' its self-intersection will be the intersection of C and C', which is non-negative. Thus having negative self-intersection means C is the isolated effective representative of its linear system.

4. Intersection theory of a blowup

For the rest of the paper, we are largely concerned with blowing up a projective surface at a point and studying the geometric qualities which change or remain the same when doing so. Whenever not stated, X is a smooth projective surface and $\widetilde{X}$ is the blowup of X at a point.

Theorem 4.1. *There is a natural isomorphism Pic* $\widetilde{X} =$ *Pic* $X \oplus \mathbb{Z}$*.*

Proof. Since if an invertible sheaf on X is trivial on $U = X \setminus p$, it is trivial, we know that Pic $X =$ Pic U, so the map $U \hookrightarrow \widetilde{X}$ induces a map $i^* : \text{Pic } \widetilde{X} \to \widetilde{X}$. This map has a section given by $\pi^* : \text{Pic } X \to \text{Pic } \widetilde{X}$, so we have that Pic $\widetilde{X} =$ Pic $X \oplus \ker i^*$. $\ker i^*$ is the divisors with support in E, which since E is irreducible is just $nE : n \in \mathbb{Z}$. Since $nE \cdot nE = -n^2$, $nE = 0$ implies $n = 0$, so $\ker i^* = \mathbb{Z}$. □

Theorem 4.2. *The intersection theory on Pic* $\widetilde{X}$ *is given by:*

(a) If $C, D \in$ *Pic* X*,* $\pi^*(C) \cdot \pi^*(D) = C \cdot D$*;*
(b) $E \cdot E = -1$*;*
(c) $\pi^*(C) \cdot E = 0$;

Proof. (a) is true because C and D are each equivalent to differences of curves meeting transversally and not intersecting P, by Bertini's theorem. We already know (b), and for (c) we can again just make C into a difference of curves away from P.

□

Remark 4.3. Stronger than the notion of rational equivalence of divisors is that of numerical equivalence. We say that a divisor D is numerically equivalent to zero, which we write $D \equiv 0$, if $D \cdot C = 0$ for all divisors C, and two divisors D and E are numerically equivalent if $D - E \equiv 0$. The group of divisors modulo numerical equivalence is denoted $\text{Num}(X)$. It is a fact, called Severi's theorem of the base, that $\text{Num}(X)$ is a finitely generated abelian group.[8] Given an element in $\text{Pic}(\widetilde{X})$ which we can write as $\pi^*C + kE$, we have that $(\pi^*C + kE) \cdot E = -k$, so a divisor is numerically equivalent to 0 if and only if it is of the form π^*C and $C \equiv 0$. Thus $\text{Num}(\widetilde{X}) = \text{Num}(X) \oplus \mathbb{Z}$, which tells us that blowing up a point increases the rank of $\text{Num}(X)$.

As an aside, we would like to see how the cohomology groups of a surface behave under a blowup.

[8] Over $\mathbb{C}$, a sketch of a proof could follow by showing that given the exact sequence $0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \to H^2(\mathbb{Z})$ induced by the exponential sheaf sequence, the quotient $H^1(\mathcal{O}_X^*)/H^1(\mathcal{O}_X) = \text{Pic}(X)/\text{Pic}^0(X)$ is the Neron-Severi group, or the group of divisors modulo algebraic equivalence. Since this is sitting inside $H^2(\mathbb{Z})$ of a compact complex manifold, we know that it is a finitely generated abelian group. It is easy to show that algebraically equivalent divisors are numerically equivalent, (see Hartshorne V Ex 1.7), so $\text{Num}(X)$ is also a finitely generated abelian group.

