CELLULAR HOMOLOGY AND THE CELLULAR BOUNDARY FORMULA

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Abstract. This paper will first go through some core concepts and results in homology, then introduce the concepts of CW complex, subcomplex and cellular homology. Thus we will state and provide an informal proof of the cellular boundary formula, which allows us to compute the cellular homology groups. In the last section of the paper, we will use the cellular boundary formula to compute the homology groups of some CW complexes.

Contents

1. Introduction

The homology groups $\{H_n(X)\}_{\{n\in\mathbb{N}\}}$ of a topological space X are introduced in order to understand the properties and the structure of the space X in relation to other spaces. This paper will introduce the concept of CW complexes and cellular homology as a tool to compute the homology of a topological space.

The reader will notice throughout the paper that dealing with CW complexes and the cellular boundary formula dramatically simplifies the computation of homology groups. Indeed, in singular and simplicial homology the number of simplices of a space can be too large to be easily computed. This is the reason behind the introduction of the theory of CW complexes.

This paper assumes the reader to understand the main concepts of simplicial and singular homology. These concepts will be used throughout the paper. The reader is thus expected to be familiar with objects such as commutative diagrams, long and short exact sequences, chains and chain complexes, free abelian groups.

2. Preliminary definitions and results

The following result comes from singular homology. It will be used in all the relevant proofs in this paper:

Proposition 2.1. *If X and Y are two topological spaces, any continuous map $f : X \to Y$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for all $n \in \mathbb{N}$.*

Proof. First we define a map $f_\# : C_n(X) \to C_n(Y)$, defined, for any n-simplex $\sigma : \Delta^n \to X$, as $f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y$. We can then extended this linearly to all of $C_n(Y)$ in the following way:

$$f(\sum_{i=1}^{n} n_i \sigma_i) = \sum_{i=1}^{n} n_1 f(\sigma_i)$$

where $n_i \in \mathbb{N}$.

We claim that $f_\#$ commutes with $\partial_n : C_n(X) \to C_{n-1}(X)$. Hence it takes the elements of $H_n(X)$ (cosets) to elements of $H_n(Y)$. This allows us to define f_* as the restriction of $f_\#$ to the cosets of $H_n(X)$. A complete proof can be found in [1], page 111. □

Corollary 2.2. *If f is a homotopy equivalence, then f_* is an isomorphism.*

This follows from the fact that if g and h are homotopic, then $g_* = h_*$. We omit the proof of this claim because it is not the core focus of this paper. If we take it to be true, and consider an homotopy equivalence $f : X \to Y$, then $\exists g$ such that $f \circ g \cong Id_Y$ and $g \circ f \cong Id_X$. Hence $f_* \circ g_* = Id_Y$ and $g_* \circ f_* = Id_X$, which means that f_* is an isomorphism. A complete proof of the claim can be found in [1], chapter 2, page 112.

We are now able to define the degree of a map between boundaries of $n+1$ dimensional cells.

Definition 2.3. Let $f : S^n \to S^n$ be a continuous map. Then, since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is an homomorphism from an infinite cyclic group to itself, thus it must be of the form $f_*([\sigma]) = m[\sigma]$ for some $m \in \mathbb{Z}$. We call m the *degree* of f, or $\deg(f) = m$.

Some useful facts about the degree of a map are that $\deg(f \circ g) = \deg(f)\deg(g)$ (the proof uses Corollary 2.2), and that if f is a reflection of a S^n about one of the $n+1$ coordinates, then $\deg(f) = -1$.

From this, it follows that if f is the antipodal map, reflecting each point of S^n about the origin, then $\deg(f) = (-1)^{n+1}$, since it is the composition of $n+1$ reflections (one for each axis).

Now we move onto the ideas of relative homology. Consider a topological space X and a subspace $A \subset X$.

We define the groups $C_n(X, A) = C_n(X)/C_n(A)$ for any $n \in \mathbb{N}$. Notice that this group is free abelian. In simple words, $C_n(X, A)$ is the set of relative chains of X modulo A. For each $n \in \mathbb{N}$, we know that since A is a subcomplex of X, $\forall \sigma \in C_n(A)$, $\partial_n(\sigma) \in C_{n-1}(A)$. We can naturally define the homomorphism $\partial_n : C_n(X, A) \to C_n(X, A)$ as the composition of the boundary map on $C_n(X)$ and the quotient map $C_n(X) \to C_n(X, A)$. Similarly to the chain complex of $C_n(X)$, it is easy to check that $\partial_{n-1}(\partial_n(\sigma)) = 0$ for any $\sigma \in C_n(X, A)$.Thus we obtain the following long exact sequence:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \cdots$$

We can then define the **relative n-th homology group** of the pair (X, A) to be $H_n(X, A) = \ker(\partial_n)/\mathrm{Im}(\partial_{n+1})$. One can easily check that for any space X, $H_n(X, \emptyset) = H_n(X)$ for all X.

