# THE FUNDAMENTAL GROUP AND SEIFERT-VAN KAMPEN'S THEOREM

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Abstract. The fundamental group is an essential tool for studying a topological space since it provides us with information about the basic shape of the space. In this paper, we will introduce the notion of free products and free groups in order to understand Seifert-van Kampen's Theorem, which will prove to be a useful tool in computing fundamental groups.

Contents



## 1. Introduction

One of the fundamental questions in topology is whether two topological spaces are homeomorphic or not. To show that two topological spaces are homeomorphic, one must construct a continuous function from one space to the other having a continuous inverse. To show that two topological spaces are not homeomorphic, one must show there does not exist a continuous function with a continuous inverse. Both of these tasks can be quite difficult as the recently proved Poincaré conjecture suggests. The conjecture is about the existence of a homeomorphism between two spaces, and it took over 100 years to prove. Since the task of showing whether or not two spaces are homeomorphic can be difficult, mathematicians have developed other ways to solve this problem. One way to solve this problem is to find a topological property that holds for one space but not the other, e.g. the first space is metrizable but the second is not. Since many spaces are similar in many ways but not homeomorphic, mathematicians use a weaker notion of equivalence between spaces – that of homotopy equivalence. In this paper we investigate a topological invariant of homotopy equivalence class that Poincaré invented – the fundamental group of a space. This invariant will serve as a means of differentiating spaces that belong to different classes. More specifically, we investigate a tool for computing the fundamental group of certain spaces: the Seifert-van Kampen's theorem. Once we have this tool, we can start to compute different fundamental groups and determine whether or not two spaces are homeomorphic or not. Additionally, we can use this tool to answer questions like "What is the fundamental group of a figure-eight space?". This theorem helps us answer that question by providing us with a simple formula to compute the fundamental group of spaces made up of components whose fundamental groups we understand. After proving this theorem, one can easily compute the fundamental group of a figure-eight space, by decomposing it into a few components whose fundamental groups we already know.

## 2. Background Definitions and Facts

**Definition 2.1.** Let $X$ be a topological space and $x_0$, $x_1 \in X$. A **path** in $X$ from $x_0$ to $x_1$ is a continuous function $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We call $x_0$ the **initial point** and $x_1$ the **final point**. The **reverse path** of $f$ is the function $\bar{f} : [0,1] \to X$ given by $\bar{f}(s) = f(1-s)$. A **loop** at $x_0$ is a path that begins and ends at $x_0$.

One can easily define a path to be a continuous function whose domain is *any* closed interval of the real line. Our definition of path naturally leads to a topological property which we state next.

**Definition 2.2.** A topological space $X$ is said to be **path connected** if every pair of points in $X$ can be joined by a path in $X$.

We will now present a useful theorem, whose proof we will omit.

**Theorem 2.3** (Pasting Lemma). *Let $X = A \cup B$, where $A$ and $B$ are closed in $X$. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then $f$ and $g$ combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.*

Now we define an operation on paths whose endpoints are compatible.

**Definition 2.4.** Let $f : [0,1] \to X$ be a path in $X$ from $x_0$ to $x_1$ and $g : [0,1] \to X$ be a path in $X$ from $x_1$ to $x_2$. We define the **product** $f * g$ of $f$ and $g$ to be the path $H : [0,1] \to X$ given by the equations

$$H(x) = \begin{cases} f(2x) & \text{for } x \in [0, \frac{1}{2}] \\ g(2x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

The map $H$ is well-defined and continuous by the Pasting Lemma. We can think of $H$ as a path that first follows $f$ twice as fast and then $g$ twice as fast. One can observe that the product essentially glues two paths together to form a new path.

The fundamental concept with which we will now be concerned is homotopy, defined next. The concept of homotopy and path homotopy is central to our study of Seifert-van Kampen's Theorem.

**Definition 2.5.** Let $f, g : X \to Y$ be two continuous functions. A **homotopy** from $f$ to $g$ is a continuous function $F : X \times [0,1] \to Y$ such that, for all $x \in X$,

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x).$$

We say that $f$ and $g$ are **homotopic** if there exists a homotopy between them, and we use the notation $f \simeq g$ if $f$ and $g$ are homotopic.

