Minimizing the Uncertainty Principle in the Weyl-Heisenberg and Affine Groups

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Abstract

We begin by defining properties of the Fourier transform leading up to a proof of Heisenberg's uncertainty inequality. Using a generalized form of this inequality, we then explore it's minimizing functions in both the Weyl-Heisenberg and Affine groups. These functions are of interest because they form the basis of the canonical form of the windowed Fourier transform and wavelet transform respectively.

Contents

1 Introduction

Heisenberg's uncertainty principle, one of the most famous results of quantum mechanics, states that one cannot simultaneously know both the position and momentum of a moving particle. However, this seemingly physical result has deeply-rooted underlying mathematical principles based in Fourier analysis. This is known as Heisenberg's inequality. Functions that cause this inequality to become an equality can be found by solving a differential equation arising from the self-adjoint operators of different groups. These functions are of interest because they are closely tied to transforms associated with different groups, for instance they form the basis of the canonical form of the windowed Fourier transform when one examines the Weyl-Heisenberg group. Following the structure Dahlke and Maas [4], in this paper we will derive the minimizing functions in the Weyl-Heisenberg and Affine groups.

2 Heisenberg's Inequality

The Fourier transform is a key tool in mathematics which decomposes a function into its frequency components. It is defined as follows.

Definition 2.1. (The Fourier Transform) For a function $f \in L^2(\mathbb{R})$ the function $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx$ is the *Fourier Transform* of f.

The inverse Fourier transform takes the frequency components of a function and recomposes them into a function of time.

Definition 2.2. (The Inverse Fourier Transform) For a function $f \in L^2(\mathbb{R})$ the *Inverse Fourier Transform* is $f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi$.

Definition 2.3. A function $f \in L^2(\mathbb{R})$ if $\int_{\mathbb{R}} |f(x)|^2dx < \infty$. $L^2(\mathbb{R})$ is a Hilbert space with a well defined inner product $\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$.

For functions $f, g \in L^2(\mathbb{R})$ we have the following results, that are key to proving Heisenberg's inequality.

Theorem 2.4. (Cauchy-Schwarz Inequality) *Let* $f, g \in L^2(\mathbb{R})$*, then* $|\langle f, g\rangle|^2 \leq \|f\|^2\|g\|^2$.

Proof. Let $f, g \in L^2(\mathbb{R})$, and let $h = f - \frac{\langle f,g\rangle}{\langle g,g\rangle}g$, then

$$\langle h, g\rangle = \langle f - \frac{\langle f, g\rangle}{\langle g, g\rangle}g, g\rangle = \langle f, g\rangle - \langle \frac{\langle f, g\rangle}{\langle g, g\rangle}g, g\rangle = \langle f, g\rangle - \langle f, g\rangle = 0,$$

so h, g are orthogonal. Thus

$$\|f\|^2 = \langle x, x\rangle = \langle h+\frac{\langle f, g\rangle}{\langle g, g\rangle}g, h+\frac{\langle f, g\rangle}{\langle g, g\rangle}g\rangle = \|h\|^2+0+|\frac{\langle f, g\rangle}{\langle g, g\rangle}|^2\langle g, g\rangle = \|h\|^2+\frac{|\langle f, g\rangle|^2}{\|g\|^2} \geq \frac{|\langle f, g\rangle|^2}{\|g\|^2}.$$

Therefore $\|f\|^2 \geq \frac{|\langle f,g\rangle|^2}{\|g\|^2}$, so $\|f\|^2\|g\|^2 \geq |\langle f, g\rangle|^2$. □

Theorem 2.5. (Parseval's Identity) *Given* $f \in L^2(\mathbb{R})$*,*

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2d\xi = \|\hat{f}\|^2$$

Proof. For $f \in L^2(\mathbb{R})$,

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2dx = \int_{\mathbb{R}} f(x)\overline{f(x)}dx = \int_{\mathbb{R}} (\int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi)\overline{(\int_{\mathbb{R}} \hat{f}(\xi')e^{2\pi i\xi x}d\xi)}$$

by taking the fourier transform. Rearranging we have

$$\begin{aligned}\int_{\mathbb{R}}(\int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi)\overline{(\int_{\mathbb{R}} \hat{f}(\xi')e^{2\pi i\xi x}d\xi)} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{f}(\xi')}d\xi \int_{\mathbb{R}} e^{2\pi i(\xi-\xi')x}d\xi \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{f}(\xi')}d\xi(\delta(\xi)) \\ &= \int_{\mathbb{R}} \hat{f}(\xi)d\xi \int_{\mathbb{R}} \overline{\hat{f}(\xi)}d\xi \\ &= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|^2.\end{aligned}$$