Theorem 4.4. *$H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^i(X, \mathcal{O}_X)$ for all $i \geq 0$.*

Proof. First, let's do the case $i \geq 1$. All we have to do is show that $R^i\pi_*\mathcal{O}_{\widetilde{X}}$ vanishes everywhere and we can then apply theorem 2.2. Since away from p, π is an isomorphism, we can cover $X \setminus p$ with affine open sets and see that $R^i(X)_{|U} = 0$. All that remains is its potential behavior around P. But this tells us that

$$\widehat{R^i\pi_*\mathcal{O}_{\widetilde{X}}} = \varprojlim H^i(E_n, \mathcal{O}_{E_n}).$$

Since $\mathcal{O}_{E_n} = \mathcal{O}_{\widetilde{X}}/\mathcal{J}^n$, we have an exact sequence

$$0 \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1} \longrightarrow \mathcal{O}_{E_{n+1}} \longrightarrow \mathcal{O}_{E_n} \longrightarrow 0$$

We know from remark 1.6 that $\mathcal{J}^n/\mathcal{J}^{n+1} = \mathcal{O}_E(n)$. We know that $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$ for $n \geq 0$, and that $H^i(\mathbb{P}^1, \mathcal{O}(n)) = 0$ for $i > 1$ because $\mathbb{P}^1$ is one-dimensional. Thus, for $i > 0$ the exact sequence in cohomology

$$0 = H^i(\mathbb{P}^1, \mathcal{O}_n) \longrightarrow H^i(\mathcal{O}_{E_{n+1}}) \longrightarrow H^i(\mathcal{O}_{E_n}) \longrightarrow H^{i+1}(\mathbb{P}^1, \mathcal{O}_n) = 0$$

tells us that all of the elements of the inverse system are isomorphic. But the first one, $H^i(E, \mathcal{O}_E)$ is trivial for $i > 0$, so the entire inverse system must be zero. Thus the higher direct images of π_* vanish and so by theorem 2.2 the theorem holds for $i > 0$.

For $i = 0$ we just use the fact that $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, which we know because the map is birational and X is normal, as in the proof of theorem 2.4. Thus $\Gamma(X, \mathcal{O}_X) = \Gamma(X, \pi_*\mathcal{O}_{\widetilde{X}}) = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. □

Remark 4.5. The equivalence of cohomology groups is a fabulous result because it immediately tells us that for surfaces, properties like Euler characteristic or arithmetic genus are invariant under blowups of points. When we see in section 6 that every birational transformation of surfaces is a sequence of blowups and blowdowns, this gives us a powerful birational invariant.

5. Canonical divisors and the proper transform of a curve

Theorem 5.1. *Adjunction formula: If C is a nonsingular curve of genus g on the surface X, and if K is the canonical divisor on X, then*

$$2g - 2 = C.(C + K)$$

Proof. We know that $\omega_C = \omega_X \otimes \mathcal{N}_{C/X} = \omega_x \otimes \mathcal{L}(C) \otimes \mathcal{O}_C$ by the normal exact sequence. But $\omega_x \otimes \mathcal{L}(C) = \mathcal{L}(C + K)$. We know from curves that the degree of the canonical divisor on a smooth curve is $2g - 2$, so we have

$$2g - 2 = \deg_C(\omega_C) = \deg_C(\mathcal{L}(K + C) \otimes \mathcal{O}_C) = C.(C + K).$$

□

Remark 5.2. The above proof is important not only because it allows us to compute information about curves when we know the arithmetic genus, but also because it gives us a definition of arithmetic genus which extends to a general divisor; $p_a(C) = \frac{1}{2}C \cdot (C + K) + 1$. We will see that within a class of birational curves, the arithmetic genus is a measure of a curve's failure to be smooth.

Theorem 5.3. *The canonical divisor of $\widetilde{X}$ is given by $K_{\widetilde{X}} = \pi^* K_+ X + E$.*

Proof. Since K_X is determined by its behavior on $U = X \setminus P$, and $K_{\widetilde{X}}$ must look like K_X when restricted to $U = \widetilde{X} \setminus E$, we must have that $K_{\widetilde{X}]} = \pi^* K_X + nE$ for some n. Using the adjunction formula above, we have that

$$\begin{aligned} p_a(E) &= E \cdot (K_{\widetilde{X}} + E) \\ -2 &= E \cdot (\pi^* K_X + (n+1)E) \\ &= -(n+1) \end{aligned}$$

□

Given a curve $C \hookrightarrow X$ containing a point P, we can blow up the point P and see what happens to C. The preimage $\pi^{-1}(C)$ will have two irreducible components: the exceptional divisor E and the closure of $\pi^{-1}(C \setminus P)$. We call this second component the proper transform of C and denote it $\widetilde{C}$.

