SOLVING AND COMPUTING THE DISCRETE DIRICHLET PROBLEM

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Abstract. We introduce simple random walks (SRW) and use them to model a gambling scenario in one dimension. Extending to higher dimensions, we introduce the discrete Dirichlet problem and show how it can be solved with SRW. We then use a bit of linear algebra to compute the probabilistic expressions that appear in the solutions to the problem.

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1. Random Walk Preliminaries

Random walks are ubiquitous in stochastic mathematics with multiple formalizations and a wide range of applications in the sciences. In the most general sense, a random walk is a series of random steps that induces a path. In this paper, however, we will restrict our attention to random walks on the d-dimensional integer lattice $\mathbb{Z}^{\mathsf{d}}$. More specifically, we will be dealing with simple random walks, in which the only possible transitions are between adjacent points and the transition probabilities are symmetric.

Definition 1.1. A simple random walk (SRW) starting at $x \in \mathbb{Z}^{\mathsf{d}}$ is an infinite sum of random variables, whose partial sums take the form

$$S_n = x + X_1 + X_2 + ... + X_n$$

where $X_1, X_2, ..., X_n$ are identical, independent random variables with probability distributions $P\{X_j = e_k\} = P\{X_j = -e_k\} = 1/(2d)$. Unless otherwise indicated, we will assume that $x = 0$ (that the walk begins at the origin).

When dealing with a probabilistic object, we are often interested in both probability distributions and expected values. In this case, we begin our study of SRW

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by looking at the probability distribution of S_n for any given n. In multiple dimensions, finding this distribution is not so simple, but in one dimension we can construct an explicit formula. The following notation will be helpful.

Notation 1.2. We abbreviate $P\{S_{n+k} = y \mid S_n = x\}$ as $P_k(x, y)$

For this notation to make sense, the defined probability must be independent of n, which is true of the time homogenous random walks we are dealing with. In addition, simple random walks exhibit spacial homogeneity, since $X_1, X_2, ..., X_n$ are independent of the location of the walk. As such,

$$P_k(x, y) = P_k(0, x - y)$$

and we can generalize our results about SRW beginning at the origin to SRW beginning at arbitrary locations.

Returning to our original question, we want to know the probability that an SRW beginning at x is at a point j after n discrete steps. Considering the one dimensional case, one observation we can immediately make is that such an occurrence is impossible if $j - x$ and n have different parities. An odd number of steps would force different parities on the number of forward and backward steps, making it impossible to achieve an even distance. An even number of steps would ensure identical parities of the number of forward and backward steps, making it impossible to achieve an odd distance. Let us then assume that the number of steps and the distance travelled are both even and find a formula for $P_{2k}(a, a+2j) = P_{2k}(0, 2j)$. This is a simple application of combinatorics. In order for the walker to reach $a+2j$ after $2k$ steps, the walker must take $k+j$ steps to the right and $k-j$ steps to the left. There are $\binom{2k}{k+j}$ ways for this to happen, each of which has probability 2^{-2k} of occurring. Thus,

$$P_{2k}(a, a+2j) = P\{S_{2k} = 2j\} = \binom{2k}{k+j} 2^{-2k}$$

In addition to probability distributions, we are also interested in expected values. The expected location of an SRW is trivially its starting point, since the probabilities of $+1$ and -1 steps are the same. Computing $\mathbf{E}[(S_n)^2]$, however, proves more interesting.

Proposition 1.3. *The expected value of $(S_n)^2$ is n.*

Proof. We have

$$\begin{aligned}\mathbf{E}[(S_n)^2] = \mathbf{E}[(\sum_{j=1}^{n} X_j)^2] &= \mathbf{E}[\sum_{j=1}^{n}\sum_{k=1}^{n} X_j X_k] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \mathbf{E}[X_j X_k] = n + \sum_{j \neq k} \mathbf{E}[X_j X_k].\end{aligned}$$

For $j \neq k$, $X_j X_k$ equals 1 with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$, so the sum appearing in the last expression is equal to 0. □

We now have a grasp of some fundamental features of SRW, and we can begin a discussion of their applications. We start in one dimension with a problem in gambling, but we soon see how this problem can be extended to multiple dimensions.

2. Gambler's Ruin

Suppose we have a game with equally likely outcomes of winning and losing and a constant bet of \$1. We bring to the table a dollars, and we continue to play until we reach a set number N dollars or we run out of money. We want to know the probability that we reach our goal of N dollars. We can model this scenario as a one-dimensional SRW S_n beginning at a. To represent the time at which we stop gambling, we give the following definition.