The following is a really simple example illustrating the concept of "relative" cycle being elements of $H_n(X, A)$. If you consider a square as a CW complex X with four 0-cells v_0, v_1, v_2, v_3, four 1-cells and one 2-cell, and let $A \subset X$ be the segment $[v_0, v_1]$, then elements of $C_0(X, A)$ are of the form $n_0 v_2 + n_1 v_3$ for $n_0, n_1 \in \mathbb{N}$. Element of $C_1(X, A)$ are of the form $n_0[v_0, v_2] + n_2[v_2, v_3] + n_3[v_3, v_0]$, and $C_2(X, A) = C_2(X)$. Notice that $\partial_1([v_1, v_2] + [v_2, v_3] + [v_3, v_0]) = v_0 - v_1$, so $([v_1, v_2] + [v_2, v_3] + [v_3, v_0]) \notin H_1(X)$, but it is an element of $H_1(X, A)$, i.e. it is not a cycle in X but it is a relative cycle of (X, A).

Proposition 2.4. *We can construct the following long exact sequence*

$$\cdots \xrightarrow{\Delta_{n+1}} H_n(A) \xrightarrow{i_n^*} H_n(X) \xrightarrow{q_n^*} H_n(X, A) \xrightarrow{\Delta_n} H_{n-1}(A) \xrightarrow{i_{n-1}^*} \cdots$$

This comes from a general construction in homological algebra. The complete, detailed proof can be found in [1] pages 115-118.

3. CW Complexes

Definition 3.1. An n-dimensional open cell is a topological space that is homeomorphic to an n-dimensional open ball B^n.

This is crucial for the definition of a CW complex

Definition 3.2. A CW complex is a space $X = \bigcup_n X^n$ constructed inductively in the following way:

(1) Let X^0 (the 0-skeleton of X) be a discrete (possibly infinite) set of points, each of which is considered a 0-dimensional open cell.

(2) Given the set X^{n-1}, we construct X^n by attaching every element of a set of n-cells $\{e_i^n\}_{i \in I}$ (where I is some index set) to X^{n-1}. More formally, we have an attaching map for each n cell e_i^n; $\phi_i : \partial B^n = S^{n-1} \to X^{n-1}$, and we define $X^n = (X^{n-1} \sqcup \{e_i^n \mid i \in I\})/\sim$, such that if $x \in X^{n-1}$ and $y \in \partial e_i^n$, $x \sim y$ if $\phi_i(y) = x$. It should be clear from the definition that no two points of X^{n-1} are "collapsed together" in their inclusion in X^n.

It is immediate to see that the concept of CW complex is more general than that of simplicial complex. This is one of the reasons that make cellular homology more appealing than simplicial homology.

Definition 3.3. Given a CW complex X, a subcomplex of X is a subspace $A \subset X$ that is a CW complex itself and its cells are some of the cells of X.

It is important to notice that for any CW complex X and any $n \in \mathbb{N}$, the n-skeleton X^n is a subcomplex of X.

4. The Homology Groups of CW Complexes

We will now define the homology of CW complexes, or cellular homology. To do so, we will use the following results, whose proofs can be found in [1]

Theorem 4.1. *(Excision Theorem) Let X be a topological space, and $B \subset A \subset X$ such that the closure of B is contained in the interior of A. Then $\forall n \in \mathbb{N}$, $H_n(X \setminus B, A \setminus B) \cong H_n(X, A)$.*

A proof can be found in [1], Chapter 2, pages 119-124.

