COMBINATORIAL GAMES AND SURREAL NUMBERS

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Abstract. We begin by introducing the fundamental concepts behind combinatorial game theory, followed by developing operations and properties of games. We then develop an equivalence relation on games and the collection of equivalence classes will then define the class of surreal numbers. We conclude by proving the surreal numbers to be a totally ordered abelian group under addition.

Contents

1. Introduction

This work aims to be an expository paper that builds the fundamentals of combinatorial game theory and their connection to surreal numbers from scratch with a direct and brief approach accompanied with precise notation and terminology by combining inspirations and certain constructive lemmas of various works on combinatorial game theory into a new and direct approach. We begin with the fundamentals of games and proceed to develop numeric games with an equivalence relation. The equivalence classes of numeric games then form the surreal numbers. Some of the statements and their proofs have been inspired by or taken from the referenced works, but many were of original construction.

Combinatorial game theory separates itself from economic game theory through the use of ordered pairs of sets rather than matrices. In addition, it restricts itself to two perfectly informed players taking discrete, alternating turns without chance. Such a game may be difficult to imagine, but a good example is the board game Go. A less traditional but more readily calculable game would be Hackenbush. Luckily, no knowledge of these games is required to understand the theory.

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2. The Fundamentals of Games

Definition 2.1. Let Ω be the collection of all *games*. Then the ordered pair $\{G^L|G^R\} \in \Omega$ if and only if $G^L \subseteq \Omega$ and $G^R \subseteq \Omega$. Thus a *game* $G = \{G^L|G^R\}$ is an ordered pair of sets whose elements are also games.

This recursive definition can often be confusing at a first glance. To clarify this definition, it is helpful to consider the empty set. A set with no elements is also a set whose elements are games. That is, $\emptyset \subset \Omega$. Thus, the empty set may act as a "base" set to construct these games.

Example 2.2. : $\{\emptyset|\emptyset\}$ is an ordered pair of the empty set with itself and thus is a game. Notice the bar, which for our purposes is the equivalent of a Cartesian ordered pair's comma. (For convention, we stay with the bar). This game is more comfortably written as $\{|\}$.

Warning 2.3. : $\{\{\emptyset\}|\{\emptyset\}\}$ is not a game because the empty set itself is not a game and thus the set of the empty set is not a set whose elements are games.

Example 2.4. : More games:

$$\{\{|\}|\} \qquad \{|\{|\}\} \qquad \{\{|\}|\{|\}\}$$

As can be seen, there are infinitely many games possible by simply combining two games already made. From here on, we will consider the general case and let G denote any game. We restrict the game to two players who alternate taking turns. These players can be called Left and Right, White and Black, Adam and Bertha, 1 and 2, etc.

Definition 2.5. : The *left options* of G are the elements of the left set, denoted $G^L_i \in G^L$, and the *right options* of G are the elements of the right set, denoted $G^R_j \in G^R$.

Remark 2.6. : All left and right options of G are games by definition 2.1.

It may be helpful to think of options as possible moves to make on a given turn, with each turn beginning a new game. In the game of chess, the first move is done by white, so we represent the first turn as a game in which the Left options are White's possible moves and each of Left's options would be a game in which Black's options are listed for the next turn. (We could also have Left play Black and Right play White if we so desired).

Definition 2.7. *(Normal Play Convention)* A game *ends* when the player whose turn it is has no options. That player *loses*. If a game ends and a player has not lost, that player *wins*.

Remark 2.8. a player can still play with no options as long as it is not their turn when they have no options).

Definition 2.9. *(The Descending Game Condition)*: There does not exist an infinite sequence of games $G_1, G_2, ...$ s.t. $\forall G_i$, G_{i+1} is a left or right option of G_i.

Remark 2.10. This can be intuitively seen as "all games eventually end" or "there are no infinite loops of games."

Definition 2.11. A game G' is a *position* of G if there exists a sequence $G_1, ...G', ..., G_n$ s.t. $\forall G_i$, G_{i+1} is a left or right option of G_i.