□

Lemma 2.6. *For a differentiable function $f \in L^2(\mathbb{R})$, $\int_{\mathbb{R}}|f(x)|^2dx = -2\int_{\mathbb{R}} xf(x)f'(x)dx$.*

Proof. This is just an example of integration by parts, since

$$\int_{\mathbb{R}}|f(x)|^2dx = x|f(x)|^2|_{-\infty}^{\infty} - 2\int_{\mathbb{R}} xf(x)f'(x)dx.$$

Since $f \in L^2(\mathbb{R})$ it must evaluate to zero at ∞ and $-\infty$, so

$$\int_{\mathbb{R}}|f(x)|^2dx = -2\int_{\mathbb{R}} xf(x)f'(x)dx.$$

□

Lemma 2.7. *For a differentiable function $f \in L^2(\mathbb{R})$, $\hat{f}'(x) = 2i\pi\xi\hat{f}(x)$.*

Proof. Let $f \in L^2(\mathbb{R})$, $\hat{f}'(x) = \int_{\mathbb{R}} f'(x)e^{-2\pi i\xi x}dx$, once again using integration by parts we have

$$\int_{\mathbb{R}} f'(x)e^{-2\pi i\xi x}dx = f(x)e^{-2\pi i\xi x}|_{\mathbb{R}} + 2i\pi\xi\int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx = 2i\pi\xi\hat{f}(x).$$

□

We are now ready to prove Heisenberg's inequality.

Theorem 2.8. (Heisenberg's Inequality) *If $f \in L^2(\mathbb{R})$, then*

$$16\pi^2(\int_{\mathbb{R}} x^2|f(x)|^2dx \int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2d\xi) \geq \|f\|^4.$$

Proof. Let $f \in L^2(\mathbb{R})$, then

$$\|f\|^4 = \int_{\mathbb{R}}|f(x)|^2\overline{|f(x)|^2}dx = 4\int_{\mathbb{R}}|xf(x)\overline{f'(x)}|^2dx$$

by the previous lemma. Then by Cauchy-Schwartz and Parseval's Identity we have

$$\int_{\mathbb{R}} |xf(x)\overline{f'(x)}|^2 dx \leq 4\int_{\mathbb{R}} |xf(x)|^2 dx \int_{\mathbb{R}} |f'(x)|^2 dx = 4\int_{\mathbb{R}} |xf(x)|^2 dx \int_{\mathbb{R}} |\hat{f}'(\xi)|^2 d\xi.$$

Finally by another previous lemma we have that

$$4\int_{\mathbb{R}} |xf(x)|^2 dx \int_{\mathbb{R}} |\hat{f}'(\xi)|^2 d\xi = 16\pi^2 \int_{\mathbb{R}} |xf(x)|^2 dx \int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2 d\xi.$$

Thus we have that $\|f\|^4 \leq 16\pi^2(\int_{\mathbb{R}} |xf(x)|^2 dx \int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2 d\xi)$.

□

Definition 2.9. For a self-adjoint operator $S, \mu_f(S) = \langle Sf, f\rangle$.

Theorem 2.10. (Generalized Heisenberg Inequality) *Let S and T be densely defined self-adjoint operators with $[S,T] = ST - TS$, then for $f \in \mathcal{D}[S,T]$ and $\|f\| = 1$ and any pair (s,t) of real numbers the following inequality holds:*

$$|\langle [S,T]f, f\rangle|^2 \leq 4\|(S-s)f\|^2 \|(T-t)f\|^2.$$

Additionally equality holds if and only if there exists a $\lambda \in i\mathbb{R}$ such that $(S-s)f = \lambda(T-t)f$ and $s = \mu_f S = \langle Sf, f\rangle$, and $t = \mu_f(T) = \langle Tf, f\rangle$ [4].

Proof. Let S and T be densely defined self-adjoint operators with $[S,T] = ST - TS$ and let $\mathcal{D}$ be the domain of S and T and let $f \in \mathcal{D}[S,T]$ with $\|f\| = 1$ and let (s,t) be a pair of real numbers. Then

$$\begin{aligned} |\langle [S,T]f, f\rangle|^2 = |\langle (ST-TS)f, f\rangle|^2 &= |\langle (STf - TSf), f\rangle|^2 \\ &= |\langle STf, f\rangle - \langle TSf, f\rangle|^2 \\ &= |\langle Tf, Sf\rangle - \langle Sf, Tf\rangle|^2 \\ = |\langle (T-t)f, (S-s)f\rangle - \langle (S-s)f, (T-t)f\rangle|^2 \end{aligned}$$