Lemma 4.2. *In a CW complex X, for each $n \in \mathbb{N}$, the map $f_n : \bigsqcup_i B_i \to X^n$ (the union of the boundary maps and the characteristic maps of the B_i) induces an isomorphism $f_* : H_m(\bigsqcup_i B_i, \bigsqcup_i S_i) \to H_m(X^n, X^{n-1})$ for all $m \in \mathbb{N}$. Here $\{B_i\}_{i \in I}$ is a set of n-dimensional balls. Each n-cell e_i^n of X is the image of $\overset{\circ}{B}_i$ under its characteristic map χ_i, and the boundary maps ϕ_i map $\partial B_i = S_i$ to X^{n-1}. f is the union of all the maps χ_i and ϕ_i.*

A proof of this lemma can be found in [2] pages 222-224. The proof uses both the excision theorem and Proposition 2.2. This is crucial to prove the following:

Proposition 4.3. *For a CW complex X, $H_m(X^n, X^{n-1}) = 0$ for $m \neq n$ and is free abelian for $m = n$ with basis in one to one correspondance with the set of n-cells $\{B_i^n\}_{i \in I}$.*

Proof. The proof follows from the preceding lemma. Notice that if $m \neq n$, $H_m(B, S) = 0$ for any n-ball B and its boundary S. Therefore, $H_m(\bigsqcup_i B_i, \bigsqcup_i S_i) = 0$ for $n \neq m$, thus $H_m(X^n, X^{n-1}) = 0$.

If $n = m$, we know from singular homology that $H_m(\bigsqcup_i B_i, \bigsqcup_i S_i)$ is free abelian and is isomorphic to a subgroup of $\mathbb{Z}^{|I|}$, hence so is $H_m(X^n, X^{n-1})$, with exactly one basis element for each n-cell e_i. □

The previous proposition tells us, for example, that if a CW complex has k n-cells (for finite k), $H_n(X^n, X^{n-1})$ will be isomorphic to $\mathbb{Z}^k$.

Now we will mention some consequences of the previous result.

Corollary 4.4. *If X is a CW complex and $m > n$, $H_m(X^n) = 0$.*

This is shown in [1] pages 137-138.

The previous result help us define the differential map from which we can compute the cellular homology groups of a CW complex.

Given a CW complex X, we can apply the maps Δ_n constructed in Proposition 2.4 to send elements of $H_n(X^n, X^{n-1})$ to $H_{n-1}(X^{n-1})$ for any $n \in \mathbb{N}$. We can thus define the quotient map, analogous to the one constructed in the proof of Proposition 2.4: $q_{n_*} : H_n(X^n) \to H_n(X^n, X^{n-1})$. Therefore we have the following chain complex:

$$\cdots \xrightarrow{q_{n+1_*}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\Delta_{n+1}} H_n(X^n) \xrightarrow{q_{n_*}} H_n(X^n, X^{n-1}) \xrightarrow{\Delta_n} H_{n-1}(X^{n-1}) \xrightarrow{q_{n-1_*}} \cdots$$

We can thus define $d_n = (q_{n-1_*} \circ \Delta_n) : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$. Thus we get another chain complex:21

$$\cdots \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-2}} \cdots$$

We call this the **cellular chain complex** of X. We thus obtain the following:

Proposition 4.5. $\ker(d_n)/Im(d_{n+1}) \cong H_n(X)$ *for all* $n \in \mathbb{N}$.

This means that the cellular homology groups of a CW complex coincide with its simplicial or singular homology groups. Thus we have a convenient, computationally easier way of deriving the homology groups of a CW complex. A proof of Proposition 4.5 can be found in [1] pages 139-140.

As we have seen that $H_n(X^n, X^{n-1})$ is a free abelian group having the n-cells of X as basis elements (Proposition 4.3), Proposition 4.5 implies that $H_n(X)$ is free abelian generated by some of the n-cells of X. If X does not have n-cells, then clearly $H_n(X) = 0$.

5. The Cellular Boundary Formula

We now derive a way to compute the map d_n, which, by the previous proposition, simplifies the computation of the homology groups of a CW complex.

Let X be a CW complex. We write $H_n(X^n, X^{n-1}) = \langle \{e_i^n\}_{i\in I}\rangle$ where $\{e_i^n\}_{i\in I}$ is the set of n-cells of X and $H_{n-1}(X^{n-1}, X^{n-2}) = \langle \{e_j^{n-1}\}_{j\in J}\rangle$ where $\{e_j^{n-1}\}_{j\in J}$ is the set of $(n-1)$-cells of X (I and J are two index sets). Recall that every $n-$cell e_i^n is attached to the $n-1$ skeleton by an attaching map $\phi_i : \partial B_i^n = S_i^{n-1} \to X^{n-1}$.