Theorem 5.4. *Let C be an effective divisor on X, and let P be a point of multiplicity r on C, and let $\widetilde{X} \xrightarrow{\pi} X$ be the blowup of X at P. Then*

$$\pi^* C = \widetilde{C} + rE.$$

Proof. Let $\mathfrak{m}$ be the sheaf of ideals of P on X, and pick an affine open set $U = \operatorname{Spec} A$ such that $\mathfrak{m}$ is generated by local parameters (x, y). Then over U we have that $\widetilde{X} = V(ux - ty) \subset U \times \mathbb{P}^1$. Let f be an equation for C on U. We know that $f = f_r(x, y) + g$, where f_r is a nonzero homogeneous polynomial in x and y of degree r, and $g \in \mathfrak{m}^{r+1}$. Now consider the open set $V = \{t \neq 0\} \subset U \times \mathbb{P}^1$. Then on this affine chart we have $y = ux$, so $E = \{x = 0\}$ and $f_r(x, y)$ becomes $f_r(x, ux) = x^r f_r(1, u)$, and g is divisible by x^{r+1} since $\mathfrak{m}^{r+1} = (x, ux)^{r+1} = (x^{r+1})$. Thus $\pi^* f = x^r(f_r(1, u) + xh)$, which vanishes with multiplicity r on E since $f_r(1, u)$ vanishes only at finitely many points of E. Since V is an open dense subset of $\widetilde{X}$ which contains an open subset of E, the multiplicity of E in $\pi^* C$ can be determined on V, so we are done.

□

Corollary 5.5. *With C, X, P as above, $\widetilde{C} \cdot E = r$, and $p_a(\widetilde{C}) = p_a(C) - \binom{r}{2}$.*

Proof. For the first part, we have

$$\begin{aligned}\widetilde{C}\cdot E &= (\pi^*C - rE)\cdot E\\ &= \pi^*C\cdot E - r(E\cdot E)\\ &= r.\end{aligned}$$

For the second part, we have

$$\begin{aligned}2p_a(\widetilde{C}) - 2 &= \widetilde{C}\cdot(\widetilde{C} + K_{\widetilde{X}})\\ &= (\pi^*C - rE)\cdot(\pi^*C - rE + \pi^*K_X + E)\\ &= \pi^*C\cdot(\pi^*C + \pi^*K_X) + (-rE)\cdot(-(r-1)E)\\ &= 2p_a(C) - 2 - (r)(r-1)\\ p_a(\widetilde{C}) &= p_a(C) - \frac{(r)(r-1)}{2}\end{aligned}$$

□

Remark 5.6. The fact that blowing up a curve at a singularity decreases the genus means that it must in some way "tame" the singularity, because it is clear that if the strict transform is still singular, we can blow up the singular point and decrease the genus again. Since $p_a(C) = \dim H^1(\mathcal{O}_C)$ is a non-negative integer, we can only repeat this process a finite number of times, which means that after a finite sequence of blowups we are left with a smooth curve. Since for a curve, normal is equivalent to smooth, and since the projection from $\widetilde{C} \to C$ is birational and finite,[9] the curve we obtain at the end of this process is the normalization of C. This tells us that if $\widetilde{C}$ is smooth it is the normalization of C, and

$$p_a(\widetilde{C}) = p_a(C) - \sum_P \binom{r_P}{2}$$

where P ranges over the singularities of $p_a(C)$ and each of its blowups, and r_P is the intersection multiplicity of C at P.