Definition 2.1. The *stopping time*, denoted T, is given by

$$T := \min\{n \mid S_n = 0 \text{ or } S_n = N\}.$$

We will later prove that T is finite with probability 1. We are then justified in defining the function $F(x) = P_T(x, N)$, and our problem amounts to evaluating this function at $x = 0, 1, ..., N$. In fact, we already know the boundary values of the function: $F(0) = 0$ and $F(N) = 1$. Moreover, a simple random walk beginning at any point x has a .5 chance of being at $x + 1$ and a .5 chance of being at $x - 1$ after a unit of time, so the function possesses the following property on the interior:

$$F(x) = \frac{1}{2}F(x+1) + \frac{1}{2}F(x-1). \tag{2.2}$$

This property, along with the given boundary values, uniquely determines the function F. Turning to the more general case in which the function takes arbitrary values a and b on the boundaries, we prove this fact.

Theorem 2.3. *Suppose $F : \{0, ..., N\} \to \mathbb{R}$ is a function satisfying (2.2) with boundary values $F(0) = a$ and $F(N) = b$. Then*

$$F(x) = a + \frac{x(b-a)}{N}.$$

Proof. Condition (2.2) can be expressed as

$$F(x) - F(x-1) = F(x+1) - F(x),$$

which, applied to each point on the interior, gives us

$$F(1) - F(0) = F(2) - F(1) = ... = F(N) - F(N-1).$$

We see that the consecutive differences in function value are all equal. The total difference, $F(N) - F(0)$, is the sum of these consecutive differences and is given by $b - a$. Consequently,

$$F(x) - F(x-1) = \frac{b-a}{N},$$

and our function is thus

$$F(x) = a + \frac{x(b-a)}{N}.$$

□

Returning to the original question of gambling, we have shown that the unique function that outputs the probability of reaching N dollars is a linear one. Our elementary model suggests the usefulness of SRWs for solving problems involving discretely defined functions. In the coming sections, we will formulate a higher dimensional analogue to this problem in gambling and see that its solution similarly involves simple random walks.

3. Harmonic Functions and Mean Value Property

Let's return our attention for the moment to equation (2.2):

$$F(a) = \frac{1}{2}F(a+1) + \frac{1}{2}F(a-1). \tag{2.2}$$

In essence, this condition states that the function evaluated at any interior point will be the average of its neighbors. Such a property of functions is called the (discrete) mean-value property, and it can easily be extended to multiple dimensions. We begin by clarifying what we mean by "interior" in higher dimensions.

Definition 3.1. Let A be a finite subset of $\mathbb{Z}^{\mathrm{d}}$. We define the *boundary* of A as

$$\partial A := \{x \in (\mathbb{Z}^d \backslash A) \mid \exists\ y \in A \text{ s.t. } |x-y| = 1\}.$$

The *closure* of A is defined as $\overline{A} = A \cup \partial A$. Finally, A is called the *interior* of $\overline{A}$.

The (discrete) mean-value property in $\mathbb{Z}^d$ is then just a generalization of equation (2.2).

Definition 3.2. Let A be a subset of $\mathbb{Z}^d$. A function $F : \overline{A} \to \mathbb{R}$ has the (discrete) *mean-value property* if for all x in A we have

$$F(x) = \frac{1}{2d} \sum_{\substack{y \in \mathbb{Z}^d \\ |x-y|=1}} F(y). \tag{3.3}$$

The continuous version of the mean-value property can be obtained by swapping sums with integrals. More specifically, we appeal to the standard metric definitions for interior, boundary, and ball and say that a function $F : \mathbb{R}^n \to \mathbb{R}$ has the mean-value property if, for all x in the interior of its domain, $F(x)$ is the average of the function values on the boundary of every ball centered at x contained in the domain of the function.

The mean-value property has been well studied due to its relationship with an important partial differential equation, known as Laplace's equation. Functions that have the mean-value property satisfy Laplace's equation; they are known as harmonic functions, and they figure prominently in much of mathematical physics. Although the proof linking the mean-value property to harmonic functions in the continuous setting is non-trivial, we will be able to see the relationship almost immediately in $\mathbb{Z}^d$. We begin with the formal definitions in the continuous setting.