Notice that to understand how d_n behaves on the group $H_n(X^n, X^{n-1})$, it is sufficient to know what $d_n(e_i^n)$ is for each $i \in I$. This leads us to the following result:

Theorem 5.1. *(Cellular Boundary Formula)*

$$d_n(e_i^n) = \sum_{j\in J} d_{ij} e_j^{n-1}$$

where for fixed (i, j), d_{ij} *is the degree of the composition of the three maps* $\phi_i : S_i^{n-1} \to X^{n-1}$ *(the attaching map for* e_i^n*),* $q : X^{n-1} \to X^{n-1}/X^{n-2}$ *(the quotient map), and* $q_j : X^{n-1}/X^{n-2} \to S_j^{n-1}$*, which collapses all the* $n-1$ *cells to the single* $n-1$ *cell* S_j^{n-1}.

Proof. We apply the following lemma, whose proof can be found both in [1] and [2]. The idea is similar to that of Lemma 4.2:

Lemma 5.2. $H_n(X^n, X^{n-1}) \cong H_n(X^n/X^{n-1})$ *for* $n \geq 1$.

For fixed $i \in I$, $j \in J$, and $n \geq 2$, consider the following diagram:

$$\begin{array}{ccccc}
H_n(B_i^n, \partial B_i^n) & \xrightarrow{\Delta_n} & H_{n-1}(\partial B_i^n) & \longrightarrow & H_{n-1}(S_j^{n-1}) \\
\downarrow \chi_{i_*} & & \downarrow \phi_{i_*} & & \uparrow q_{j_*} \\
H_n(X^n, X^{n-1}) & \xrightarrow{\Delta_n} & H_{n-1}(X^{n-1}) & \xrightarrow{q_*} & H_{n-1}(X^{n-1}/X^{n-2}) \\
& \searrow d_n & \downarrow q_{n-1_*} & \nearrow \cong & \\
& & H_{n-1}(X^{n-1}, X^{n-2}) & &
\end{array}$$

Here χ_i is the characteristic map of B_i^n and the top right horizontal arrow is defined so that the top right square commutes. The other maps have been previously defined.

We know that the lower left triangle commutes from the definition of d_n. Commutativity of the top left square is immediate from the fact that χ_i and ϕ_i act identically on the interior of B_i^n and that both horizontal arrows represent the map Δ_n. Showing that the bottom right triangle commutes amounts to showing that the maps q_* and q_{n-1_*} are homotopic. This follows from the fact that the map $q_\# : C_n(X^{n-1}) \to C_n(X^{n-1}/X^{n-2})$, induced by $q : X^{n-1} \to X^{n-1}/X^{n-2}$ (as done in Proposition 2.1), commutes with the map $q_{n-1} : C_n(X^{n-1}) \to C_n(X^{n-1})/C_n(X^{n-2})$. Hence the induced maps q_* and q_{n-1_*} also commute. Thus the whole diagram commutes.

Now first notice that the generator $[B_i^n]$ of $H_n(B_i^n, \partial B_i^n)$ corresponds to a generator e_i^n of $H_n(X^n, X^{n-1})$. Commutativity of the diagram gives us $d_n(e_i^n) = q_{n-1}^*(\phi_{i_*}(\Delta_n([B_i^n])))$, i.e. a linear combination of the $\{e_j^{n-1}\}_{j \in J}$. Notice that the map q_j, collapsing all the $n-1$ cells of X to the $n-1$ cell S_j^{n-1}, induces the map q_{j_*}, which projects linear combinations of $\{e_j^{n-1}\}_{j \in J}$ onto its summand of e_j^{n-1}. Therefore the value of $d_n(e_i^n)$ is going to be the sum of the projections q_{j_*} on the $n-1$ dimensional cells e_j^{n-1}.

In other words,

$$d_n(e_i^n) = \sum_{j \in J} q_{j_*}(q_{n-1_*}(\phi_{i_*}([\partial B_i^n])))$$

Since we defined d_{ij} to be the degree of the map $q_j \circ q \circ \phi_i$, we can conclude that

$$d_n(e_i^n) = \sum_{j \in J} d_{ij} e_j^{n-1}$$

□

The proof above works for all $n \geq 1$, but it can be easily extended to $n = 0$ by introducing the concept of **reduced Homology groups**. These, for a given space X, are obtained by considering the modified long exact sequence:

$$\cdots \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where ϵ sends a linear combiantion of 0-cells to the sum of its coefficients. This clearly does not affect the Homology groups for $n \geq 1$. Using the derived homology group $\tilde{H}_0(X)$, our previous claims are valid for each $n \in \mathbb{N}$.