since T and S are self-adjoint and since $[S,T] = ST - TS$, so

$$[(S-s),(T-t)] = (S-s)(T-t) - (T-t)(S-s) = ST - TS$$

since s,t are just real numbers. Thus

$$\begin{aligned} |\langle (T-t)f, (S-s)f\rangle - \langle (S-s)f, (T-t)f\rangle|^2 &= |\langle (T-t)f, (S-s)f\rangle - \overline{\langle (T-t)f, (S-s)f\rangle}|^2 \\ &= |2\text{im}\langle (S-s)f, (T-t)f\rangle|^2 \end{aligned}$$

where im denotes the imaginary part. Then by Cauchy-Schwarz we have,

$$|2\text{im}\langle (S-s)f, (T-t)f\rangle|^2 \leq 4|\langle (S-s)f, (T-t)f\rangle|^2 \leq 4\|(S-s)f\|^2 \|(T-t)f\|^2.$$

By Cauchy-Schwarz equality holds when $(S-s)f = \lambda(T-t)f$, where $\lambda = \text{im}\frac{\langle Sf,Tf\rangle}{\langle Tf,Tf\rangle}i$, thus λ must be imaginary. Additionally since $\|f\| = 1$, then we have

$$\begin{aligned}\langle Sf, f\rangle - s &= \langle Sf, f\rangle - \langle s, f\rangle \\ &= \langle (S-s)f, f\rangle \\ &= \lambda\langle (T-t)f, f\rangle \\ &= \lambda(\langle Tf, f\rangle - t).\end{aligned}$$

Therefore $\langle Sf, f\rangle - s = \lambda(\langle Tf, f\rangle - t)$. However since λ is imaginary and $\langle Sf, f\rangle, \langle Tf, f\rangle$ must be real, then $\langle Sf, f\rangle - s = \lambda(\langle Tf, f\rangle - t) = 0$, so $\langle Sf, f\rangle = \mu_f(S) = s$ and $\langle Tf, f\rangle = \mu_f(T) = t$.

□

Definition 2.11. The variance of a densely defined self-adjoint operator S is $var(S) = \langle (S-\mu_f(S))^2 f, f\rangle$.

Corollary 2.12. *For $f \in \mathcal{D}[S,T]$ and $\|f\| = 1$, the following inequality holds:*

$$|\mu_f([S,T])|^2 \leq 4var(S)var(T).$$

Proof. By the generalized uncertainty principle we have

$$|\mu_f([S,T])|^2 = |\langle [S,T]f, f\rangle|^2 \leq 4\|(S-s)f\|^2\|(T-t)f\|^2 = 4\|(S-\mu_f(S))f\|^2\|(T-\mu_f(T))f\|^2.$$

Additionally

$$\|(S-\mu_f(S))f\|^2 = \langle S-\mu_f(S), S-\mu_f(S)\rangle = \langle (S-\mu_f(S))^2 f, f\rangle = var(S),$$

so

$$|\mu_f([S,T])|^2 \leq 4\|(S-\mu_f(S))f\|^2\|(T-\mu_f(T))f\|^2 = 4var(S)var(T).$$

□

We will now turn our focus to the behavior of the generalized Heisenberg's inequality in different groups, and the functions that minimize it. These minimizing functions are interesting because they are related to the windowed Fourier transform in the Weyl-Heisenberg group, and the wavelet transform in the Affine group. These transforms arise from the inner product of a function $f \in L^2(\mathbb{R})$ and the group representation. However, first we must provide some important definitions.

3 Minimizing Heisenberg's Inequality in the Weyl-Heisenberg Group

Definition 3.1. A *unitary operator* is a bounded linear operator $U : H \to H$ on a Hilbert space, such that $U^*U = UU^* = I$.

Remark 3.2. It is easy to see that the operators $T_a f(x) = f(x-a), a \in \mathbb{R}$, $E_a f(x) = e^{2\pi iax} f(x), a \in \mathbb{R}$, and $D_a f(x) = |a|^{-\frac{1}{2}} f(x/a), a \in \mathbb{R} \setminus \{0\}$ are all unitary operators on $L^2(\mathbb{R})$ with the inner product defined as $\langle f, g \rangle = \int f(x)\overline{g(x)}dx$.

Lemma 3.3. *The product of two unitary operators is a unitary operator*

Proof. Let U and V be unitary operators. Then $(UV)^*(UV) = V^*U^*UV = V^*IV = V^*V = I$ by the definition above, similarly $(UV)(UV)^* = I$. Therefore by definition the product of two unitary operators is unitary. □

Definition 3.4. The left Haar measure of a group G is the unique (up to a constant) measure μ that is *left-invariant* i.e., for every integrable function f on G and every $y \in G$, $\int_G f(y \cdot x)d\mu(x) = \int_G f(x)d\mu(x)$.