Remark 2.12. A game can have infinitely many *positions* and this will not imply an infinite *sequence* of games. Similarly, a player can have infinitely many options and that will not imply an infinite sequence of games either.

Theorem 2.13. *(Conway's Multiple Induction Principle). Assuming the axiom of choice, let $P(G_1, ..., G_n)$ be a property of any n-tuple of games. Suppose this property holds if $\forall G_i \in \{G_1, ..., G_n\}$, the left and right options of G_i also satisfy this property. Then $P(G_1, ..., G_n)$ holds for every n-tuple of games.*

Proof. Notice that any such property is vacuously true for $\{|\}$ because it is the ordered pair of the empty set with the empty set. Now suppose there is such a property P that is false for an n-tuple of games, $G_1, ..., G_n$. Then, by definition of P, $\exists G_i \in \{G_1, ..., G_n\}$ that does not satisfy P. Then once again by the definition of P, $\exists G_i'$, an option of G_i. s.t. $P(G_1, ..., G_i', ..., G_n)$ is false. Thus we repeat the argument again. Since the property is always true for the $\{|\}$, this violates the descending game condition as it implies an infinite sequence of games.

□

3. Relations of Games

Having introduced some basic games and their structure, it is now time to analyze games with respect to the most common interest: deciding who will win and how.

Every game has Left and Right as players, but who starts may vary. Either Left or Right may play first, but the outcome of the game may change dramatically based on who starts. In the $\{|\}$ game, the second player will win because all they have to do is wait for the first player to make a move, which is impossible because neither player has any options. In the $\{\{|\}|\}$ game, Left will win regardless of whether they are first or second because if Left plays second they win by default and if they play first they choose $\{|\}$ and pass the turn to Right, who loses due to lack of options. Similarly, the $\{|\{|\}\}$ game has Right always winning. In the game of $\{\{|\}|\{|\}\}$, the first player always wins.

Definition 3.1. We establish the following relations:

$$G > 0 \text{ if there is a winning strategy for Left}$$

$$G < 0 \text{ if there is a winning strategy for Right}$$

$$G \sim 0 \text{ if there is a winning strategy for the second player}$$

$$G \mid\mid 0 \text{ (G is fuzzy to zero) if there is a winning strategy for the first player.}$$

We can combine notation to say:

a) $G \gtrsim 0$: if Right starts, there is a winning strategy for Left.
b) $G \lesssim 0$: if Left starts, there is a winning strategy for Right
c) $G \vartriangleright 0$: if Left starts, there is a winning strategy for Left.
d) $G \vartriangleleft 0$: if Right starts, there is a winning strategy for Right.

Remark 3.2. Notice also that $G \gtrsim 0$ and $G \lesssim 0$ does indeed imply $G \sim 0$.

A quick commentary on notation. Many of the referenced works use a symbol other than $\sim$. Conway uses $=$, which can lead to confusion as it does not always imply direct equality. Bajnok uses $\approx$, which also works as an equivalence relation. $\equiv$ would also be acceptable.

Theorem 3.3. *(The Fundamental Theorem of Combinatorial Game Theory) Any game G can be classified by at least one of the above four relations.*

Proof. Notice that this statement is equivalent to saying all games satisfy the following two conditions:

1) $G \lesssim 0$ or $G \vartriangleright 0$.
2) $G \gtrsim 0$ or $G \vartriangleleft 0$.

We proceed by Conway's Multiple Induction Principle. That is, if we prove any n-tuple of games satisfies these two conditions if their left and right options satisfy these conditions, then all games satisfy these two conditions. Choosing $n = 1$ for our n-tuple, we consider a single arbitrary game and assume that its left and right options satisfy the above conditions. (Once again, there is no need for a base case because the zero game's options satisfy any property vacuously.)