Even though the proof of the cellular boundary formula appears to be abstract, its application is straight forward. The following section provides various examples through which it will be easier to visualize the maps used in the previous proof.

6. Some examples

6.1. **The n-sphere S^n.** The sphere is one of the simplest examples of CW complex. As we know that $S^n \cong B^n/(\partial B^n)$ or B^n/S^{n-1}, we can see S^n as a CW complex with one n-cell (the interior of B^n) attached to a point, or zero cell. If we write S_m^n to denote the m skeleton of S^n, we thus have that $H_m(S_m^n, S_{m-1}^n) = 0$ for $m \neq n, 0$, while $H_n(S_n^n, S_{n-1}^n) \cong \mathbb{Z}$ and $H_0(S_0^n) \cong \mathbb{Z}$. We thus obtain the chain complex:

$$\cdots \longrightarrow 0 \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} 0 \longrightarrow \cdots \longrightarrow 0 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

Understanding how the differential maps behave is straight forward, since clearly d_m is the zero map for all $m \in \mathbb{Z}$. Thus we have that:

$$H_m(S^n) = 0 \text{ when } m \neq n, 0 \text{ and}$$
$$H_n(S^n) = \ker(d_{n-1})/\operatorname{Im}(d_n) \cong \mathbb{Z}/0 = \mathbb{Z}, \quad H_0(S^n) \cong \mathbb{Z}$$

6.2. **2-Torus.** As a CW complex, the two dimensional torus T can be seen as in picture below, i.e. a two cell t, two one cells a and b, and a single zero cells v_0 (as all four points in the figure are collapsed together). Proposition 4.5 thus tells us that, to the derive the homology groups of we can simply consider the the groups $H_0(T^0) \cong \mathbb{Z}$ (spanned by the zero cell v_0), $H_1(T^1, T^0) \cong \mathbb{Z}^2$ (spanned by the one-cells a, b), $H_2(T^2, T^1) \cong \mathbb{Z}$ (spanned by the two cell t). Finally, we have $H_n(T^n, T^{n-1}) = 0$ for $n \geq 2$, as we do not have higher dimensional cells. Thus we obtain the following sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

Next, we want to understand how d_2 and d_1 behave. For this, we use the cellular boundary formula applied to each basis element.

First we look at $d_2(t)$. We consider the map $S^1 \to S^1$ obtained by composing the attaching map of t, $(\partial t \cong S_1 \to X^{n-1})$ with the collapsing map $T^1 \to S^1 \cong (a/\partial a)$ (and then do the same for b). Observing the picture, we see that this map has degree 0, as it takes ∂t to $a - a$ (due to reversed orientation). Similarly, when applied to b, the map has degree zero. Therefore, the cellular boundary formula yields $d_2(t) = 0 \cdot a + 0 \cdot b = 0$, hence d_2 is the zero map.

Looking at d_1, we quickly realize that $d_1(a) = d_1(b) = 0$, because $T^0 = \{v_0\}$ (contains only one element. Therefore both a and b are cycles. Hence d_1 is the zero map too.

We can now compute the homology groups. We have that

$$H_0(T) = \mathbb{Z}/\operatorname{Im}(d_1) = \mathbb{Z}/\{0\} = \mathbb{Z}$$
$$H_1(T) = \ker(d_1)/\operatorname{Im}(d_2) = \mathbb{Z}^2/\{0\} = \mathbb{Z}^2$$

and

$$H_2(T) = \ker(d_2)/\{0\} = \mathbb{Z}/\{0\} = \mathbb{Z}$$

while clearly $H_n(T) = 0$ for $n \geq 3$.

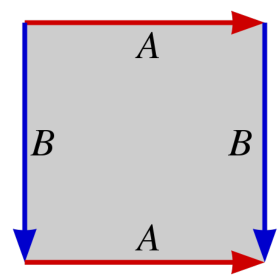


Image from [3]

6.3. **3-Torus.** The 3-dimensional torus T^3 is defined as the cartesian product $S^1 \times S^1 \times S^1$. It cannot be visualized in 3 dimensions, but the picture below portrays it as a CW Complex with one 0-cell v_0 (all the vertices of the cube are collapsed to one), three 1-cells a, b, c, three 2-cells ab, bc and ac (ab is the face whose sides are only a and b, etc.), one 3-cell abc, and no cells of higher dimensions.