Definition 3.5. Let H be a Hilbert Space. Then a representation π on G on H is a mapping $\pi : G \to L(H)$ such that $\pi(x \cdot y) = \pi(x)\pi(y)$ for all $x, y \in G$.

Definition 3.6. A vector $v \in H$ is *admissible* if $\int |\langle v, \pi(x)v \rangle|^2 d\mu(x) < \infty$, where μ is the left Haar measure on G, it is *cyclic* if the only $f \in H$ such that $\langle f, \pi(x)v \rangle = 0$ for all $x \in G$ is $f = 0$.

Definition 3.7. π is *unitary* if the map $\pi(x) : H \to H$ is unitary for every $x \in G$. π is *irreducible* if every $v \in H \setminus \{0\}$ is cyclic. π is *square-integrable* if it is irreducible and there exists an admissible $v \in H \setminus \{0\}$.

Theorem 3.8. (The Weyl-Heisenberg Group) *Let G be defined as*

$$G = \{(a, b, t) | a, b \in \mathbb{R}, t \in \mathbb{C}, |t| = 1\}$$

with representation

$$U(a, b, t)f(x) = te^{2\pi ib(x-a)}f(x-a),$$

then G is a group with the group operation $(a, b, t) \cdot (a', b', t') = (a + a', b + b', tt'e^{2\pi iba'})$.

Proof. Let G be defined as above with representation $U(a, b, t)f(x) = te^{2\pi ib(x-a)}f(x-a)$, let $(a, b, t), (a', b', t') \in G$, then

$$U(a, b, t)U(a', b', t')f(x) = U(a, b, t)t'e^{2\pi ib'(x-a')}f(x-a') = tt'e^{2\pi ib'(x-a')}e^{2\pi ib(x-a'-a)}f(x-a'-a)$$

by definition. Rearranging the variables we have

$$tt'e^{2\pi iba'}e^{2\pi i(b+b')(x-a'-a)}f(x-a'-a) = U(a + a', b + b', tt'e^{2\pi iba'}).$$

Therefore the group operation must be $(a, b, t)(a', b', t') = (a + a', b + b', tt'e^{2\pi iba'})$.

In order to make sure that this operation actually defines a group we must check that it is associative and has an identity element and inverses. It is associative since

$$\begin{aligned}((a, b, t)(a', b', t'))(a'', b'', t'') &= (a + a', b + b', tt'e^{2\pi iba'})(a'', b'', t'') \\ &= (a + a' + a'', b + b' + b'', tt't''e^{2\pi iba'}e^{2\pi i(b+b')a''}),\end{aligned}$$

and

$$
\begin{aligned}
(a,b,t)((a',b',t')(a'',b'',t'')) &= (a,b,t)(a'+a'',b'+b'',t't''e^{2\pi i b'a''}) \\
&= (a+a'+a'',b+b'+b'',t(t't''e^{2\pi i b'a''})e^{2\pi i(b(a'+a''))} \\
&= (a+a'+a'',b+b'+b'',tt't''e^{2\pi i b a'}e^{2\pi i(b+b')a''}).
\end{aligned}
$$

Therefore $((a,b,t)(a',b',t'))(a'',b'',t'') = (a,b,t)((a',b',t')(a'',b'',t''))$.

Additionally $(0,0,1)$ is the identity element since

$$(0,0,1)(a,b,t) = (a,b,te^{0}) = (a,b,t).$$

Lastly it has an inverse $(-a,-b,t^{-1}e^{2\pi i ab})$ since

$$(a,b,t)(-a,-b,t^{-1}e^{2\pi i ab}) = (a-a,b-b,tt^{-1}e^{2\pi i b(-a)}e^{2\pi i ab}) = (0,0,1).$$

Therefore since the operation $(a,b,t)(a',b',t') = (a+a',b+b',tt'e^{2\pi i ba'})$ is associative with an identity element and inverses, then it is a group operation. □

Lemma 3.9. *The product measure dadbdt is the left Haar measure on G*

Proof. Following the proof by Heil and Walnut [1], let f be integrable on G and let $T = \{t \in C, |t| = 1\}$, then

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} f((a,b,t),(a',b',t'))dadbdt &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} f(a+a',b+b',tt'e^{2\pi i ba'})dadbdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{0}^{1} f(a+a',b+b',e^{2\pi i ba'}e^{2\pi i s}e^{2\pi i x})dadbds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{0}^{1} f(a+a',b+b',e^{2\pi i u})dadbdu \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} f(v,w,t)dvdwdt.
\end{aligned}
$$