Now the proof splits into two cases depending which player's turn it is. If Left's turn, left either has an option $G^L \gtrsim 0$, which if they pick leads to victory and implies $G \vartriangleright 0$, or has every $G^L \vartriangleleft 0$, which gives right the power to force a win and implies $G \lesssim 0$. Thus, condition 1 is satisfied. If Right's turn, they either have a winning option ($G^R \lesssim 0$), implying $G \vartriangleleft 0$, or they only have options that, when picked, lead to Left winning and implying $G \gtrsim 0$. Thus condition 2 is satisfied. Since this is true for any game whose options satisfy those conditions, Conway's Induction Principle gives us that all games satisfy this property.

□

4. Adding Games

As the reader probably has guessed, the relations $\gtrsim$ and $\lesssim$ imply that games have some sort of ordering. In fact, the collection of games is partially ordered. The games that prevent it from being totally ordered are the "fuzzy" games, $G \mid\mid 0$. However, games need not be totally ordered for us to add them.

Addition can be thought of as two players playing two separate games with a legal move for a player existing as making one move in one of the games of that player's choice. Formally, we have the following recursive definition:

Definition 4.1. *(Addition)* Let $G = \{G^L | G^R\}$ and $H = \{H^L | H^R\}$ be any two games. Then

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$$

where

$$\{G^L + H, G + H^L\} = \{G^L_i + H : G^L_i \in G^L\} \cup \{G + H^L_m : H^L_m \in H^L\}$$

Note:

$$\{G^L + H, G + H^L\} \neq \{\{G^L_i + H : G^L_i \in G^L\}, \{G + H^L_i : H^L_i \in H^L\}\}.$$

This definition holds for the right side analogously.

Remark 4.2. Note $G + H$ is also a game, and thus the collection of all games, Ω, is closed under addition.

The recursive nature of the definition of addition may be puzzling. This definition has games adding games defined as the addition of other games. This may seem odd as it is intuitive to want to compute such addition, but it is not necessary to simplify the addition at all. For now, instead of computing $G + H$ as we do with numbers, we simply understand $G + H$ as G being played alongside H and their sum representing another game.

Definition 4.3. *(Negation and Subtraction)*
a) for any game $G = \{G^L|G^R\}$, $-G = \{-G^R| - G^L\}$.
b) for any games, G, H, $G - H = G + (-H)$.

Some interesting and playful properties arise immediately from experimentation with this addition:

Recalling Definition 3.1, we can expand even further to relate games to *each other*.

Definition 4.4. Let G and H be any two games. Then:
a) $G > H$ If $G - H$ has a winning strategy for Left ($G - H > 0$).
b) $G < H$ If $G - H$ has a winning strategy for Right ($G - H < 0$)
c) $G \sim H$ If $G - H$ has a winning strategy for the second player ($G - H \sim 0$).
d) $G \mid\mid H$ if $G - H$ has a winning strategy for the first player ($G - H \mid\mid 0$).

Corollary 4.5. *For any two games, G, H, at least one of the above relations holds.*

Proof. This follows directly from the Fundamental Theorem of Combinatorial Game Theory because for any two games, $G - H$ is also a game and therefore falls into one of the four categories with respect to 0.

□

5. Games as an Abelian Monoid under Addition

Proposition 5.1. *(Additive Identity of Games and the Pseudo-additive Inverse of games)*
a) $G + \{|\} = G$.
b) $G + (-G) \sim 0$. *($G \sim G$).*

Proof. Part *a*) is quite easy to show using Conway's induction principle. Suppose a game $G = \{G^L|G^R\}$ in which for every left option G_i^L, $G_i^L + \{|\} \sim G_i^L$ and for every right option, G_j^R, $G_j^R + \{|\} \sim G_j^R$. Then:

$$G + \{|\} = \{(G^L + \{|\}), \emptyset|(G^R + \{|\}), \emptyset\}$$

. This is the case because there is no $H_i^L \in H^L$ when $H^L = \emptyset$. Remembering that $\{(G^L + \{|\}), \emptyset|(G^R + \{|\}), \emptyset\} = \{(G^L + \{|\}) \cup \emptyset|(G^R + \{|\}) \cup \emptyset\}$, we are left with $\{G^L + \{|\}|G^R + \{|\}\}$. Based on our starting assumptions, this gives us $\{G^L|G^R\}$. Thus by Conway's Induction Principle, *a*) is true for every G.