6. Birational transformations

Given two projective varieties X, Y, a birational transformation T from X to Y is a map from η_X to η_Y, where η_X and η_Y are the generic points of X and Y, respectively, which induces an isomorphism of function fields $\mathcal{O}_{X,\eta_X} \cong \mathcal{O}_{Y,\eta_Y}$. Equivalently, it is an open set $U \subset X$ and a map $\phi : U \to Y$ which induces an isomorphism of function fields. If we chose a different open set V, giving the same isomorphism of function fields, the two maps would agree on $U \cap V$, so we can glue them into a map on $U \cup V$. Thus there is some maximal open set U on which the birational map

[9] It is birational because, for each blowup, it is an isomorphism away from the exceptional divisor, and the composition of birational maps is birational. It is finite because it is projective and has finite fibres.

is represented by a morphism, and we say the map is defined at these points. The points $X \setminus U$ where T is not defined are called the fundamental points of T.

We know from the valuative criterion of properness that since T is defined at the generic point of X it is defined at every point corresponding to a discrete valuation ring of $K(X)$. Thus, if X is normal, we know that every codimension 1 point corresponds to a discrete valuation ring, so T is defined at all of them. Therefore, the fundamental points of T have codimension ≥ 2 on a normal variety. In particular, if X is a smooth surface there can be only finitely many fundamental points.

Definition 6.1. A birational transformation without fundamental points, meaning a morphism of varieties which induces birational equivalence, is called a birational morphism.

Since a birational transformation $X \xrightarrow{T} Y$ is defined by a map $\phi : U \to Y$, we can take the graph $\Gamma_\phi \hookrightarrow U \times Y$. Then we can set $\Gamma_T \hookrightarrow X \times Y$ to be the closure of Γ_ϕ in $X \times Y$. Then for a given point $P \in X$, we can set $T(P) = p_2 \circ p_1^{-1}(P)$, where p_1 and p_2 are the projections from Γ_T to X and Y. If P is not a fundamental point of T then clearly $T(P)$ is a single point, but at a fundamental point P, $T(P)$ cannot be a single point, or else T could be extended to P. For example, if we consider the map $X \xrightarrow{\pi^{-1}} \widetilde{X}$,where $\widetilde{X}$ is the blowup of P, the total transform of P will be the exceptional divisor E.

The key ingredient we need to prove that a birational morphism factors into a sequence of blowdowns is the following strengthening of theorem 2.4.

Theorem 6.2. *Zariski's Main Theorem. Let $X \xrightarrow{T} Y$ be a birational tranformation of projective varieites with X normal. If P is a fundamental point of T, the total transform $T(P)$ is connected and of dimension ≥ 1.*

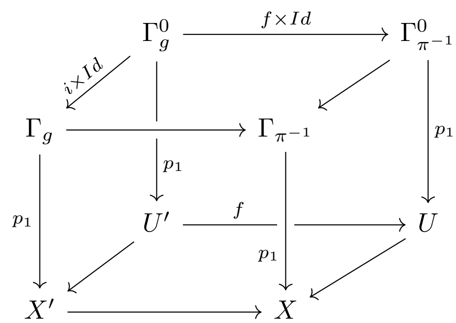
Proof. Since $U \cong \Gamma_\phi$, the morphism $\Gamma_T \xrightarrow{p_1} X$ is birational and projective, so by theorem 2.4, $p_1^{-1}(P)$ is connected. If it has dimension 0, then all the fibres have dimension 0 in a neighborhood V of P by the upper-semicontinuity of dimension of fibres. If that were true $p_1^{-1}(V) \to V$ would be projective with finite fibres, and therefore finite, and since V is normal, therefore an isomorphism, contradicting that $p_1^{-1}(P)$ must consist of multiple points. Thus $p_1^{-1}(P)$ must be of positive dimension, and since p_2 maps this isomorphically onto Y, we are done. □

Theorem 6.3. *Let $X' \xrightarrow{f} X$ be a birational morphism between smooth projective varieties. Let P be a fundamental point of f^{-1}. Then there exists a morphism $g : X' \to Bl_P(X) = \widetilde{X}$ such that*

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{g}{\nearrow} & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

commutes.