Therefore $\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} f((a,b,t),(a',b',t'))dadbdt = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} f(v,w,t)dvdwdt$ where $v = a+a'$, $w = b+b'$, and $t = u$, so by the definition above dadbdt is left-invariant, so it is the left Haar measure on G. □

Theorem 3.10. *Following the proof by Heil and Walnut [1], if $f, g \in L^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} |\langle f, U(a,b,t)g\rangle|^2 dadbdt = \|f\|^2\|g\|^2.$$

Proof. Let $f, g \in L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} |\langle f, U(a,b,t)g\rangle|^2 dadbdt = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T} |\bar{t} f(x)e^{-2\pi i bx}\overline{g(x-a)}dx|^2 dadbdt$$

integrating over $\int_T$ and by Parseval's identity we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{T}|\bar{t}f(x)e^{-2\pi ibx}\overline{g(x-a)}dx|^2 dadbdt = \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x)\overline{\hat{g}(x-a)}dx|^2 dadx = \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x)\overline{g(x-a)}dx|^2 dadx$$

$$= \int_{\mathbb{R}}|f(x)|^2 dx \int_{\mathbb{R}}|g(a)|^2 da = \|f\|_2^2\|g\|_2^2$$

□

Corollary 3.11. *Every $g \in L^2(\mathbb{R})$ is admissable.*

Proof. By the previous lemma it is clear that for all $g \in L^2(\mathbb{R})$, $\int|\langle g, U(x)g\rangle|^2 dadbdt = \|g\|^4 < \infty$, therefore all g are admissable. □

Theorem 3.12. *U is a unitary and square-integrable representation of G on $L^2(\mathbb{R})$.*

Proof. Since $U = te^{2\pi ib(x-a)}f(x-a)$, then U is the product of unitary operators, therefore U is unitary. Additionally if $\langle f, U(a,b,t)g\rangle = 0$, then by the previous lemma $\|f\|^2\|g\|^2 = 0$, so $f = 0$. Thus all g are cyclic. Therefore since there exists an admissible g and all g are cyclic, U is square-integrable. □

Theorem 3.13. (Stone's Theorem) *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a unique self-adjoint operator A such that $U_t = e^{itA}$ [5].*

A proof of Stone's theorem is outside of the scope of this paper. However, from Stone's theorem we can define two self-adjoint operators T_a, T_b. By the proof of the left Haar measure it is clear that we can ignore the toroidal component of the representation. Thus let $U'(a,b) = e^{ibx}f(x-a)$, then we can define

$$(T_a f)(x) = i\frac{\delta}{\delta a}(e^{2\pi ibx}f(x))|_{a=0,b=0} = -if'(x)$$

$$(T_b f)(x) = i\frac{\delta}{\delta b}(e^{2\pi ibx}f(x))|_{a=0,b=0} = -xf(x)$$

Theorem 3.14. (Heisenberg's Inequality for Weyl-Heisenberg Group) *For any function $f \in \mathcal{D}([T_b, T_a])$ with $\|f\| = 1$ the following inequality holds:*

$$|\langle [T_b, T_a]f, f\rangle|^2 \leq 4(\|xf\|^2 - \mu_f(T_b)^2)(\|f'\|^2 - \mu_f(T_a)^2)$$

Proof. Let T_a, T_b be defined as above, then by the generalized Heisenberg uncertainty inequality we have that

$$|\mu_f([T_b, T_a])|^2 = |\langle [T_b, T_a]f, f\rangle|^2 \leq 4\|(T_a - \mu_f(T_a))f\|^2\|(T_b - \mu_f(T_b))f\|^2.$$

Now,

$$\|(T_a - \mu_f(T_a))f\|^2 = \|T_a f\|^2 + |\mu_f(T_a)|^2\|f\|^2 - 2(\mu_f(T_a))\langle T_a f, f\rangle$$
$$= \|T_a f\|^2 + \|f\|^2(|\mu_f(T_a) - \frac{\langle T_a f, f\rangle}{\|f\|^2}|^2 - \frac{|\langle T_a f, f\rangle|^2}{\|f\|^4}).$$