Part *b*) requires more computation, but can also be done.

$$G - G = G + (-G) = \{G^L - G, G - G^R|G^R - G, G - G^L\}.$$

If we prove in this game that the second player always wins, then by definition we have similarity to 0. If each of these four options are victories for the first player, then the game holding those options is a victory for the second player. So, Assume a game G in which $G^L - G^L \sim 0$ and $G^R - G^R \sim 0$. Computing the sum one step further, we find

$$G^L - G = \{G^{L^L} - G, G^L - G^R|G^R - G, G^L - G^L\}$$

and

$$G - G^R = \{G^L - G^R, G - G^{R^R}|G^R - G^R, G - G^{R^L}\}.$$

If left goes first, then $G^L - G$ and $G - G^R$ both have a zero game for Right in the form of $G^L - G^L$ and $G^R - G^R$. Similarly, if Right goes first, we have the negatives of those games, which have the same zero games for Left. Thus the second player always wins. so $G - G \sim 0$. By Conway's Induction Principle, this is true for any G.

□

Remark 5.2. It is pivotal that Proposition 5.1(b) not be misunderstood to say for any game there is an additive inverse, for this is not true. For any G, $G - G \sim 0$, but this DOES NOT mean $G - G = \{|\}$, only that $G - G \sim \{|\}$. In fact, If $G^L \neq \emptyset$, then $(G-G)^L \neq \emptyset$ and if $G^R \neq \emptyset$, then $(G-G)^R \neq \emptyset$. This can be seen intuitively from our definition of addition.

Example 5.3. Using Conway's "real-world" analogy for the definition of game addition as multiple games being played side by side by two players, we also have an informal but intuitive example proof to help realize the concept of playing games side-by-side.

"Intuitive" Proof: $\{|\}$ is the additive identity of games because playing $\{|\}$ alongside any other game does not change the options available to the players. Any player has no choice but to make moves in the game next to $\{|\}$. Furthermore, every game $G = \{G^L|G^R\}$ has a game, $-G = \{-G^R| - G^L\}$ that creates a game $G - G$ similar to 0. This can be shown by once again seeing the game $G - G$ as two players playing two games side by side, one where Left plays first and the other where Left plays second. Whenever Left makes a move on a board, Right copies that move with the same color on the other board. Thus Left will keep making moves and Right will keep copying on the other board and the game continues until Left runs out of moves on the board they play first. Thus the second player (Right in this example) always wins. By our previous definition, $G - G \sim 0$.

Theorem 5.4. *The collection of all games Ω is an abelian monoid under addition.*

Proof. The three requirements for a monoid are closure under the operation, associativity, and an identity element. It is abelian if it also satisfies commutativity. Closure is true by our definition of addition. An identity element exists by proposition 2.16. Therefore we must only prove associativity and commutativity. Let $G = \{G^L|G^R\}, H = \{H^L|H^R\}, J = \{J^L|J^R\}$ be any arbitrary games. First we remember to employ Conway's Induction Theorem with $n = 3$ for our n-tuple of games and assume the following:

$$G^{L/R} + H = H + G^{L/R} \quad \text{(commutativity of the options)}$$
$$(G^{L/R} + H) + J = G^{L/R} + (H + J) \quad \text{(associativity of the options)}$$

(L/R means the statements are assumed for both left and right options.) In order for our induction to work, it is also necessary to assume this property analogously for the options of H and J. Hence $n = 3$ for our n-tuple of games in this proof.

a) $G + H = H + G$**.** (commutativity).

$$G + H = \{G^L + H, G + H^L|G^R + H, G + H^R\}.$$

and

$$H + G = \{H^L + G, H + G^L|H^R + G, H + G^R\}.$$

By our inductive assumption, $G^{L/R} + H = H + G^{L/R}$, and $H^{L/R} + G = G + H^{L/R}$. Therefore $G + H = H + G$.

b) $(G+H)+J = G+(H+J)$**.** (associativity).