We can think of a homotopy as a continuous deformation of one function to another function parametrized by time $t \in [0,1]$. At time $t = 0$, we have our function $f(x)$, while at time $t = 1$, we have our function $g(x)$. If we fix a time $t \in [0,1]$, then $F(x,t)$ is some continuous, single-variable function in the variable $x \in X$.

Given this definition of a homotopy between to continuous maps, we can specialize to the case where our continuous maps are paths.

**Definition 2.6.** Let $f, g$ be paths in $X$ with initial point $x_0$ and final point $x_1$. A **path homotopy** from $f$ to $g$ is a homotopy $F$ from $f$ to $g$ such that for all $t \in [0,1]$

$$F(0,t) = x_0 \text{ and } F(1,t) = x_1.$$

We say that $f$ and $g$ are **path homotopic** if there exists a path homotopy between them, and we use the notation $f \simeq_p g$ if $f$ and $g$ are path homotopic.

**Proposition 2.7.** *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

This result is easy to verify. For reflexivity, choose the constant homotopy. For symmetry, choose the homotopy with reverse time. Finally, for transitivity, consider the homotopy given by following each of the hypothesized homotopies, one after the other, twice as fast.

For the rest of the paper, we shall denote the path homotopy class of a path $f$ by $[f]$. The product operation that was defined earlier creates a well defined operation on path homotopy classes:

**Proposition 2.8.** *Let $f$ and $g$ be continuous functions such that their product $f * g$ is defined. Then*

$$[f] * [g] = [f * g].$$

*Proof.* Let $F$ be a path homotopy between $f$ and $f'$ and let $G$ be a path homotopy between $g$ and $g'$. Define

$$H(x) = \begin{cases} F(2s,t) & \text{for } s \in [0, \frac{1}{2}] \\ G(2s-1,t) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

The function $H$ is well-defined since $F(1,t) = G(0,t)$ for all $t$, and it is continuous by the pasting lemma. We leave it as an exercise to the reader to verify that $H$ is the required path homotopy between $f * g$ and $f' * g'$.

□

**Definition 2.9.** The **fundamental group** of a topological space $X$ relative to a base point $x_0$, denoted by $\pi_1(X, x_0)$, is the set of all path homotopy classes of loops based at $x_0$ with multiplication given by $*$.

As its name suggests, the fundamental group is indeed a group. The identity element is the constant path at $x_0$, the inverse of a path is given by the reverse path, and a proof of associativity can be found in [3, 51.2].

A first example one considers is the fundamental group of the unit circle $S_1 \subset \mathbb{R}^2$.

**Fact 2.10.** *Let $s_0 \in S^1$ be any point. The fundamental group of the unit circle $S^1$ with basepoint $s_0$ is isomorphic to the additive group of integers, i.e.*

$$\pi_1(S^1, s_0) \cong (\mathbb{Z}, +).$$

Explicitly, the equivalence class corresponding to $k \in Z$ is represented by the loop which maps $t \in [0,1]$ to $(\cos(2\pi kt), \sin(2\pi kt))$; it is the loop which winds around the circle $k$ times in the counter-clockwise direction.

We next give a definition that relates continuous maps and homomorphisms. This definition is one of the main reasons the fundamental group is the central invariant that it is, and it will be useful in our formulation of Seifert-van Kampen's Theorem.

**Definition 2.11.** Let $g : (X, x_0) \to (Y, y_0)$ be a continuous map. Then the **homomorphism induced by g** relative to the base point $x_0$, denoted by $g_*$, is the map

$$g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

given by the equation

$$g_*([f]) = [g \circ f].$$

We leave it as an exercise for the reader to show that this definition is well-defined.

In order to eventually compute fundamental groups via Seifert-van Kampen's Theorem, we need the following basic group-theory fact whose proof we will omit:

**Theorem 2.12** (First Isomorphism Theorem)**.** *Let $G, G'$ be groups and $\varphi : G \to G'$ be a surjective group homomorphism. Then, $G/\ker(\varphi) \cong G'$.*

The goal of this paper is to prove Seifert-van Kampen's Theorem, which is one of the main tools in the calculation of fundamental groups of spaces. Before we can formulate the theorem, we will first need to introduce some terminology from group theory, which we do in the next section.