Proof. To begin, we set $g = f \circ \pi^{-1}$ as a birational transformation, and we want to prove that it a morphism, or in other words that it has no fundamental points. So, suppose that g has a fundamental point $P' \in X'$. Since the only fundamental point of π^{-1} is P and f is a morphism, any fundamental points of g must be contained in $f^{-1}(P)$. Let U' be $X' \setminus f^{-1}(P)$ and let $U = X \setminus P$ be the domain of π^{-1}. Then we have



Since the open sets U' and U are irrerducible and so are their graphs, the map $f \times Id$ taking Γ_g^0 to $\Gamma_{\pi^{-1}}^0$ extends to a map of the closures, such that the above diagram commutes. The second projection from $\Gamma_{\pi^{-1}}$ to $\widetilde{X}$ takes $p_1^{-1}(P)$ to the exceptional divisor E, hence the total transform of P' is E.

Now, we know that g^{-1} only has finitely many fundamental points (remember that everything is smooth and therefore normal) so pick Q in E such that $g^{-1}(Q) = P'$. The isomorphisms of function fields between the three varieties means we can view $\mathcal{O}_P$, $\mathcal{O}'_P$ and $\mathcal{O}_Q$ as subrings of the same function field. In this case, $\mathcal{O}_Q$ dominates $\mathcal{O}_{P'}$ and $\mathcal{O}_P$. This is because any rational function which is defined at P' is defined at E and therefore Q, and if a rational function vanishes at $\mathcal{Q}$ and is well-defined at P' then this must mean it takes the same value on all of E, so vanishes on E, so vanishes on P'. The same argument works for $\mathcal{O}_P$.

Locally around $P \in X$, we can pick local parameters x and y and an open set $V \ni P$ such that $\pi^{-1}(V) = Z(yt - xu) \subset V \times \mathbb{P}^1$. We can linearly change coordinates so that $Q = (0,0) \times [0:1]$. So, E is cut out by $y = 0$, and Q is cut out by $t = 0, y = 0$.

Since $g(P') = E = Z(y)$, the image of any element in $\mathfrak{m}_{P'}$ is contained in (y).

Since P is a fundamental point of f^{-1}, its preimage $C = f^{-1}(P)$ is one-dimensional and contains P', so we can say it's cut out by the single variable z in $\mathcal{O}_{P'}$. Since $f(C) = P$, $f^\#$ maps x and y into (z), so $x = az$ and $y = bz$. But since y is a local generator of $\mathcal{O}_Q$ it is contained in $\mathfrak{m}_Q$ and not in $\mathfrak{m}_Q^2$, and therefore its image in $\mathcal{O}_{P'}$ is also contained in $\mathfrak{m}_{P'}$ and not in $\mathfrak{m}_{P'}^2$. Since z is already contained in $\mathfrak{m}_{P'}$ this means that b must be a unit in $\mathcal{O}_{P'}$, so $t = x/y = a/b$ is contained in $\mathcal{O}_{P'}$. But by above this means that $t \in (y)$, a contradiction since t and y are local parameters of $\mathcal{O}_Q$.

□

What this factoring theorem tells us is that for any curve C in X' which is contracted to a point in X, we can factor the map f into $X' \xrightarrow{g} \widetilde{X} \to X$, where our new map g takes C to a curve in $\widetilde{X}$. Theorem 6.2 tells us that if a birational morphism does not contract any curves to a point, it is an isomorphism (since it is an injective morphism between two proper varieties), so the above result tells us that each contraction of a biratonal morphism can be resolved by a blowdown. This leads us to the following theorem:

Theorem 6.4. *Let $X' \xrightarrow{f} X$ be a birational morphism of surfaces. Let $n(f)$ be the number of irreducible curves $C' \hookrightarrow X'$ such that $f(C')$ is a point. Then $n(f)$ is finite and f can be factored into a composition of exactly $n(f)$ blowdowns.*