Definition 3.4. Let F be a real-valued, twice continuously differentiable function defined on an open subset of $\mathbb{R}^n$. The *Laplacian* of F, denoted ΔF, is the sum of all the unmixed second partial derivatives of f:

$$\Delta F := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

Laplace's equation is the second-order partial differential equation

$$\Delta F = 0.$$

F is *harmonic* if it satisfies Laplace's equation.

We now translate these definitions to the discrete setting by first constructing a satisfactory analog for derivatives – in particular, unmixed second partial derivatives. For a real-valued, twice-differentiable function F defined on a subset of $\mathbb{Z}^d$, we write the forward partial derivative with respect to the variable x_i as

$$\frac{\partial^+ F}{\partial x_i} := F(x+e_i) - F(x)$$

and the backward partial derivative with respect to the same variable as

$$\frac{\partial^- F}{\partial x_i} := F(x) - F(x-e_i)$$

We can then obtain a sensible definition for an unmixed partial second derivative; take a forward partial derivative and then take the backward partial derivative with respect to the same variable:

$$\begin{aligned}\frac{\partial^2 F}{\partial x_i^2} := \frac{\partial^-}{\partial x_i}\frac{\partial^+ F}{\partial x_i} &= \frac{\partial^-}{\partial x_i}(F(x+e_i) - F(x))\\ &= (F(x+e_i) - F(x)) - (F(x) - F(x-e_i))\\ &= F(x+e_i) + F(x-e_i) - 2F(x)\end{aligned}$$

One can see that the same function is obtained if we swap the order of forward and backward differentiation.

Lastly, we extend the definition for the Laplacian and harmonic functions to the discrete setting.

Definition 3.5. For $A \subseteq \mathbb{Z}^d$ and $F : \overline{A} \to \mathbb{R}$, the *Laplacian* is defined on A and is given by

$$\Delta F := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = \sum_{i=1}^{d}(F(x+e_i) + F(x-e_i) - 2F(x))$$

F is a discrete *harmonic function* if

$$\Delta F = \sum_{i=1}^{d}(F(x+e_i) + F(x-e_i) - 2F(x)) = 0 \tag{3.6}$$

We can now easily relate harmonic functions to the mean-value property.

Theorem 3.7. *Let A be a subset of $\mathbb{Z}^d$. $F : A \to \mathbb{R}$ is harmonic if and only if it has the mean-value property.*

Proof. From equations (3.3) and (3.6), we see that being harmonic and having the mean-value property are both equivalent to the condition

$$2dF(x) = \sum_{i=1}^{d} F(x+e_i) + F(x-e_i)$$

□

4. Dirichlet Problem

Having established a link between the mean-value property and harmonic functions, we are now ready to introduce the higher dimensional analog to the gambler's ruin problem. To find the probability of reaching our goal of N dollars given our starting value, we required a function that satisfied condition (2.2) on the interior of its domain and was equal to 0 or 1 on its boundary. We later saw that condition (2.2) was the one-dimensional version of the mean-value property, a property possessed by a group of functions known as harmonic. We can thus generalize gambler's ruin to multiple dimensions as follows: We are given a finite set $A \subset \mathbb{Z}^d$ and a real-valued function F defined on ∂A. We wish to find an extension of F to $\overline{A}$ that is harmonic on the interior. The continuous version of this problem is known as the Dirichlet problem – named after the German mathematician Peter Gustav Lejeune Dirichlet – and the existence and uniqueness of its solution depend on properties of the domain, as well as a certain smoothness of the boundary data. The discrete version, however, can always be solved uniquely.

For the rest of the section, we let A be a finite subset of $\mathbb{Z}^d$. To prove uniqueness, we will use the following property of harmonic functions, known as the weak maximum principle.

Lemma 4.1 (Weak Maximum Principle)**.** *A harmonic function $F : \overline{A} \to \mathbb{R}$ achieves its extrema on ∂A.*

Proof. Suppose for contradiction that F does not achieve its maximum on its boundary. Then the function achieves its maximum value M on some interior point $x \in A$. In particular, for all $y \in \mathbb{Z}^d$ such that $|x - y| = 1$, we have $F(y) \leq F(x)$. But by the mean-value property,

$$F(x) = \frac{1}{2d} \sum_{\substack{y \in \mathbb{Z}^d \\ |x-y|=1}} F(y).$$

We can then conclude that for any such neighbor, say $x+e_i$, $M = F(x) = F(x+e_i)$. In other words, the function achieves its maximum at $x+e_i$, and so, by assumption, $x + e_i$ is an interior point. We can then compare $x + e_i$ to its neighbors to show that $x + 2e_i$ is also an interior point, and, inductively, $x, x + e_i, x + 2e_i, ...$ are all interior points. This is impossible, since A is finite, and thus, F achieves its maximum on its boundary. We can similarly show that the minimum is achieved on the boundary. □

Just as in the one dimensional case, solving the general Dirichlet problem will involve simple random walks and stopping times, which we now define for multiple dimensions.