$(G+H)+J = \{G^L+H, G+H^L|G^R+H, G+H^R\}+J =$
$\{(G^L+H)+J, (G+H^L)+J, (G+H)+J^L|(G^R+H)+J, (G+H^R)+J, (G+H)+J^R\}$.

Now taking a look at $G+(H+J)$, we have

$$G+\{H^L+J, H+J^L|H^R+J, H+J^R\} =$$

$\{G^L+(H+J), G+(H^L+J), G+(J+H^L)|G^R+(H+J), G+(H^R+J), G+(J+H^R)\}$.

By our inductive assumption of the associativity of the options, these two games are equal. Therefore by Conway Induction, $(G+H)+J = G+(H+J)$. Ω forms an abelian monoid under addition.

□

One might wonder why games are not quite an abelian group but rather a monoid. Recall Proposition 5.1 and Remark 5.2 discussing the "pseudo-additive inverse" of games. Adding a game and its corresponding negative results in a game similar to $\{|\}$, but not equal to $\{|\}$. Thus it is only a monoid. The next section investigates how to craft a Group from the equivalence classes of numerical games.

6. Numerical Games and Equivalence Classes

Lemma 6.1. *For any games, G, H,*
1) If $G \gtrsim 0$ and $H \gtrsim 0$, then $G+H \gtrsim 0$
2) If $G \gtrsim 0$ and $H \rhd 0$, then $G+H \rhd 0$.

Proof. 1) For Conway Induction, assume G, H to be games such that $G \gtrsim 0$, $H \gtrsim 0$, and that *both parts* of the lemma holds for their options. $G \gtrsim 0$ implies no $G_j^R \lesssim 0$ and $H \gtrsim 0$ implies no $G_i^L \lesssim 0$. By the fundamental theorem of combinatorial game theory, $\forall G_j^R \in G^R$, $G_j^R \rhd 0$.

Similarly, $\forall H_n^R \in H^R$, $H_n^R \rhd 0$.

Therefore by our inductive hypothesis, any right option in

$$G+H = \{G+H^L, G^L+H|G+H^R, G^R+H\}$$

will be $\rhd 0$. Therefore $G+H \gtrsim 0$.

2) Similarly, assume $G \gtrsim 0$ and $H \rhd 0$. Then $\exists H_m^L \in H^L$ such that $H_m^L \gtrsim 0$. By our inductive hypothesis, $G+H_m^L \gtrsim 0$. Since $G+H$ has $G+H_m^L$ as a left option, $G+H \rhd 0$.

□

Remark 6.2. This holds analogously for $G \lesssim 0$ and $H \lesssim 0$ as well as $H \lhd 0$.

Remark 6.3. We use separate indices i, j, m, n to remind the reader that there is no necessary correspondence between the different sets of options within any game or between any two games in these statements.

Lemma 6.4. *Let G be any game and let G_0 be any game such that $G_0 \sim 0$. Then $G+G_0 \sim G$.*

Proof. This is equivalent to proving $G - G + G_0 \sim 0$. Since $G - G \sim 0$, $G - G \gtrsim 0$ and $G - G \lesssim 0$. Similarly $G_0 \gtrsim 0$ and $G_0 \lesssim 0$. Thus, $G - G + G_0 \gtrsim 0$ and $G - G + G_0 \lesssim 0$. Therefore $G - G + G_0 \sim 0$. By commutativity, $G + G_0 - G \sim 0$. Therefore, $G + G_0 \sim G$. □

Proposition 6.5. $\sim$ *is an equivalence relation on games.*

Proof. The three requirements for an equivalence relation are for the relation to be reflexive, transitive, and symmetric

a) $G \sim G$ (reflexive)