However, since $\mu_f(T_a) = \langle T_a f, f\rangle$ and $\|f\|^2 = 1$, then $|\mu_f(T_a) - \frac{\langle T_a f, f\rangle}{\|f\|^2}|^2 = 0$. Therefore we have,

$$\|(T_a - \mu_f(T_a))f\|^2 = \|T_a f\|^2 - \|f\|^2(\frac{|\langle T_a f, f\rangle|^2}{\|f\|^4}) = \|T_a f\|^2 - |\langle T_a f, f\rangle|^2 = \|f'\|^2 - \mu_f(T_a)^2.$$

Similarly we have that $\|(T_b - \mu_f(T_b))f\|^2 = \|xf\|^2 - \mu_f(T_b)^2$. Thus,

$$|\mu_f([T_b, T_a])|^2 \leq 4\|(T_a - \mu_f(T_a))f\|^2\|(T_b - \mu_f(T_b))f\|^2 = 4(\|xf\|^2 - \mu_f(T_b)^2)(\|f'\|^2 - \mu_f(T_a)^2).$$

□

Corollary 3.15. *Functions of the form $f(x) = Ce^{ibt}e^{-dt^2}$ minimize Heisenberg's inequality for the Weyl-Heisenberg group.*

Proof. By the generalized uncertainty principle we have that equality holds when $(S-s)f = \lambda(T-t)f$ where $\lambda \in i\mathbb{R}$. Substituting in with the self-adjoint operators T_a and T_b for the Weyl-Heisenberg group we have the differential equation $-xy - sy = ic(-i\frac{dy}{dx} - ty) =>$ $y(tic - x - s) = c\frac{dy}{dx}$. Separating the variables and integrating we have $(tic-s)x - \frac{x^2}{2} = \ln(y) + c$. Therefore we have $f(x)$ of the form $f(x) = Ce^{(tic-s)x-\frac{x^2}{2}} = Ce^{ibt}e^{-dt^2}$.

□

Note: For this proof we are assuming that f is differentiable, however it does work in general.

The windowed Fourier transform is defined as

$$G_\varphi f = \int_{\mathbb{R}} f(x)\overline{\varphi(x-a)e^{ibx}} = \langle f, U'(a,b)\varphi\rangle.$$

The canonical choice of φ is $\varphi(x) = \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}$, thus taking $C = \pi^{-\frac{1}{4}}, b = 0$ and $d = \frac{1}{2}$, we have that φ is a function that minimizes the uncertainty inequality for the Weyl-Heisenberg group. Thus the windowed Fourier tranform is the inner product of a function $f \in L^2\mathbb{R}$ and the square-integrable representation of G applied to the minimizing function of the uncertainty relation.

Similarly, the wavelet transform is the inner product of a function $f \in L^2\mathbb{R}$ and the square irreducible representation of $\mathcal{A}$ applied to the minimizing function of the uncertainty relation, where $\mathcal{A}$ is the Affine Group defined below. The square-integrable unitary representation of the Affine Group is $[U(a,b)f](x) = |a|^{-\frac{1}{2}}f(\frac{x-b}{a})$, and the wavelet transform is defined as

$$W_\varphi f = \int_{\mathbb{R}} f(x)\overline{|a|^{-\frac{1}{2}}\varphi(\frac{x-b}{a})} = \langle f, U(a,b)\varphi\rangle.$$

Therefore, similarly to the case of the Weyl-Heisenberg group, we can find the canonical form of φ by computing the minimizing function for the uncertainty principle of the Affine Group based on the self-adjoint operators derived from the representation.

4 Minimizing Heisenberg's Inequality in the Affine Group

Definition 4.1. (The Affine Group) The affine group, $\mathcal{A}$ is a locally compact group with a left Haar measure $dadb$ given by

$$\mathcal{A} = \{(a,b)|(a,b) \in \mathbb{R}^2, a \neq 0\},$$

with the group law

$$(a,b) \circ (a',b') = (aa', ab' + b).$$

Definition 4.2. The unitary representation of the Affine Group in $L^2(\mathbb{R})$ is

$$[U(a,b)f](x) = |a|^{-\frac{1}{2}} f(\frac{x-b}{a}).$$

Lemma 4.3. *Let $f, g \in L^2(\mathbb{R})$. Then*

a. $(f(at))\hat{} = \frac{\hat{f}(\frac{\gamma}{a})}{|a|}$

b. $(g(t-b))\hat{} = \hat{g}(\gamma)e^{-\gamma 2\pi i b}$

Proof. a. Let $f \in L^2(\mathbb{R})$. Then $(f(at))\hat{} = \int_{\mathbb{R}} f(at)e^{-2i\pi\gamma t}dt$. Substituting we have

$$\int_{\mathbb{R}} f(at)e^{-2i\pi\gamma t}dt = \int_{\mathbb{R}} \frac{f(u)}{a} e^{-2i\pi\gamma\frac{u}{a}}du = \frac{\hat{f}(\frac{\gamma}{a})}{|a|}.$$

b. Let $g \in L^2(\mathbb{R})$. Then $(g(t-b))\hat{} = \int_{\mathbb{R}} g(t-b)e^{-2i\pi\gamma t}dt$. Substituting we have

$$\int_{\mathbb{R}} g(u)e^{-2i\pi\gamma(u+b)}du = e^{-2i\pi\gamma b}\int_{\mathbb{R}} g(u)e^{-2i\pi\gamma u}du = \hat{g}(\gamma)e^{-2i\pi\gamma b}.$$