Proof. The fact that the number of fundamental points of f^{-1} is finite, and that the preimage of each one is a subvariety of X' and therefore has finitely many irreducible components tells us that $n(f)$ is finite. We choose some fundamental point P of f^{-1} and factor f into $\pi \circ g$ as above. Since if $g(C)$ is a point, necessarily $f(C)$ is a point, we can see that $n(g) \leq n(f)$. If f maps a curve D to a point $Q \neq P$, then $\pi^{-1}(Q)$ is also a point, so necessarily g contracts the curve to a point also. We know that so the preimage of $E = \pi^{-1}(P)$ is a finite set of curves in X', and since g is surjective it must map at least one of these curves surjectively onto E, and since by Zariski's main theorem the preimage of any point in E is connected, it can only map one curve surjectively. This gives us exactly one curve that f contracts to a point but g doesn't, so $n(g) = n(f) - 1$. After applying this process $n(f)$ times we are left with an isomorphism. □

Theorem 6.5. *Let $X \xrightarrow{T} X'$ be a birational transormation of surfaces. Then it is possible to factor ϕ into a finite sequence of blowups and blowdowns.*

Proof. The bulk of this argument is the following lemma:

Lemma 6.6. *Let $X \xrightarrow{T} X''$ be a birational map of projective surfaces. Pick a very ample line bundle H on X' and pick a smooth irreducible curve $C' \in |2H|$ such that C' avoids the fundamental points of T^{-1}. Set $C = T^{-1}(C')$. If $p_a(C) = p_a(C)$ then T is a morphism.*

Proof. The reason we want $C' \in |2H|$ is that, since a line bundle has positive intersection with any curve E, $H \cdot E \geq 1$, so $C \cdot E \geq 2$. We will use this to force a curve to become singular.

Before going further, we should check that any of what we wrote is well-defined. If C' avoids the fundamental points of T^{-1}, C is also an irreducible curve, and if we picked a different C'_1 such that $C'_1 - C' = (f)$ then $T^{-1}(C'_1) - T^{-1}(C') = (f)$, since T^{-1} is birational.

We know that C' will be the normalization of C, because it is a normal curve which has a finite birational map to C, so has a map to the normalization, and since the curves are smooth this map is an isomorphism. Thus, by remark 5.6, $p_a(C) = p_a(C')$ implies that $C \cong C'$, and in particular C is smooth. Now, if T has a fundamental point $P \in X$, then $T(P)$ contains an irreducible curve D in X'. But then $C' \cdot D \geq 2$ by definition of C'. Wiggling C' so that the intersection is transverse, we see that C' intersects D at two distinct points. This means that $T^{-1}(C')$ has a double point at P, contradicting that C is smooth. □

Proof of theorem 6.5: now that we know this, we can take $X \xrightarrow{T} X'$, H, C', C as in the lemma. If $m(T) = p_a(C) - p_a(C')$. If $m(T) = 0$, we know that T is a birational morphism, so we apply theorem 6.4 and we are done. If $m(T) > 0$, necessarily C is singular at a point P. Blow up X at P and get $\widetilde{X} \xrightarrow{\pi} X$. Then $\widetilde{C}$ is linearly equivalent to $(T \circ \pi)^{-1}(C')$ after we wiggle C' so it avoids hitting P. Since $p_a(\widetilde{C}) < p_a(C)$, $m(T \circ \pi) < m(T)$, so we can continue blowing up X until we have a surface X'' and maps $X'' \xrightarrow{f} X$ and $X'' \xrightarrow{T \circ f} X'$ such that $m(T \circ f) = 0$. by the lemma $T \circ f$ is a morphism, and so is a succession of blowdowns, and f is succession of blowdowns by construction. □

Remark 6.7. Coupled with remark 4.5, the above theorem tells us that arithmetic genus is a birational invariant. This is a useful tool in the minimal model program.

Most of the material in the paper has been leading up to this theorem: we have proven that every blowup produces an exceptional divisor with self-intersection -1. Castelnuovo's criterion provides a converse, telling us that in fact any rational curve with self-intersection -1 is the result of a blowup, or more tellingly, can be blown down.