Definition 4.2. For every point x in A, we define the stopping time as

$$T_x := \min\{n \mid S_0 = x, S_n \in \partial A\}.$$

If the origin of the random walk is clear from context, we omit the subscript.

Earlier, we stated without proof that the stopping time of a random walk in one dimension is finite with probability 1. We now prove that this is true in all dimensions.

Proposition 4.3. *For all $x \in A$, T_x is finite with probability* 1.

Proof. For every $x \in \overline{A}$, we define $f(x) = P\{T_x \text{ is finite}\}$. T_x is finite if and only if the random walk beginning at x reaches a boundary point in finite time. To reach a boundary point in finite time, the random walk must first go through one of the $2d$ points neighboring x. Since

$$P\{S_n = a \mid S_0 = y\} = P\{S_{n+1} = a \mid S_1 = y\},$$

we have

$$P\{T_x \text{ is finite} \mid S_1 = y\} = P\{T_y \text{ is finite}\} = f(y),$$

and thus, f satisfies the mean value property:

$$f(x) = \frac{1}{2d} \sum_{\substack{y \in \mathbb{Z}^d \\ |x-y|=1}} f(y).$$

Moreover, for all x in the boundary of A, $f(x) = 1$ since the stopping time is finitely 0. By the maximum principle, f achieves both its maximum and minimum on its boundary, so $f(x) = 1$ for all $x \in \overline{A}$. $\square$

Finally, the following observation will be used in our solution.

Lemma 4.4. *Let F and G be harmonic functions defined on A and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha F + \beta G$ is harmonic.*

This is clear from the fact that the mean-value property is retained under linear combination.

We are now ready to construct the unique solution to the Dirichlet problem.

Theorem 4.5. *(Discrete Dirichlet Problem) Let $F : \partial A \to \mathbb{R}$. Then*

$$U(x) = \mathbf{E}[F(S_T) \mid S_0 = x],$$

is the unique harmonic function for which $U(x) = F(x)$ for all $x \in \partial A$.

Proof. Let's consider the solution to a simplified problem in which a given function f is only non-zero at a single point p in the boundary, for which $f(p) = 1$ (see Figure 2). The function $u_p(x) := P_T(x, p)$ satisfies the mean value property on the interior, since the probability of a walk starting at x leaving the region at p is just the sum of the probabilities of the walk being at a given neighbor at $n = 1$ and then exiting at p, i.e.

$$P_T(x, p) = \frac{1}{2d} \sum_{\substack{y \in \mathbb{Z}^d \\ |x-y|=1}} P_T(y, p).$$

In addition, the constructed function agrees with f on the boundary. To find a solution for the arbitrary function F, we take a linear combination of the solutions to each such simplified problem, allowing p, the point at which the function equals 1, to span the boundary. The particular linear combination we take is

$$U(x) = \sum_{p \in \partial A} F(p) P_T(x, p) = \mathbf{E}[F(S_T) \mid S_0 = x].$$

By Lemma 4.4, this function is harmonic. Moreover, it agrees with F on the boundary. We now prove that this is the unique solution: Suppose G is a solution. By Lemma 4.4, $F - G$ is harmonic, and so, by the weak maximum principle, $F - G$ achieves its extrema at its boundary. But F and G are solutions to the same Dirichlet problem and therefore they agree on the boundary. Thus, for all x in the boundary, $(F-G)(x) = 0$, so the maximum and minimum are equal. As such, $(F-G)$ is identically 0, implying that $F = G$.

□

5. Explicit Solutions with Linear Algebra

With this theorem, we have shown that there exists a unique solution to every Dirichlet problem. Nevertheless, the solution we constructed is not quite satisfying, as it is given in terms of probabilities of random walk positions. We can compute the solution more explicitly by solving a system of linear equations for the interior values of the function.