## 3. Free Groups and Free Products

**Definition 3.1.** Let $G$ be a group and let $\{G_\alpha\}_{\alpha \in J}$ be a family of subgroups of $G$. We say that these subgroups **generate** $G$ if every element $x \in G$ can be written as the finite product of elements of the subgroups $G_\alpha$. That is, there exists a sequence $(x_1, \ldots, x_n)$ where $x_i \in G_{\alpha_i}$ such that $x = x_1 \cdot \ldots \cdot x_n$.
We call this sequence $(x_1, \ldots, x_n)$ a **word** of length $n$ that **represents** $x$.

Immediately, one questions whether the representation of any element $x \in G$ is unique. That is, do there exist two distinct words that still represent the same element $x$? This question gives rise to the following remarks.

**Remarks:**
1. Since we do not necessarily have an abelian group, we cannot necessarily rearrange the factors, i.e. the elements of each $G_\alpha$ of $x$.
2. We can, however, combine the factors $x_i$ and $x_{i+1}$ if they belong to the same subgroup $G_\alpha$. If that is the case, then we obtain the word of length $n-1$ given by $(x_1, ..., x_{i-1}, x_i x_{i+1}, x_{i+2}, ..., x_n)$, which still represents the same element $x$. This operation of combining factors is one of the operations to which we will refer to as *reduction*.
3. We can delete any element $x_i$ that equals the identity element 1. Like before, we obtain a shorter word that still represents the original element $x$. We will also refer to this operation of deleting identity elements as *reduction*.

Note that *reduction* refers to both the operation of combining factors and the operation of deleting identity elements.

**Definition 3.2.** A word $(x_1, \dots, x_n)$ is called *reduced* if for each $i$ the elements $x_i$ and $x_{i+1}$ do not belong to the same group $G_\alpha$ and $x_i \neq 1$.
Note that by applying reduction operations to any word $(x_1, \dots, x_n)$, we can find a reduced word that represents the same element. Furthermore, the reduced word that represents the identity element is the empty set, which is the word of length zero.

Now, one might question whether the concatenation of two reduced words is another reduced word. This question leads to the following remark.

**Remark**:
If $(x_1, ..., x_n)$ and $(y_1, .., y_m)$ are reduced words representing $x, y \in G$, then their juxtaposition $(x_1, ..., x_n, y_1, ..., y_m)$ is a word representing $xy$. However, this representation of $xy$ is not necessarily reduced since $x_n$ and $y_1$ could belong to the same subgroup $G_\alpha$. When this is the case, we can use our reduction operations to obtain a shorter word.

**Definition 3.3.** Let $G$ be a group and $\{G_\alpha\}_{\alpha \in J}$ be a family of subgroups of $G$ that generate $G$. We say that $G$ is the **free product** of the subgroups $G_\alpha$ if, for $\alpha \neq \beta$, the intersection $G_\alpha \cap G_\beta$ contains only the identity element, and if each element $x \in G$ has a unique representation by a reduced word in the groups $\{G_\alpha\}_{\alpha \in J}$.
We denote this by

$$G = \prod_{\alpha \in J}^{*} G_\alpha.$$

In the finite case, we write $G = G_1 * \cdots * G_n$.

A key characteristic of the free product is that it satisfies this universal property, whose proof we shall omit:

**Proposition 3.4.** *A group $G$ is the free product of a family of subgroups $\{G_\alpha\}_{\alpha\in J}$ if and only if for any group $H$ and any family of homomorphisms $h_\alpha : G_\alpha \to H$, there exists a unique homomorphism $h : G \to H$ whose restriction to $G_\alpha$ equals $h_\alpha$ for each $\alpha$.*

Also, we will use one useful fact regarding free products in our formulation of Seifert-van Kampen's Theorem:

**Fact 3.5.** *For every two groups $G$ and $H$ there exists a larger group $K$ that includes $G$ and $H$ as subgroups and such that $K$ is the free product of its subgroups $G$ and $H$. A group $K$ with this property is unique up to a unique isomorphism. We call such a group $K$* the free product of $G$ and $H$ *and denote it by $G * H$.*

The free product of $G$ and $H$ is essentially the set of reduced formal words from $G$ and $H$ with multiplication given by juxtaposition followed by word reduction. 2) you can say is one line what this group is:

**Definition 3.6.** Let $G$ be a group and let $\{a_\alpha\}_{\alpha\in J}$ be a set of elements of $G$. We say the set $\{a_\alpha\}_{\alpha\in J}$ **generates** $G$ if every element of $G$ can be written as the product of the elements $a_\alpha$.

