The Hopf invariant one problem

Ishan Banerjee

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Abstract

This paper will discuss the Adams-Atiyah solution to the Hopf invariant problem. We will first define and prove some identities concerning the Adams operations. Then we will look at the proof of the ordinary Hopf invariant one problem. Finally we will look at some results concerning the pth cup power mod p and the mod p version of the Hopf invariant problem.

Contents

1 Introduction

Given a map $f : S^{2n-1} \to S^n$, one can assign to it a number, which we call the Hopf invariant, whose definition involves only ordinary cohomology theory.

One can ask the following question: for which values of n do we have a map f with Hopf invariant one? This question turns out to have interesting consequences in geometry. It is related to both the parallelizability of spheres and the existence of division algebra structures on $\mathbb{R}^n$.

Adams had already answered the question in 1960 using only ordinary cohomology, but there is a much simpler proof of the theorem using K-theory due to Adams and Atiyah.

In this paper we will go through the proof using K-theory. We assume the reader is familiar with the basics of K-theory and ordinary cohomology theory. It may be possible to understand the paper without this background if the reader is willing to assume some basic facts on cohomology on faith, as we do not require that the reader knows the proofs of any nontrivial theorem about cohomology theory or K-theory.

2 Preliminaries on the Hopf invariant one problem

The Hopf invariant may be defined as follows: Take a map $f : S^{2n-1} \to S^n$ and use it as an attaching map to glue a $2n$ cell e^{2n} to S^n. This gives us a space which we call X_f. The cohomology of the resulting space is easily computable, and it turns out that $H^0(X_f) = H^n(X_f) = H^{2n}(X_f) = \mathbb{Z}$. Let α be a generator of $H^n(X_f)$ and β a generator of $H^{2n}(X_f)$. Then, α^2 is an element of $H^{2n}(X_f)$ and hence $\alpha^2 = k\beta$ for some integer k. We call this integer the Hopf invariant of f and denote it by $H(f)$. $H(f)$ depends on the choice of α and β, but only up to sign.

The Hopf invariant one problem is the following question: for which values of n does there exist an f such that $H(f) = \pm 1$?

As a preliminary result, one can see that if n is odd, then $\alpha^2 = -\alpha^2$ by the graded commutativity of the cup product. This implies that $\alpha^2 = 0$ and hence $H(f)$ is always zero. We henceforth assume n is even.

This problem is interesting because of the following result:

Theorem 2.1. *If any of the following are true, then there exists a map from $S^{2n-1} \to S^n$ of Hopf invariant one.*

1. *S^{n-1} is parallelizable.*
2. *$\mathbb{R}^n$ is a division algebra.*
3. *S^{n-1} is an H-space.*

In his 1960 paper Adams shows that the only possible values of n for which there can be an element of Hopf invariant one are $1, 2, 4$ and 8. For these 4 values of n it was already known that S^{n-1} was parallelizable and that $\mathbb{R}^n$ was a division algebra. In fact, the division algebras on $\mathbb{R}^n$ in these 4 cases are the ones that we are all familiar with: the real numbers, the complex numbers, the quaternions and octonions. In Adams first proof, he used secondary cohomology operations in ordinary cohomology. However in 1964 Adams and Atiyah came up with a much simpler proof of the result. The proof involved only primary cohomology operations, but in K-theory as opposed to ordinary cohomology theory. In this paper we will talk about the K-theory solution to the Hopf invariant one problem. We will then talk about the solution to the Hopf invariant problem mod p, which is a related question.

3 Adams Operations

The proof of the Hopf invariant one problem will, as mentioned earlier involve cohomology operations on K-theory. The cohomology operations we will use are called the Adams operations. It is a bit difficult to a priori motivate the Adams operations, except in a kind of general way that motivates all cohomology operations: it is a way of making cohomology a *sharper* invariant, which places more constraints on the kinds of maps between spaces. However, the Adams operations have very natural properties and hopefully the properties themselves somewhat justify the following definitions.

An Adams operation, typically denoted ψ^k with k an integer, is a natural transformation from the K theory functor to itself such that :

1. $\psi^k : K(X) \to K(X)$ is a ring homomorphism.
2. $\psi^k(L) = L^k$ when L is a line bundle.

In fact, it turns out that these two properties characterize ψ^k uniquely.