Since $G - G \sim 0$, $G \sim G$.

b) If $G \sim H$, $H \sim G$ (symmetric)

Suppose $G \sim H$, then $G - H \sim 0$. Since $\forall G, G - G \sim 0$, $G - H - (G - H) \sim 0$. Since $G - H \sim 0$, $-(G - H) = H - G \sim 0$. Thus, $H \sim G$.

c) If $G \sim H$ and $H \sim J$, then $G \sim J$. (Transitive)

Suppose $G \sim H$ and $H \sim J$. Then $G - H \sim 0$ and $H - J \sim 0$. By Lemma 6.1, $G - H + H - J \sim 0$. Since $-H + H \sim 0$, $G - J \sim 0$. □

$||$ is not an equivalence relation because G is never fuzzy to itself. Therefore we construct a subclass of Ω in which every game is in an equivalence class.

Definition 6.6. We say Γ is the class of all *numeric games*. A *numeric game* is a game G such that $\forall G_i^L$ and $\forall G_j^R$, $G_i^L < G_j^L$. To distinguish between generalized games and numeric games, we use X, Y, Z instead of G, H, J.

Lemma 6.7. Γ *is closed under addition.*

Proof. Consider any two numeric games $X = \{X^L|X^R\}$ and $Y = \{Y^L|Y^R\}$. By definition of addition, $X+Y = \{X+Y^L, X^L+Y|X+Y^R, X^R+Y\}$. By definition of numeric game, for any left option Y_m^L, $Y_m^L < Y$ as well as $X_i^L < X$.

By Lemma 6.1, $X+Y_m^L < X+Y$. Similarly, for any option X_i^L, $X_i^L+Y < X+Y$. Once more by definition of numeric game, for any right option Y_n^R, $Y < Y_n^R$ and for any right option X_j^R, $X < X_j^R$. Therefore by Lemma 6.1, $X + Y < X + Y_n^R$ and $X + Y < X_j^R + Y$. By definition, $X + Y$ is a numeric game. □

Proposition 6.8. *Let* $X = \{X^L|X^R\}$ *and* $Y = \{Y^L|Y^R\}$ *be numeric games such that* $X < Y$. *Then* $-Y < -X$.

Proof. By definition, $X - Y < 0$. Recall $-X = \{-X^R| - X^L\}$. For Conway Induction we assume $-(-X_j^R) = X_j^R$ and $-(-X_i^L) = X_i^L$, which yields $-(-X) = X$. Therefore $-Y - (-X) = -Y + X$. Since games are commutative, $-Y + X = X - Y < 0$. Therefore $-Y - (-X) < 0$, so $-Y < -X$. □

Corollary 6.9. *Let* $X = \{X^L|X^R\}$ *be a numeric game. Then* $-X = \{-X^R|-X^L\}$ *is a numeric game.*

Proof. By definition of numeric game, for any left and right option, $X_i^L < X$ and $X < X_j^R$. By Proposition 6.8, $-X_j^R < -X$ and $-X < -X_i^L$. Therefore $-X$ is a numeric game by Definition 6.5. □

Theorem 6.10. *$\lesssim$ is an ordering on Γ and addition respects the ordering.*

That is, $\lesssim$ satisfies the following three properties:

a) (Transitivity) For any numeric games X, Y, Z if $X \lesssim Y$ and $Y \lesssim Z$, then $X \lesssim Z$.

b) (Trichotomy) For any two numeric games $X = \{X^L|X^R\}, Y = \{Y^L|Y^R\}$, exactly one of the following holds: $X < Y$, $X > Y$ or $X \sim Y$.

c) (Addition respects the ordering) If $X \lesssim Y$, then $X + Z \lesssim Y + Z$ where $X, Y, Z \in \Gamma$.