As of now, we have two different meanings of the word *generate*. These two different meanings are interconnected: the group is generated by elements if and only if it is generated by the cyclic subgroups that these elements generate. Hence, the two different meanings of to "generate" are naturally the same.

**Definition 3.7.** Let $\{a_\alpha\}_{\alpha\in J}$ be a set of elements of a group $G$ and suppose that each $a_\alpha$ generates an infinite cyclic subgroup $G_\alpha$ of $G$. If $G$ is the free product of the groups $\{G_\alpha\}_{\alpha\in J}$, then $G$ is said to be a **free group**. The set $\{a_\alpha\}_{\alpha\in J}$ is called a **set of free generators**.

Essentially, a free group is a group with no restrictions on it (hence the name). Explicitly, a free group is simply the set of reduced words whose letters are taken from the generating set and their inverses and whose multiplication is given by the concatenation followed by word reduction.

**Example 3.8.** The group of integers $(\mathbb{Z}, +)$ is free and generated by the set $\{1\}$.

## 4. Seifert-van Kampen Theorem

We now state and prove two versions of the Seifert-van Kampen Theorem. Throughout this section, we always assume that $X = U \cup V$, where $U$ and $V$ are open in $X$. We also assume that $U$, $V$, and $U \cap V$ are path connected. The proof of the two versions of Seifert-van Kampen's Theorem requires a preliminary result.

**Lemma 4.1.** *Let $X$ be a topological space and let $x_0 \in U \cap V$. Then every loop in $X$ based at $x_0$ is homotopic to a product of loops based at $x_0$ contained in either $U$ or $V$.*

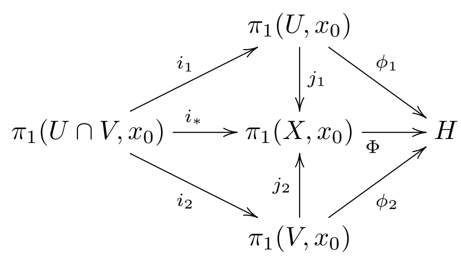
*Proof.* Let $f : [0,1] \to X$ be a loop with basepoint $x_0$. By compactness of $[0,1]$, there exists a partition $0 = s_0 < \cdots < s_m = 1$ of $[0,1]$ such that each subinterval $[s_{i-1}, s_i]$ is mapped into either $U$ or $V$. Assume $U$ contains $f[s_{i-1}, s_i]$ and let $f_i$ be the path obtained by restricting $f$ to $[s_{i-1}, s_i]$. Then $f$ is the composition of $f_1 * \cdots * f_m$ where $f_i$ is a path in either $U$ or $V$. Since $U$ and $V$ are path-connected, we can choose a path $g_i$ in $U$ or $V$ from $x_0$ to the point $f(s_i)$ in $U$ or $V$. Consider the loop

$$(f_1 * \bar{g_1}) * (g_1 * f_2 * \bar{g_2}) * (g_2 * f_3 * \bar{g_2}) * \cdots * (g_{m-1} * f_m)$$

which is homotopic to $f$. Note that this loop is a composition of loops based at $x_0$ lying in either $U$ or $V$. Hence, every loop in $X$ is homotopic to a product of loops contained in either $U$ or $V$.

□

**Theorem 4.2** (Seifert-van Kampen Theorem Version 1)**.** *Let $X$ be a topological space and let $x_0 \in U \cap V$. Let $\phi_1 : \pi_1(U, x_0) \to H$ and $\phi_2 : \pi_1(V, x_0) \to H$ be homomorphisms. Let $i_1, i_2, j_1, j_2$ be the homomorphisms indicated in the following diagram, each induced by inclusion.*




*If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there exists a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.*

*Proof.* First we show uniqueness of $\Phi$. By Lemma 4.1, every loop is homotopic to a product of loops that reside in either $U$ or $V$. Thus the fundamental group $\pi_1(X, x_0)$ is generated by the images of $j_1$ and $j_2$. Because $\Phi$ is determined by $\phi_1$ and $\phi_2$ on these images, it follows that it is determined on every product of elements from these images. But these products include all of the elements, and so $\Phi$ is completely determined by $\phi_1$ and $\phi_2$. Therefore, $\Phi$ is unique.