□

Lemma 4.4. *Let $f, g \in L^2(\mathbb{R})$. Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle f, U(a,b)g\rangle|^2 dadb = \int_0^\infty |\hat{f}(\omega)|^2 d\omega \int_0^\infty \frac{|\hat{g}(\xi)|^2}{\xi}d\xi + \int_{-\infty}^0 |\hat{f}(\omega)|^2 d\omega \int_{-\infty}^0 \frac{|\hat{g}(\xi)|^2}{\xi}d\xi$$

Proof. Following the proof by Heil and Walnut [1], let $f, g \in L^2(\mathbb{R})$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle f, U(a,b)g\rangle|^2 dadb = \int_{\mathbb{R}}\int_{\mathbb{R}} |\int_{\mathbb{R}} f(x)\overline{g(\frac{x-b}{a})}|a|^{-\frac{1}{2}}dx|^2 dadb = \int_{\mathbb{R}}\int_{\mathbb{R}} |\int_{\mathbb{R}} f(at)\overline{g(t-b)}|a|^{\frac{1}{2}}dt|^2 dadb.$$

By Parseval's identity we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\int_{\mathbb{R}} f(at)\overline{g(t-b)}|a|^{\frac{1}{2}}dt|^2 dadb = \int_{\mathbb{R}}\int_{\mathbb{R}} |\int_{\mathbb{R}} (f(at))\hat{}\,\overline{(g(t-b))\hat{}}|a|^{\frac{1}{2}}dt|^2 dadb.$$

By the previous lemma, we have $(f(at))\hat{} = \frac{\hat{f}(\frac{\gamma}{a})}{|a|}$ and $(\overline{g(t-b)})\hat{} = \overline{\hat{g}(\gamma)}e^{\gamma 2\pi i b}$. Therefore we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|\int_{\mathbb{R}}(f(at))\hat{}(\overline{g(t-b)})\hat{}|a|^{\frac{1}{2}}dt|^2 dadb = \int_{\mathbb{R}}\int_{\mathbb{R}}|\int_{\mathbb{R}}\hat{f}(\frac{\gamma}{a})\overline{\hat{g}(\gamma)}e^{\gamma 2\pi i b}|a|^{-\frac{1}{2}}d\gamma|^2 dadb$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}|\hat{f}(\frac{\gamma}{a})|^2|\overline{\hat{g}(\gamma)}|^2|a|^{-1}d\gamma da = \int_{\mathbb{R}}|\hat{f}(\omega)|^2\int_{\mathbb{R}}|\overline{\hat{g}(a\omega)}|^2 d\omega da$$

by variable substitution.

$$\int_{\mathbb{R}}|\hat{f}(\omega)|^2\int_{\mathbb{R}}|\overline{\hat{g}(a\omega)}|^2 d\omega da = \int_0^{\infty}|\hat{f}(\omega)|^2 d\omega\int_{\mathbb{R}}|\hat{g}(a\omega)|^2 d\omega da + \int_{-\infty}^{0}|\hat{f}(\omega)|^2 d\omega\int_{\mathbb{R}}|\hat{g}(a\omega)|^2 d\omega da$$

$$= \int_0^{\infty}|\hat{f}(\omega)|^2 d\omega\int_0^{\infty}\frac{|\hat{g}(\xi)|^2}{\xi}d\xi + \int_{-\infty}^{0}|\hat{f}(\omega)|^2 d\omega\int_{-\infty}^{0}\frac{|\hat{g}(\xi)|^2}{\xi}d\xi.$$

□

Theorem 4.5. *Every function f satisfying $\int_{\mathbb{R}}\frac{|\hat{f}(\xi)|^2}{|\xi|}d\xi < \infty$ is admissable.*