Proof. a) Suppose $X, Y, Z \in \Gamma$ such that $X \lesssim Y$ and $Y \lesssim Z$. Then, by definition, $X - Y \lesssim 0$ and $Y - Z \lesssim 0$. By Lemma 6.3, $X - Y + Y - Z \sim X - Z \lesssim 0$. Therefore, $X \lesssim Z$.

b) It suffices to prove that for any numeric games X, Y, $X - Y \lesssim 0$ or $X - Y \gtrsim 0$. By definition of addition, $X - Y = X + (-Y) = \{X^L - Y, X - Y^R | X^R - Y, X - Y^L\}$. By Lemma 6.7, $X - Y$ is a numeric game. Therefore by transitivity, every left option of $X - Y$ is less than every right option $X - Y$.

By Definition 4.5 this implies that the first player does not necessarily have a winning strategy because there cannot be both a left and right option similar to zero and there cannot be both a left option greater than zero and a right option less than zero. By Definition 3.1, $X - Y$ is not fuzzy to 0. By the fundamental theorem of combinatorial game theory, $X - Y \lesssim 0$ or $X - Y \gtrsim 0$.

c) This follows precisely from Lemma 6.1. Suppose $X, Y, Z \in \Gamma$ such that $X \lesssim Y$. Then by definition, $X - Y \lesssim 0$. By Proposition 5.1, for any numeric game Z, $Z - Z \sim 0$. By Lemma 6.1, $X - Y + Z - Z \lesssim X - Y \sim 0$. Since games are associative, $X - Y + Z - Z = X + Y - Y - Z$. Therefore, $X + Z - Y - Z \lesssim 0$. By definition, $X + Z \lesssim Y + Z$.

□

Corollary 6.11. *Γ is a totally ordered abelian monoid under addition.*

Proof. Notice that $\{|\}$ is a numeric game. Therefore Γ has an additive identity by Proposition 5.1. By Lemma 6.7, Γ satisfies closure under addition. Since $\Gamma \subset \Omega$, Γ satisfies commutativity and associativity by Theorem 5.4. Similarly to Ω, not every numeric game has an additive inverse. By Theorem 6.10, Γ is totally ordered. Since $X \lesssim Y$ and $Y \lesssim X$ implies $X \sim Y$, antisymmetry holds. □

Now that we have devoted much effort to investigating numeric games, it is time to use them to develop the surreal numbers.

Definition 6.12. The *Class of Surreal Numbers*, denoted *No*, is the collection of equivalence classes of numeric games. A *surreal number* is an equivalence class of numerical games and can be written in two ways:

a) the number assigned to its equivalence class, such as 0.

b) any numeric game in that surreal number's equivalence class.

Remark 6.13. Conway used the abbreviation "No." in *On Numbers and Games* to denote the surreal numbers, which he referred to as simply "numbers" and thus called the class "No." as shorthand.

Definition 6.14. Addition and Ordering on No.

a) *(Addition of equivalence classes of numeric games)* Let A and B be equivalence classes of numeric games. Then $A + B = X + Y$ where X is any numeric game such that $X \sim A$ and Y is any numeric game such that. $Y \sim B$.

b) *(Ordering of equivalence classes of numeric games)* Let A and B be equivalence classes of numeric games. Then $A > B$, $A < B$, or $A \sim B$ if and only if $\forall X \in \Gamma$ such that $X \sim A$ and $\forall Y \in \Gamma$ such that $Y \sim B$, $X > Y$, $X < Y$, or $X \sim Y$, respectively.

(Intuitively, we compare equivalence classes by comparing their numeric games).

Remark 6.15. By Definition 6.14, $\lesssim$ is an ordering on No. and addition respects the ordering. Furthermore, by Lemma 6.7, No. is closed under addition. Also to be noted, Definition 6.14 allows us to immediately imply that any numeric game is either $\lesssim 0$ or $\gtrsim 0$.