We now introduce some useful notation: given a path $f$ in $X$, we will use $[f]$ to denote its path-homotopy class in $X$. If $f$ lies in $U$, then $[f]_U$ will denote its path-homotopy class in $U$. The notations $[f]_V$ and $[f]_{U \cap V}$ are defined similarly.

To show the existence of $\Phi$ we will define several different maps, each building on the previous. Furthermore, we will divide this proof into different parts for clarity.

**PART 1**: First, we extend $\phi_1$ and $\phi_2$ to a set map $\rho$ defined on all loops in $X$ that are contained in either $U$ or $V$. Let $f$ be a loop based at $x_0$ that lies in either $U$ or $V$. Define an element of the group $H$ by

$$\rho(f) = \phi_1([f]_U) \text{ if } f \text{ lies in } U,$$
$$\rho(f) = \phi_2([f]_V) \text{ if } f \text{ lies in } V.$$

Note that $\rho$ is well-defined, for if $f$ lies in both $U$ and $V$, then

$$\phi_1([f]_U) = \phi_1(i_1([f]_{U\cap V})) \text{ and } \phi_2([f]_V) = \phi_2(i_2([f]_{U\cap V})),$$

and these two elements are equal since $\phi_1 \circ i_1 = \phi_2 \circ i_2$ by hypothesis. We claim that $\rho$ satisfies the following conditions:

(1) If $[f]_U = [g]_U$ or if $[f]_V = [g]_V$, then $\rho(f) = \rho(g)$.

(2) If both $f$ and $g$ lie in $U$ or if both $f$ and $g$ lie in $V$, then $\rho(f * g) = \rho(f) \cdot \rho(g)$.

Indeed, the first property follows from the definition of $\rho$, and the second property follows since $\phi_1$ and $\phi_2$ are homomorphisms.

**PART 2**: We extend $\rho$ to a map $\sigma$ that assigns to each *path* $f$ lying in $U$ or in $V$, an element of $H$ such that the map $\sigma$ satisfies condition (1) of $\rho$ and satisfies condition (2) whenever $f * g$ is defined. This step is necessary as our proof will involve paths that are not necessarily closed loops.

For each $x \in X$, choose a path $\alpha_x$ from $x_0$ to $x$ as follows:

- If $x = x_0$, let $\alpha_x$ be the constant path at $x_0$.
- If $x \in U \cap V$, let $\alpha_x$ be a path in $U \cap V$.
- If $x \in U$ or $x \in V$, but $x \notin U \cap V$, let $\alpha_x$ be a path in $U$ or $V$, respectively.

Then, for any path $f$ in $U$ or in $V$ from $x$ to $y$, we define a *loop* $L(f)$ based at $x_0$ by the equation

$$L(f) = a_x * (f * \bar{\alpha}_y).$$

By our choice of $\alpha_x$ and $\alpha_y$, if $f$ were a path in $U$, then $L(f)$ is a loop in $U$, and smilarly for $V$.

Finally, we define,

$$\sigma(f) = \rho(L(f)).$$

We now show that $\sigma$ is indeed an extension of $\rho$. If $f$ is a *loop* based at $x_0$ lying in either $U$ or $V$, then

$$L(f) = \alpha_{x_0} * (f * \bar{\alpha}_{x_0})$$

where $\alpha_{x_0}$ is the constant path at $x_0$. It follows that $L(f)$ is path homotopic to $f$ in either $U$ or $V$, so that $\rho(L(f)) = \rho(f)$ by property (1) of $\rho$. Hence, $\sigma(f) = \rho(f)$.

We now show properties (1) and (2) hold for $\sigma$. To check condition (1), let $f$ and $g$ be *paths* that are path homotopic in $U$. If $F$ is a path homotopy in $U$ from $f$ to $g$, then the homotopy $L(F)$ is a path homotopy n $U$ from $L(f)$ to $L(g)$. Hence, $L(f)$ and $L(g)$ are path homotopic in $U$ so condition (1) for $\rho$ applies. The proof for when $f$ and $g$ are path homotopic in $V$ is similar. Hence, $\sigma$ indeed satisfies

condition (1).
To check condition (2), let $f$ be and $g$ be arbitrary paths in $U$ or $V$ such that

$$f(0) = x, f(1) = g(0) = y, \text{ and } g(1) = z.$$

We have

$$L(f) * L(g) = (\alpha_x * (f * \bar{\alpha_y})) * (\alpha_y * (g * \bar{\alpha_z})).$$

It follows that $L(f) * L(g)$ is path homotopic to $L(f * g)$ since multiplication is associative up to path homotopy and multiplication with the opposite path is homotopic to the constant path.
So

$$\rho(L(f * g)) = \rho(L(f) * L(g)) = \rho(L(f)) \cdot \rho(L(g))$$

by condition(2) for $\rho$. Hence, $\sigma(f * g) = \sigma(f) \cdot \sigma(g)$ and so properties (1) and (2) also hold for $\sigma$.