Theorem 6.8 (Castelnuovo's Criterion). *If E is a curve on a surface such that $E \cong \mathbb{P}^1$ and $E^2 = -1$, then there exists a morphism $X \xrightarrow{f} X_0$ to a smooth projective surface such that $f(E)$ is equal to a point, p, and $X \xrightarrow{f} X_0$ is the blowup of X_0 at p.*

Proof. What we want to do is find a line bundle M on X which is generated by its global sections, very ample when restricted to $X \setminus E$, and such that $M \cdot E = 0$. This means that M will induce a morphism to some projective space, be a closed immersion on $X \setminus E$, and map E to a point.

Pick a very ample line bundle H on X such that $H^1(X, H) = 0$.[10] Now let $k = H \cdot E$, and then set $M = H + kE$, so $M \cdot E = 0$.

Now, for $i = 0, \dots k$, we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{L}(H + (i-1)E) \longrightarrow \mathcal{L}(H + iE) \longrightarrow \mathcal{O}_E \otimes \mathcal{L}(H + iE) \longrightarrow 0$$

We know that $\mathcal{O}_E \otimes \mathcal{L}(H+iE)$ has degree $k-i$ on E, and since $E \cong \mathbb{P}^1$ it is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(k-i)$. We have the long exact sequence

[10]By Serre vanishing, we can just pick any very ample line bundle and take a sufficiently high multiple. See Hartshorne III.5.3

$$H^0(X, H+(i-1)E) \xrightarrow{\cdot t} H^0(X, H+iE) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(k-i)) \to H^1(X, H+(i-1)E) \longrightarrow H^1(X, H+iE) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(k-i))$$

where t is some fixed element in $\mathcal{L}(E)$ which vanishes on E.

We know by Serre duality, or from even more elementary methods in the case of $\mathbb{P}^1$, that

$$\dim H^1(\mathbb{P}^1, \mathcal{O})(k-i)) = \dim H^0(\mathbb{P}^1, \mathcal{O}(-2+i-k),$$

so for $i \leq k$ this will be zero. Thus, if $H^1(X, H+(i-1)E) = 0$ then so is $H^1(X, H+iE)$ for $i \leq k$. Since by our choice of H the claim is true for $i = 0$, it is true for all $i \leq k$. Thus for all $i \leq k$ we have the exact sequence

$$0 \longrightarrow H^0(X, H+(i-1)E) \xrightarrow{\cdot t} H^0(X, H+iE) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(k-i)) \longrightarrow 0.$$

If our original H had sections $s_0, \ldots, s_n$, $H+E$ will have sections $ts_0, \ldots, ts_n$, plus the pulbacks of k sections of $\mathcal{O}(k-1)$, and $H+kE$ will have a basis of global sections

$$\langle t^k s_0, \ldots, t^k s_n, t^{k-1} v_0^{(k-1)}, \ldots t^{k-1} v_{k-1}^{(k-1)}, \ldots, tv_0^{(1)}, tv_1^{(1)}, v \rangle$$

where $v_0^{(r)}, \ldots v_r^{(r)}$ are pullbacks of sections of $\mathcal{O}_E(r)$ and v is the pullback of a section of $\mathcal{O}_E$. Thus all of these sections vanish on E except the final one. We know that $t^k s_0, \ldots, t^k s_n$ seperate points and tangent vectors away from E, and on E we know that around any point we have one nonzero section and two independent sections in the maximal ideal, so we know that $H+kE$ is generated by global sections. We can see that the map induced by $H+kE$ sends E to a point because all but one of the sections vanish identically on it. and away from E it is a closed embedding.

Call the map induced by $H+kE$ f_1 and set $f_1(X) = X_1$. We don't know much about what X_1 looks like, but to make things easier we can take its normalization $X_0 \xrightarrow{\pi} X_1$, and by the universal property of normalization get a map $X \xrightarrow{f} X_0$. Since f has finite fibres, the preimage of P must be a finite set. Since $X_1 \setminus P$ is smooth, $X_0 \setminus \pi^{-1}(P) \cong X_1 \setminus P$, so the image of X contains $X_0 \setminus \pi^{-1}(P)$, so f is dominant, and since X is projective f must be surjective. But E is irreducible so f can only take E to a point, so therefore $f : X \to X_0$ is still a map taking E to a point and an isomorphism elsewhere, only now we know that X_0 is normal.