We will construct ψ^k in terms of the exterior power operations and prove that what we construct has the required properties, using the following lemma (which we do not prove).

Lemma 3.1 (Splitting Lemma)**.** *Given a vector bundle $E \to X$ with X compact Hausdorff, there is a compact Hausdorff space $F(E)$ and a map $p : F(E) \to X$ such that $p^* : K(X) \to K(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.*

A proof of this is found in [1] .

Now we will construct the Adams operations. We wish to define the Adams Operations as some polynomial function of the exterior powers λ^i. In other words, we will define $\psi^k(E) = Q_k(\lambda^0 E, \lambda^1 E. \cdots)$ for some choice of Q_k.

We will need the following identities about the exterior power operations. We will also need to extend the exterior power operations from Vect(X) to $K(X)$.

Lemma 3.2. *The following identities are true for any two vector bundles E and F over some space X.*

1. $\lambda^k(E) = 0$ *if* $k > \dim E$.
2. $\lambda^k(E \oplus F) = \oplus_{i+j=k} \lambda^i(E) \otimes \lambda^j(F)$.

This immediately follows from the corresponding identities for vector spaces (which we will not prove). □

We can now define $\lambda(E)$ to be the formal sum $\sum_{i\in\mathbb{N}} \lambda^i(E)t^i$. It is an element of $K(X)[[t]]$(however, using part (i) of the lemma, the sum is finite). Using part (ii) of the lemma we can conclude that $\lambda(E \oplus F) = \lambda(E)\lambda(F)$. This makes λ a morphism (of abelian monoids) from Vect(X) to formal power series in $K(X)$ with constant term 1 (as $\lambda^0(E)$ is always 1). As this is a group, the map λ

factors through $K(X)$, and we obtain a map from $K(X)$ to $K(X)[[t]]$ which we will also denote λ. We can define $\lambda^i(E)$ to be the coefficient of t^i in $\lambda(E)$. This extends the exterior power operations to $K(X)$

Suppose $L_1, \ldots, L_n$ are line bundles.We then know that

$$\lambda(L_1 + \cdots + L_n) = \prod \lambda L_i = \prod (\lambda^0 L_i + \lambda^1 L_i t) = \prod (1 + L_i t)$$

. Hence $\lambda^i(L_1 + \ldots + L_n) = s_i(L_1, \ldots, L_n)$ where s_i is the ith elementary symmetric polynomial.

Now we can proceed to figure out which polynomial Q_k we should choose. First, we will consider the case $E = L_1 + \cdots + L_n$, when all of the L_i are line bundles. In this case, we know that $\lambda^i(E) = s_i(L_1, \cdots, L_n)$. We want $\psi^k(E) = L_1^k + \cdots + L_n^k$. How shall we go about doing this? Well, we know that purely in terms of algebra, there is a unique polynomial such that $x_1^k + \ldots + x_n^k = Q_k(s_1, \ldots, s_n)$. This Q_k is called the Newton polynomial. We define $\psi^k(E) = Q_k(\lambda^0 E, \lambda^1 E. \cdots)$ with the chosen Q_k. Note that with this definition, ψ_k does indeed satisfy the identity $\psi_k(E) = L_1^k \oplus \cdots \oplus L_n^k$.

One notices that our definition of Q_k seems to depend on n and hence, one might worry if this construction is well-defined. However, one immediately observes by a degree argument that Q_k cannot involve the elementary symmetric polynomial of degree more than k, regardless of the value of n. We will in fact get the same polynomial for all $n \geq k$. Hence, the Newton polynomial Q_k can be defined in terms of the first k elementary symmetric polynomials.

Now we will show that the ψ^k have the required properties. It is clear that $\psi^k(L) = L^k$ when L is a line bundle. It is also clear that ψ^k is natural, as we constructed it using fixed polynomials in the λ^i's which were themselves natural. Therefore, it remains to show that ψ^k is a ring homomorphism.