**PART 3**: Finally, we will extend $\sigma$ to a set map $\tau$ that assigns to an *arbitrary* path $f$ of $X$ an element of $H$. Given any path $f$ in $X$, the compactness of the interval $[0, 1]$ implies that we can choose a subdivision $0 = s_0 < \cdots < s_n = 1$ of $[0, 1]$ such that $f$ maps each of the subintervals $[s_{i-1}, s_i]$ into $U$ or $V$. Let $f_i$ denote the path obtained by restricting $f$ to $[s_{i-1}, s_i]$. Then $f_i$ is a path in $U$ or $V$, and

$$[f] = [f_1] * \cdots * [f_n].$$

We now define $\tau$ by:

$$\tau(f) = \sigma(f_1) \cdots \sigma(f_2) \cdots \sigma(f_n). \tag{4.3}$$

We claim that the map $\tau$ will satisfy similar conditions to $\rho$ and $\sigma$:
(1) If $[f] = [g]$, then $\tau(f) = \tau(g)$.
(2) If $f * g$ is defined, then $\tau(f * g) = \tau(f) \cdot \tau(g)$.
But before we prove this claim, we shall show that this definition of $\tau$ is independent of the choice of subdivision. It suffices to show that the value of $\tau(f)$ remains unchanged if we add one additional point $p$ to the subdivision. Let $i$ be the index such that $s_{i-1} < p < s_i$. If we compute $\tau(f)$ using this new subdivision, the only change in (4.3) is that the factor $\sigma(f_i)$ is replaced by the product $\sigma(f_i') \cdot \sigma(f_i'')$ where $f_i'$ and $f_i''$ are the paths obtained by restricting $f$ to $[s_{i-1}, p]$ and $[p, s_i]$, respectively. But, $f_i$ is path homotopic to $f_i' * f_i''$ in $U$ or $V$, so we have $\sigma(f_i) = \sigma(f_i') \cdot \sigma(f_i'')$, by conditions (1) and (2) for $\sigma$. Thus, $\tau$ is independent of our choice of subdivsion and hence well-defined.

It immediately follows that $\tau$ is an extension of $\sigma$: if $f$ already lies in $U$ or $V$, then one can use the trivial partition of $[0, 1]$ to define $\tau(f)$ and so $\tau(f) = \sigma(f)$ by definition.

Now, we prove that $\tau$ satisfies condition (1): If $[f] = [g]$, then $\tau(f) = \tau(g)$.
Let $f$ and $g$ be paths in $X$ from $x$ to $y$ and let $F$ be the path homotopy between them. We first prove the claim under the additional assumption that there exists a subdivision $s_0, ..., s_n$ of $[0, 1]$ such that $F$ carries each rectangle $R_i = [s_{i-1,s_i}] \times [0, 1]$ into either $U$ or $V$.
Given $i$, let $f_i$ and $g_i$ be the paths obtained by restricting $f$ and $g$ to $[s_{i-1}, s_i]$, respectively. The restriction of $F$ to the rectangle $R_i$ gives us a homotopy between

$f_i$ and $g_i$ that takes place in either $U$ or $V$, but it is not necessarily a path homotopy since the end points of the paths may move during the homotopy, i.e. the paths $\beta_i(t) = F(s_i, t)$. Consider the paths $\beta_i = F(s_i, t)$ in $X$ from $f(s_i)$ to $g(s_i)$ traced out by these end points during the homotopy. Note that the paths $\beta_0$ and $\beta_n$ are the constant paths at both $x$ and $y$, respectively.
We show that for each $i$,