Applying Proposition 4.5 we simply need to calculate the homology of the sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

We use the cellular boundary formula to compute $\ker(d_n)$ and $\operatorname{Im}(d_n)$ for $n = 1, 2, 3$. As in the two dimensional case, since there is only one vertex v_0, it follows that d_1 is the zero map, thus $\ker(d_1) \cong \mathbb{Z}^3$ and

$\text{Im}(d_1) = \{0\}$. It is also easy to notice that for each 2-cell, d_2 is the same map as in the 2-dimensional case, thus $d_2 = 0$ on both ab, bc and ac. Thus again $\ker(d_2) \cong \mathbb{Z}^3$ and $\text{Im}(d_2) = \{0\}$.

Moving onto $d_3(abc)$, we look at the map $S^2 \to S^2$ composed by the attaching map of abc and the collapsing of the two skeleton onto ab; we notice that the 2-cell ab appears twice in reversed orientation. The same happens for bc and ac. Therefore, we can conclude that d_3 is also the zero map, so $\ker(d_3) \cong \mathbb{Z}$ and $\text{Im}(d_3) = \{0\}$. We get:

$$H_0(T^3) \cong \mathbb{Z}, \quad H_1(T^3) \cong \mathbb{Z}^3, \quad H_2(T^3) \cong \mathbb{Z}^3, \quad H_3(T^3) \cong \mathbb{Z}$$

while clearly $H_n(T) = 0$ for $n \geq 4$.

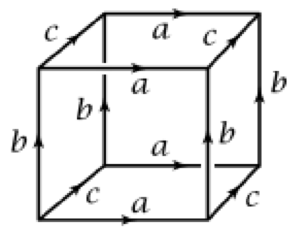


Image from [1], page 142

6.4. **Klein Bottle.** The Klein Bottle K as a CW complex is shown in the picture below. Thus our analysis will be similar our previous ones.

K has 0-cell v_0, two 1-cells a (in red) and b (in blue), and a 2-cell ab. Hence we are considering the same as that of the 2-Torus. However, as we will see, the maps d_n behave differently.

Let us first notice that d_1 is the zero map since there is only one vertex v_0. Thus $\ker(d_1) \cong \mathbb{Z}^2$, and $\text{Im}(d_1) = \{0\}$. Then let us look at d_2. Notice that $q_{a_*}(d_2(ab)) = 0$ as a appears twice in opposite orientations, while $q_{b_*}(d_2(ab)) = 2$ since b appears twice in the same orientation. Therefore, $d_2(ab) = 2b$. It follows that $\ker(d_2) \cong \{0\}$ and $\text{Im}(d_2) \cong \{(0, 2z) \in \mathbb{Z} \mid z \in \mathbb{Z}\} = \{0\} \times 2\mathbb{Z}$. We obtain:

$$H_0(K) \cong \mathbb{Z}, \quad H_1(K) \cong \mathbb{Z}^2/(\{0\} \times 2\mathbb{Z}) = (\mathbb{Z}^2/\{0\}) \times (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2, \quad H_2(K) \cong \{0\}/\{0\} = \{0\}$$

Thus we can notice how the different pattern in the attaching map of the two cell ab leads to different homology groups from those of the 2-torus.

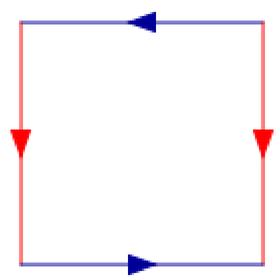


Image from [4]

6.5. **The Real Projective Spaces.** The space $\mathbb{R}P^n$ (for some $n \in \mathbb{N}$) is defined as the space of all lines in $\mathbb{R}^{n+1}$ going through the origin. We claim that as a CW complex, $\mathbb{R}P^n$ can be seen as the union of one k cell for each $k \leq n$. We prove this by induction. First we notice that the statement is trivially true for $n = 0$. Then consider that $\mathbb{R}P^n \cong S^n/\sim$ where $v \sim -v$ for all $v \in S^n$. Note that this is the same as saying that $\mathbb{R}P^n \cong D^n/\sim$ where $v \sim -v$ for all $v \in \partial D^n = S^{n-1}$. In other words, $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$. Thus $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$. Notice that for each k, the k-skeleton of $\mathbb{R}P^n$ is $\mathbb{R}P^k$.