Proof. By the previous lemma we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|\langle f, U(a,b)f\rangle|^2 dadb = \int_0^{\infty}|\hat{f}(\omega)|^2 d\omega\int_0^{\infty}\frac{|\hat{f}(\xi)|^2}{\xi}d\xi + \int_{-\infty}^{0}|\hat{f}(\omega)|^2 d\omega\int_{-\infty}^{0}\frac{|\hat{f}(\xi)|^2}{\xi}d\xi,$$

so as long as $\int_{\mathbb{R}}\frac{|\hat{f}(\xi)|^2}{|\xi|}d\xi < \infty$, $\int_{\mathbb{R}}\int_{\mathbb{R}}|\langle f, U(a,b)f\rangle|^2 dadb < \infty$, so f is admissable. □

We can define the self-adjoint operators for the Affine Group, the same way as we did for the Weyl-Heisenberg group. We have $[U(a,b)f](x) = |a|^{-\frac{1}{2}}f(\frac{x-b}{a})$ is a unitary representation, so

$$(T_a f)(x) = i\frac{\delta}{\delta a}(|a|^{-\frac{1}{2}}f(\frac{x-b}{a}))|_{a=1,b=0} = -i\frac{1}{2}f(x) - ixf'(x)$$
$$(T_b f)(x) = i\frac{\delta}{\delta b}(|a|^{-\frac{1}{2}}f(\frac{x-b}{a}))|_{a=1,b=0} = -if'(x)$$

Theorem 4.6. (The Affine Uncertainty Principle) *Let T_a and T_b be self-adjoint operators, then for all $f \in \mathcal{D}([T_a, T_b])$ with $\|f\|^2 = 1$ the following inequality holds*

$$\mu_f([T_a,T_b])^2 = |\mu_f(T_b)|^2 \leq 4(\|f'\|^2 - \mu_f(T_b)^2)(\|xf'\|^2 - \frac{1}{2}\|f\|^2 - \mu_f(T_a)^2).$$

Proof. Let T_a and T_b be self-adjoint operators, and let $f \in \mathbb{R}$ with $\|f\| = 1$, then

$$|\mu_f([T_a,T_b])|^2 = |\langle [T_a,T_b]f, f\rangle|^2$$
$$= |\langle (T_a(-if'))(x) - (T_b(-\frac{1}{2}f - xf'))(x), f\rangle|^2$$
$$= |\langle (\frac{1}{2}f' - xf'')(x) - (-\frac{1}{2}f' - xf'')(x), f\rangle|^2$$
$$= |\langle f', f\rangle|^2 = |\langle i(T_b f)(x), f\rangle|^2 = |\mu_f(T_b)|^2.$$

By the same proof as for the Weyl-Heisenberg group we have that

$$\begin{aligned}|\mu_f([T_a,T_b])|^2 = |\langle [T_a,T_b]f,f\rangle|^2 &\le 4\|(T_a-\mu_f(T_a))f\|^2\|(T_b-\mu_f(T_b))f\|^2\\ &= 4(\|T_af\|^2-|\langle T_af,f\rangle|^2)(\|T_b\|^2-|\langle T_bf,f\rangle|^2)\\ &= 4(\|xf'\|^2-\frac{1}{2}\|f\|^2-\mu_f(T_a)^2)(\|f'\|^2-\mu_f(T_b)^2).\end{aligned}$$

Therefore we have that

$$|\mu_f([T_a,T_b])|^2 = |\mu_f(T_b)|^2 \le 4(\|f'\|^2-\mu_f(T_b)^2)(\|xf'\|^2-\frac{1}{2}\|f\|^2-\mu_f(T_a)^2).$$

□

Theorem 4.7. *For a given set of numbers $s,t\in\mathbb{R}$, $\lambda\in i\mathbb{R}$, then $f(x)=c(x-\lambda)^\alpha$ with $\alpha=-\frac{1}{2}-i\lambda t+is$ is a minimizing function of the affine uncertainty principle.*

Proof. Similarly to the Weyl-Heisenberg group, by the generalized uncertainty principle we have that equality holds when $(S-s)f=\lambda(T-t)f$ where $\lambda\in i\mathbb{R}$. Substituting in with the self-adjoint operators T_a and T_b for the Affine group we have the differential equation $-\frac{1}{2}iy-ix\frac{dy}{dx}=\lambda(-i\frac{dy}{dx}-ty) => \frac{idx}{\lambda-x}=\frac{dy}{y(t\lambda-s-\frac{i}{2})}$. Separating the variables and integrating we have $f(x)$ of the form $f(x)=c(x-\lambda)^\alpha$, where $\alpha=-\frac{1}{2}-it\lambda+is$. Thus we have that the function φ in the wavelet transform is of the form $\varphi=c(x-\lambda)^\alpha$.

□

Note: For this proof we are assuming that f is differentiable, however it does work in general.

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