$$f_i * \beta_i \simeq_p \beta_{i-1} * g_i$$

with the path homotopy taking place in $U$ or in $V$.
In the rectangle $R_i$, take the edge path that runs along the bottom and right edges of $R_i$, from $[s_{i-1}, 0]$ to $[s_i, 0]$ to $[s_i, 1]$; if we compose this path by the map $F$, we obtain the path $f_i * \beta_i$. Similarly, if we take the edge path along the left and top edges of $R_i$, and compose it by $F$, we obtain the path $\beta_{i-1} * g_i$. Because $R_i$ is convex, there is a path homotopy in $R_i$ between these two edge paths. If we compose this path homotopy by $F$ we obtain a path homotopy between $f_i * \beta_i$ and $\beta_{i-1} * g_i$ that takes place in either $U$ or $V$, as desired.
It follows from properties (1) and (2) for $\sigma$ that

$$\sigma(f_i) \cdot \sigma(\beta_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i),$$

so

$$\sigma(f_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i) \cdot \sigma(\beta_i)^{-1} \tag{4.4}$$

It follows similarly that since $\beta_0$ and $\beta_n$ are constant paths, $\sigma(\beta_0) = \sigma(\beta_n) = 1$ (this is because the equality $\beta_0 * \beta_0 = \beta_0$ implies that $\sigma(\beta_0) \cdot \sigma(\beta_0) = \sigma(\beta_0)$ and similarly for $\beta_n$).
We now compute as follows:

$$\tau(f) = \sigma(f_1) \cdot \sigma(f_2) \cdots \sigma(f_n).$$

Substituting (4.4) into this equation and simplifying gives us the equation

$$\begin{aligned} \tau(f) &= \sigma(g_1) \cdot \sigma(g_2) \cdots \sigma(g_n) \\ &= \tau(g) \end{aligned}$$

Thus, we have proved condition (1) in our special case.
Now we prove condition (1) in the general case. Given $f$ and $g$ and a path homotopy $F$ between them, let us choose (by compactness of $[0,1]^2$) subdivisions $s_0 < \ldots < s_n$ and $t_0 < \ldots < t_m$ of $[0,1]$ such that $F$ maps each subrectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into either $U$ or $V$. Let $f_j$ be the path $f_j(s) = F(s, t_j)$; then $f_0 = f$ and $f_m = g$. The pair of paths $f_{j-1}$ and $f_j$ satisfy the requirements of our special case from which we deduce that $\tau(f_{j-1}) = \tau(f_j)$ for each $j$. It follows that $\tau(f) = \tau(g)$.

Now we prove condition (2) for the set map $\tau$. Given a composition of paths $f * g$ in $X$, let us choose a subdivision $s_0 < \cdots < s_n$ of $[0,1]$ containing the point $\frac{1}{2}$ as a subdivision point such that $f * g$ carries each subinterval into either $U$ or $V$. Let $k$ be the index such that $s_k = \frac{1}{2}$.
For $i = 1, ..., k$, the increasing linear map of $[0,1]$ to $[s_{i-1}, s_i]$ followed by $f * g$, is the same as the increasing linear map of $[0,1]$ to $[2s_{i-1}, 2s_i]$ followed by $f$; call this map $f_i$. Similarly, for $i = k+1, ..., n$, the postitive linear map of $[0,1]$ to $[s_{i-1}, s_i]$

followed by $f*g$ is the same as the increasing linear map of $[0,1]$ to $[2s_{i-1}-1, 2s_i-1]$ followed by $g$; call this map $g_{i-k}$. Using the subdivision $s_0, ..., s_n$ for the domain of the path $f*g$, we have

$$\tau(f*g) = \sigma(f_1)\cdots\sigma(f_k)\cdot\sigma(g_1)\cdots\sigma(g_{n-k}).$$

Using the subdivision $2s_0, ..., 2s_k$ for the path $f$, we have (by definition of $\tau$)

$$\tau(f) = \sigma(f_1)\cdots\sigma(f_k).$$

Finally, using the subdivision $2s_k - 1, ..., 2s_n - 1$ for the path $g$, we have

$$\tau(g) = \sigma(g_1)\cdots\sigma(g_{n-k}).$$

Thus, (2) clearly holds.

The theorem follows. For each loop $f$ in $X$ based at $x_0$ we define

$$\Phi([f]) = \tau(f).$$

Conditions (1) and (2) show that $\Phi$ is a well-defined homomorphism.
Now, we show $\Phi \circ j_1 = \phi_1$. If $f$ is a loop in $U$, then

$$\Phi(j_1([f]_U)) = \Phi([f]) = \tau(f) = \rho(f) = \phi_1([f]_U)$$

as desired. The proof that $\Phi \circ j_2 = \phi_2$ is similar.