All that remains in the proof is to prove that X_0 is smooth at P. We want to show that $\widehat{\mathcal{O}_{X_0,P}} = k[[x,y]]$, since if a completion of a local ring is regular then so is the local ring. [11] The theorem of formal functions tells us that

$$\widehat{\mathcal{O}_{X_0,P}} = \varprojlim H^0(E_n, \mathcal{O}_{E_n})$$

[11] Hartshorne I, theorem 5.4A

where E_n is the closed subscheme $f^{-1}\mathfrak{m}_P^n \cdot \mathcal{O}_X$. Since $f^{-1}(P) = Y$, we know that for some n $\mathcal{J}_Y^n \subset f^{-1}\mathfrak{m}_P \cdot \mathcal{O}_X$, and conversely $f^{-1}\mathfrak{m}_P^m \cdot \mathcal{O}_X \subset \mathcal{J}_Y$ for some m. Thus the sequences of ideals are cofinal with respect to one another so they will result in the same completion.[12] We want to show

$$H^0(E_n, \mathcal{O}_{E_n}) = k[x,y]/(x,y)^n = k[[x,y]]/(x,y)^n.$$

We have the exact sequence

$$0 \longrightarrow \mathcal{J}_E^n/\mathcal{J}_E^{n+1} \longrightarrow \mathcal{O}_{E_{n+1}} \longrightarrow \mathcal{O}_{E_n} \longrightarrow 0$$

By remark 1.5, $\mathcal{J}_E^n/\mathcal{J}_E^{n+1} = \mathcal{J}_E^n \otimes \mathcal{O}_E$ is equal to $\mathcal{O}_E(n)$, so $H^0(\mathcal{O}_{\mathbb{P}^1}(n)) = \langle x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\rangle$. we have the exact sequence in cohomology

$$0 \longrightarrow \langle x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\rangle \longrightarrow H^0(\mathcal{O}_{E_{n+1}}) \longrightarrow H^0(\mathcal{O}_{E_n}) \longrightarrow 0$$

So by induction $H^0(\mathcal{O}_{E_{n+1}}) = k[x,y]/(x,y)^{n+1}$.

So to recap, we have our map $f : X \to X_0$ which contracts E to a point and is an isomorphism away from E, and we know that X_0 is smooth. Since there is only one irreducible curve which is contracted to a point, this must be a blowdown, and we are done. □

7. Minimal models

Given any surface X, we know that birational maps will naturally factor into a sequence of blowups followed by a sequence of blowdowns. We know that if the birational map is a morphism, the map is in fact only a sequence of blowdowns, so is a sequence of contractions of the original surface. This gives us a partial ordering on the set of surfaces birationally equivalent to a given surface. A relatively minimal model is a surface in this set which has no surfaces below it, meaning that any birational morphism away from the surface is an isomorphism. Castelnuovo's criterion tells us that if a surface has a -1 rational curve, known as an exceptional curve of the first kind, it is not relatively minimal, as we can blow down the curve. Conversely, if a surface X is not relatively minimal, it has a birational morphism $X \xrightarrow{\phi} X'$ which is not an isomorphism. Since ϕ can be factored into a series of blowdowns, the first blowdown must necessarily contract a -1 rational curve on X, so X contains an exceptional curve of the first kind. This tells us that being a relatively minimal model is precicely equivalent to having no exceptional curves of the first kind.

Theorem 7.1. *Every surface has a relatively minimal model.*

Proof. Pick some representative of the surface X. If X has no exceptional curves, we are done. If it does, choose one, and let $X \xrightarrow{\phi} X'$ be the blowdown. By remark 4.3 $\mathrm{Num}(X) = \mathrm{Num}(X') \oplus \mathbb{Z}$, which tells us that the rank of $\mathrm{Num}(X')$ is less than that of $\mathrm{Num}(X)$. Since $\mathrm{Num}(X)$ is a finitely generated abelian group, this process must terminate in a finite number of blowdowns. □

[12] Hartshorne II, remark 9.3.1.

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