Example 6.16. 0 is an equivalence class. Notice that $\{\{|\}|\} \not\sim 0$, so $\{\{|\}|\}$ is in another equivalence class, which Conway puts equal to 1 (for reasons we won't get into). Then $0 + 1 = X + Y$ where X is any surreal number such that $X \sim 0$ and Y is any numerical game such that $Y \sim \{\{|\}|\} \sim 1$. As expected, $1 > 0$.

Theorem 6.17. *No. is a totally ordered abelian group under addition.*

Proof. Let A, B, and C be equivalence classes of numeric games. Then:

a) *(Totally Ordered)* Antisymmetry, transitivity, and trichotomy hold between A, B, and C.

We begin by choosing numeric games X, Y, Z such that $X \sim A$, $Y \sim B$, and $Z \sim C$. By Theorem 6.10, trichotomy and transitivity hold between X, Y, and Z. By Definition 4.5, antisymmetry also holds. That is, if $A \leq B$ and $B \leq A$, $A = B$. Therefore by Definition 6.14, No. is totally ordered.

b) *(Commutativity)* $A + B = B + A$.

Consider two numeric games, $X_1 \sim A$ and $Y_1 \sim B$. Next, consider two, not necessarily the same numeric games, $X_2 \sim A$ and $Y_2 \sim B$. By Lemma 6.1, $X_1 + Y_1 \sim X_2 + Y_2$. Therefore the game $X_1 + Y_1$ and the game $X_2 + Y_2$ are in the same equivalence class and thus $A + B = B + A$.

c) *(Associativity)* $(A + B) + C = A + (B + C)$

Now consider numeric games $X_1, X_2, \sim A$, $Y_1, Y_2 \sim B$, and $Z_1, Z_2 \sim C$. We want to prove that $(X_1 + Y_1) + Z_1 \sim X_2 + (Y_2 + Z_2)$. Since numeric games are themselves associative by Theorem 5.4, $X_2 + (Y_2 + Z_2) = (X_2 + Y_2) + Z_1$. Since $X_1 \sim X_2$, $Y_1 \sim Y_2$, and $Z_1 \sim Z_2$, $(X_1 + Y_1) + Z_1 \sim (X_2 + Y_2) + Z_2$ by Lemma 6.1. Therefore $(A + B) + C = A + (B + C)$.

d) *(Additive Identity)* $A + 0 = A$.

Consider the numeric game $X \sim A$. By Lemma 6.3, any game $G_0 \sim 0$ satisfies $X + G_0 \sim X$. Therefore adding any equivalence class with the 0 equivalence class will yield the same original equivalence class.

e) *(Additive Inverse)* $\forall A$, $\exists -A$ such that $A + (-A) = 0$.

Consider a numeric game $X = \{X^L | X^R\}$ such that $X \sim A$. By Proposition 5.1, $\exists - X = \{-X^L | X^R\}$ such that $X + (-X) \sim 0$. Furthermore by Corollary 6.9, $-X$ is a numeric game. Consider any numeric game $-Y \sim -X$, and we find $X + (-Y) \sim 0$ as well. Therefore the entire equivalence class of numeric games similar to $-X$, denoted by $-A$, satisfies the property $A + (-A) = 0$. □

7. Conclusion and Acknowledgments

We have now completed the introduction to surreal numbers from combinatorial games. Beginning with defining fundamental concepts and the addition of games, we then proved games to be an abelian monoid with a special subset of games known as numeric games. Numeric games are important for understanding surreal numbers because they allow us construct the equivalence classes that are the surreal numbers. Finally, we finished with the proof of the surreal numbers being an abelian group under the addition defined in combinatorial games.

Of course, there is more to the story. Conway in *On numbers and Games* goes so far as to define a special form of multiplication that preserves the order between the surreal numbers and from there he proves the surreal numbers to be an ordered field. Furthermore, the surreal numbers bear remarkable similarity to the hyperreal numbers in that they contain infinite and infinitesimal numbers, all of which can be represented as games. Interestingly enough, applications of the surreal numbers are being investigated. Surreal analysis is a nascent area of mathematics and it is of great hope that many wonderful insights come from such explorations.

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