As review, we are given a set $A \in \mathbb{Z}^d$ of finite cardinality n and a real-valued function F defined on ∂A, and our goal is to find an extension of F to $\overline{A}$ that is harmonic on the interior. We can easily represent this problem as a system of n linear equations in n unknowns by applying the mean-value property to each point on the interior. The solution to this system is the solution to the given Dirichlet problem and so it is unique. To solve this system, we move all the unknown terms to the left hand side of the equations, and represent the system with the matrix equation $Ax = B$, where $A \in M_n(\mathbb{R})$, $x \in \mathbb{R}^n$ is an unknown column vector consisting of the function values on the interior, and $B \in \mathbb{R}^n$ is a column vector that depends on the values of F.

The existence of a unique solution to this equation implies that the linear transformation described by A is bijective and therefore invertible. We can then invert the matrix A, and our solution is given as a simple product of matrices:

$$x = A^{-1}(Ax) = A^{-1}B.$$

We will now use this method to find solutions to specific Dirichlet problems. To allow for graphical representation, we choose regions in $\mathbb{Z}^2$, and to avoid computational drudgery, our regions do not contain too many points. For the same reason, we make use of the computational engine Wolfram Alpha to invert matrices and perform matrix multiplication. Our first example will be a row of three points; we then turn to a more complex shape.

Example 5.1. Let $A = \{(-1,0), (0,0), (1,0)\}$ (Figure 1).

We will first compute the solution for this region in terms of non-numeric boundary function values and then choose specific values to plug in. We call the solution function u and the given boundary function F. Applying the mean-value property, we can set up the following system of linear equations:

$$\begin{cases} u(x) = \frac{1}{4}u(y) + \frac{1}{4}(F(a) + F(b) + F(h)) \\ u(y) = \frac{1}{4}u(x) + \frac{1}{4}u(y) + \frac{1}{4}(F(c) + F(g)) \\ u(z) = \frac{1}{4}u(y) + \frac{1}{4}(F(f) + F(e) + F(d)) \end{cases} .$$

Moving all the values of u to the left side and multiplying by 4, we obtain

$$\begin{cases} 4u(x) - u(y) = F(a) + F(b) + F(h) \\ 4u(y) - u(x) - u(z) = F(c) + F(g) \\ 4u(z) - u(y) = F(f) + F(e) + F(d) \end{cases} ,$$

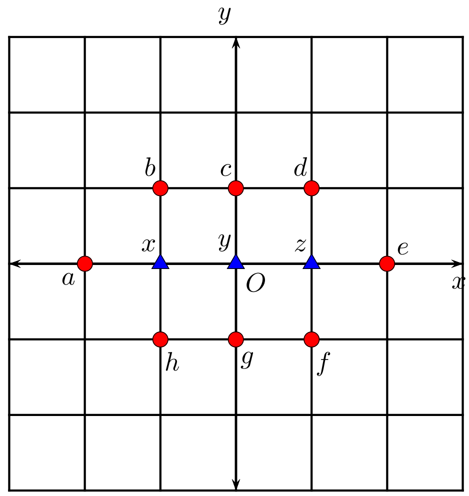

FIGURE 1

which can be represented by the matrix equation

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u(x) \\ u(y) \\ u(z) \end{bmatrix} = \begin{bmatrix} F(a)+F(b)+F(h) \\ F(c)+F(g) \\ F(f)+F(e)+F(d) \end{bmatrix}.$$

WolframAlpha allows us to invert the square matrix appearing in the above equation. We can then solve for u by multiplying from the left:

$$\begin{bmatrix} u(x) \\ u(y) \\ u(z) \end{bmatrix} = \begin{bmatrix} \frac{15}{56} & \frac{1}{14} & \frac{1}{56} \\ \frac{1}{14} & \frac{2}{7} & \frac{1}{14} \\ \frac{1}{56} & \frac{1}{14} & \frac{15}{56} \end{bmatrix} \begin{bmatrix} F(a)+F(b)+F(h) \\ F(c)+F(g) \\ F(f)+F(e)+F(d) \end{bmatrix}.$$