□

The above theorem allows us to give a concrete presentation of $\pi_1(X, x_0)$; this presentation will be the content of the next theorem.

**Theorem 4.5** (Seifert-van Kampen Theorem Version 2)**.** *Let $X$ be a topological space and let $x_0 \in U \cap V$. Let $j : \pi_1(U, x_0) * \pi(V, x_0) \to \pi_1(X, x_0)$ be the homomorphism of the free product that extends the homomorphisms $j_1$ and $j_2$. Then, $j$ is surjective, and its kernel is the least normal subgroup $N$ of $\pi_1(U, x_0) * \pi(V, x_0)$ that contains all elements represented by words of the form $(i_1(g)^{-1}, i_2(g))$ for $g \in \pi_1(U \cap V, x_0)$.*

*Proof.* First, note that Lemma 4.1 implies that $j$ is surjective.

Next we show $N \subseteq \ker j$. Since the kernel of $j$ is normal, it suffices to show that $i_1(g)^{-1}i_2(g)$ belongs to the kernel for each $g \in \pi_1(U \cap V, x_0)$.
If $i : U \cap V \to X$ is the inclusion mapping, then

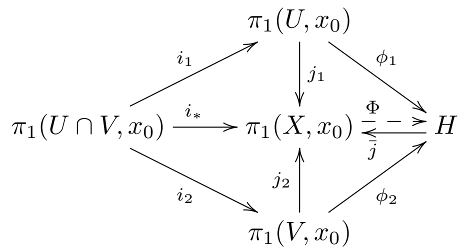
$$ji_1(g) = j_1i_1(g) = i_*(g) = j_2i_2(g) = ji_2(g).$$

This implies that $ji_1(g) = ji_2(g)$ and so $i_1^{-1}(g)i_2(g)$ is mapped to the identity element. Hence, $i_1^{-1}(g)i_2(g)$ belongs to the kernel of $j$. It immediately follows that $j$ induces an epimorphism

$$\bar{j} : \pi_1(U, x_0) * \pi_1(V, x_0)/N \to \pi_1(X, x_0).$$

Now to show that $N = \ker j$, we show $\bar{j}$ is injective.
Denote $H = \pi_1(U, x_0) * \pi_1(V, x_0)/N$ and let $\phi_1 : \pi_1(U, x_0) \to H$ equal the inclusion of $\pi_1(U, x_0)$ into the free product followed by the projection of the free product onto its quotient by $N$. Likewise, let $\phi_2 : \pi_1(V, x_0) \to H$ equal the inclusion of $\pi_1(V, x_0)$ into the free product followed by the projection of the free product onto

its quotient by $N$.
Consider the diagram:




We will show that $\phi_1 \circ i_1 = \phi_2 \circ i_2$. Let $g \in \pi_1(U \cap V, x_0)$, then $\phi_1(i_1(g))$ is the coset $i_1(g)N \in H$. Similarly, $\phi_2(i_2(g))$ is the coset $i_2(g)N \in H$. But, since $i_1(g)^{-1}i_2(g) \in N$, we have that $i_1(g)N = i_2(g)N$.

By Theorem 4.2 there exists a homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$. We will show that $\Phi$ is a left inverse for $\bar{j}$.
It suffices to show that $\Phi \circ \bar{j}$ acts as the identity on any generator of $H$, i.e. on any coset of the form $gN$ where $g \in \pi_1(U, x_0)$ or $g \in \pi_1(V, x_0)$.
If $g \in \pi_1(U, x_0)$, then

$$\bar{j}(gN) = j(g) = j_1(g).$$

So,

$$\Phi(\bar{j}(gN)) = \Phi(j_1(g)) = \phi_1(g) = gN.$$

The proof for $g \in \phi_1(V, x_0)$ is similar. Thus, $\Phi$ is a left inverse for $\bar{j}$ and so $\bar{j}$ is injective. It immediately follows that $\bar{j}$ is bijective. Hence, the kernel of $j$ is $N$ as claimed.

□

Now that we have Seifert-van Kampen's Theorem, we can get an exact formula for the fundamental group of a space $X$ if we know the fundamental groups of a decomposition of $X$ into $U$,$V$, and their intersection $U \cap V$. This theorem often comes up when gluing familiar spaces together along a common and familiar subspace.

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