Thus we analyze the following chain complex:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

To compute the boundary map d_k we analyze the projection $\phi_k : \partial e^k \cong S^{k-1} \to \mathbb{R}P^{k-1}$ composed with the quotient map $q : \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1}$ (from the proof of the cellular boundary formula).

Notice that the preimage of $\mathbb{R}P^{k-2}$ under the map ϕ_k is clearly $S^{k-2} \subset S^{k-1}$. Moreover, we have that $S^{k-1} \setminus S^{k-2} = B_1^{k-1} \sqcup B_2^{k-1}$, two $k-1$ open balls. These, under ϕ_k, are mapped homeomorphically to $\mathbb{R}P^{k-1} \setminus \mathbb{R}P^{k-2}$. One easily notices that the image B_1^{k-1} under ϕ_k is the "top" part of $\mathbb{R}P^k$, i.e. ϕ_k restricted to B_1^{k-1} is the identity map (which has degree 1).

On the other hand, ϕ_k restricted to B_2^{k-1} is the antipodal map, as the second open ball is mapped to the "lower part" of $\mathbb{R}P^k$. As we saw in the remarks after Definition 2.3, the degree of the antipodal map of S^{k-1} is $(-1)^k$. Therefore the map $\phi_k \circ q$ can be seen as the sum of the identity and the antipodal map, hence we can conclude that its degree is $1 + (-1)^k$.

Thus we get that $d_k(e^k) = 2e^{k-1}$ if k is even and $d_k(e^k) = 0$ if k is odd. Therefore, if k is even $\ker(d_k) = \{0\}$ and $\text{Im}(d_k) \cong 2\mathbb{Z}$; while when k is odd, $\ker(d_k) \cong \mathbb{Z}$ and $\text{Im}(d_k) = \{0\}$. Thus, when n is even,

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd} \\ \{0\} & \text{if } k \text{ is even} \\ \mathbb{Z} & \text{if } k = 0 \end{cases}$$

when n is odd,

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd, and } , k \neq n \\ \{0\} & \text{if } k \text{ is even} \\ \mathbb{Z} & \text{if } k = n, 0 \end{cases}$$

6.6. **The connected sum $\mathbb{R}P^2 \# \mathbb{R}P^2$.** We define $\mathbb{R}P^2 \# \mathbb{R}P^2$ as $((\mathbb{R}P^2 \setminus U) \sqcup (\mathbb{R}P^2 \setminus U))/\sim$, where U is some open neighborhood and $\sim$ identifies the boundaries of the two copies of U. The picture below shows how this can be represented graphically. c represents the boundary of the open neighborhood U. The four vertex of the square belong to the same equivalence class.

It is clear from the picture that $\mathbb{R}P^2 \# \mathbb{R}P^2$ is a CW complex with one 0-cell v_0, three one cells a, b and c and two 2-cells ac and bc. We thus have the sequence:

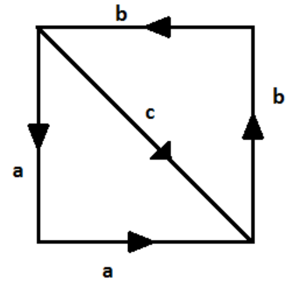
$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

We immediately notice that d_1 is the zero map because we only have one 0-cell. Then we compute $d_2(ab) = 2a - c$ by collapsing the 1-skeleton and noticing that the boundary of ab goes through a twice in the same direction and once through c in the opposite direction. Similarly, we obtain that $d_2(bc) = 2b + c$. Hence the $\ker(d_2) = \{0\}$ and the image of d_2 contains the element of $\mathbb{Z}^3$ of the form $(2n_1, 2n_2, n_2 - n_1)$ for $n_1, n_2 \in \mathbb{Z}$. Thus $\text{Im}(d_2) \cong 2\mathbb{Z} \times \{0\} \times \mathbb{Z}$. Therefore we have that

$$H_0(\mathbb{R}P^2 \# \mathbb{R}P^2) \cong \mathbb{Z}, \quad H_1(\mathbb{R}P^2 \# \mathbb{R}P^2) \cong \mathbb{Z}^3/(2\mathbb{Z} \times \{0\} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2,$$

$$H_2(\mathbb{R}P^2 \# \mathbb{R}P^2) = \{0\}$$

Notice that this is the same result as for the Klein Bottle (in fact, the two spaces are homeomorphic).



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