Let's see what happens when we plug in some numeric values. If the function on the boundary is identically 0, we can see that u is 0, as well. What happens if we change the value at one point, say c, to 1, and keep the rest 0? We then have:

$$\begin{bmatrix} u(x) \\ u(y) \\ u(z) \end{bmatrix} = \begin{bmatrix} \frac{15}{56} & \frac{1}{14} & \frac{1}{56} \\ \frac{1}{14} & \frac{2}{7} & \frac{1}{14} \\ \frac{1}{56} & \frac{1}{14} & \frac{15}{56} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{14} \\ \frac{2}{7} \\ \frac{1}{14} \end{bmatrix}.$$

We can verify that this is indeed the solution by checking that it satisfies the mean-value property. Since the solution is unique, our function must be equal to the one we constructed in Theorem 4.5. Recall that this function was

$$U(x_o) = \mathbf{E}[F(S_T) \mid S_0 = x_o].$$

In this case,

$$\mathbf{E}[F(S_T) \mid S_0 = x_o] = \sum_{p \in \partial A} F(p) P_T(x_o, p) = P_T(x_o, c).$$

But by the uniqueness of the solution,

$$u(x_0) = U(x_0) = P_T(x_o, c),$$

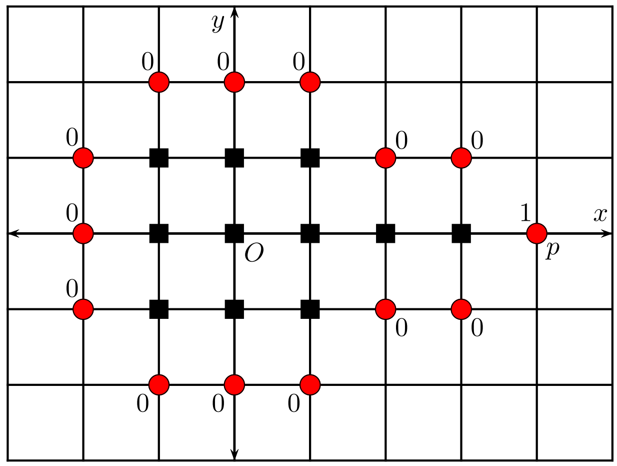

Figure 2

and so, by computing the solution to this particular Dirichlet problem, we have calculated, for each point in A, the probability that an SRW beginning at that point first exits A at c. For example, an SRW beginning at z has a $\frac{1}{14}$ probability of first exiting the region at c. Moreover, summing the interior values and dividing by 3, we obtain the probability, $\frac{1}{7}$, that an SRW with an unknown starting point in A exits A at c.

As we can see, the solution to the particular Dirichlet problem in which the boundary function is set at 0 at all but one point, at which it equals 1, has a nice interpretation in terms of random walks. In addition, as we saw in the proof of Theorem 4.5, the solution for any other boundary function is simply a linear combinations of solutions to this special case. For these reasons, we choose the same sort of boundary function for our next example.

Example 5.2. Let B be the 3 by 3 grid centered at the origin, along with two points arranged horizontally to the right. We let the boundary function equal 0 at all but the right-most point, p, at which we set it equal to 1 (Figure 2).

We now go straight to the solution, rounding to the third decimal spot:

.002	.005	.007		
.003	.009	.024	.073	.268
.002	.005	.007		

As we would expect, it is highly unlikely for a random walk beginning at $(-1, 1)$ or $(-1, -1)$ to exit the region through $(4, 0)$. It is perhaps surprising, however, that even a simple random walk beginning at the origin has less than a 1% chance of reaching (4,0) before it hits another point on the boundary.

6. Conclusion

As was mentioned at the outset of the paper, random walks have a great deal of applications. In physics, for example, we are often interested in the limit of random walks as the unit time goes to 0, a formalization known as Brownian motion. The movement of a single gas particle as it collides with other particles and moves in space can be modeled by Brownian motion, although simple random walks on a region of the integer lattice can offer a decent approximation of this and other processes. In this paper, we found that the unique solution to the Dirichlet problem with all 0's and one 1 on the boundary is given by the probability that a simple random walk first leaves the region through the point with the value 1. We can solve the Dirichlet problem with basic linear algebra and thereby ascertain this probability. As such, linear algebra can tell us where a gas particle will likely exit a confined space. That we can apply methods in linear algebra to learn about fundamentally random processes is a surprising feature of this rich branch of mathematics.

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My exposition follows closely the presentation of two sources: Professor Greg Lawler's book *Random Walk and the Heat Equation* [1] and the previous REU paper "Random Walks in $\mathbb{Z}^d$ and the Dirichlet Problem" [2] by Eric Guan.

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