Let E and E' be two bundles. We wish to prove that ψ^k is additive, i.e. $\psi^k(E+E') = \psi^k(E) + \psi^k(E')$. First we will prove this in the case where E and E' both split as a sum of line bundles. Suppose

$$E = L_1 + \ldots + L_n$$

and

$$E' = L'_1 + \ldots + L'_m.$$

Then

$$\psi^k(E + E') = \psi^k(L_1 + \ldots + L_n + L'_1 + \ldots L'_m)$$
$$= L_1^k + \ldots L_n^k + L_1'^k + \ldots L_m'^k = \psi^k(E) + \psi^k(E').$$

For the general case, we will use the splitting principle. Note that if $f : X' \to X$ is such that f^* is injective, then to show that $\psi^k(E+E') = \psi^k(E) + \psi^k(E')$, it suffices to show that

$$f^*(\psi^k(E + E')) = f^*(\psi^k(E) + \psi^k(E')) = f^*\psi^k(E) + f^*\psi^k(E').$$

Using the naturality of ψ_k the previous equation is equivalent to

$$\psi^k(f^*E + f^*E') = \psi^k(f^*E) + \psi^k(f^*E').$$

We can use the splitting principle twice in order to construct an X' and a map f such that f^* is injective and the two bundles both split as a sum of line bundles. This reduces the general problem to the case of a sum of line bundles and we are done.

Now we just have to prove that $\psi^k(E \otimes E') = \psi^k(E) \otimes \psi^k(E')$. We will again prove this in stages. When E and E' are both line bundles the equation becomes $(EE')^k = E^k E'^k$ which is true. If both E and E' split as a sum of line bundles,we can use algebra to reduce the problem to the previous case. In the general case of two arbitrary bundles, we can again appeal to the splitting principle and reduce it to the case when both are sums of vector bundles, in a similar manner to how we showed additivity.

Hence we've constructed and established the basic properties of the Adams operations. We will need a few more identities about these Adams operations.

Lemma 3.3.

$$\psi^p(E) \equiv E^p mod\, pK(X)$$

.

Proof: If $E = L_1 + \ldots + L_n$, the equation becomes

$$L_1^p + \ldots + L_n^p \equiv (L_1 + \ldots + L_n)^p \text{mod } pK(X).$$

In the general case, we can again appeal to the splitting principle and reduce the problem to the case of a sum of line bundles. □

Lemma 3.4. $\psi^k \circ \psi^l = \psi^{kl} = \psi^l \circ \psi^k$

Proof: The proof is very similar to that of the previous lemma and we shall skip it. Again the essential idea is to prove the result for sums of line bundles, and use the splitting principle to transfer the result to the case of general bundles. □

This lemma is very important as it imposes relations on the Adams operations, similar to the Adem relations for the Steenrod operations. These relations will play an important role in the solution of the Hopf invariant one problem.

Lemma 3.5. $\psi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ *is given by* $\psi^k(x) = k^n x$

If $n = 1$, then $\tilde{K}(S^2)$ is generated by $b = H - 1$, where H is the canonical line bundle over S^2 viewed as $\mathbb{C}P^1$. Furthermore, $b^2 = 0$ (a proof of this fact may be found in [1]). In this case,

$$\psi^k(b) = H^k - 1 = (1 + b)^k - 1 = 1 + kb - 1 = kb$$

(we have used the fact that $b^2 = 0$). Therefore the result is true when $n = 1$. Now we will induct on n. Assume that the result is true for n. We know by Bott periodicity that $\tilde{K}(S^2) \otimes \tilde{K}(S^{2n}) \cong \tilde{K}(S^{2(n+1)})$ with the isomorphism being given by the exterior tensor product with b. The group $\tilde{K}(S^{2(n+1)})$ is generated by $b \star x$ where x is a generator of S^{2n} and $\star$ is the exterior tensor product. But then $\psi^k(b \star x) = \psi^k(b) \star \psi^k(x) = kb \star k^n x = k^{n+1} b \star x$. This completes the induction step and the result follows. $\square$

We note that $K(X)$ is a filtered ring when X is a CW complex. It has a descending filtration $F_q K(X) = \ker(K(X) \to K(X^{q-1})$, where X^q is the q skeleton of X. Now we will prove some facts about the filtration and its relations to the Adams operations, which we will be using later. Note that for any m , we have an exact sequence that goes as follows :

$$0 = K(\vee_i S^{2m+1}) = K(X^{2m+1}/X^{2m}) \to K(X^{2m+1}) \to K(X^{2m})$$

Hence the map $K(X^{2m+1}) \to K(X^{2m})$ is injective and hence $F_{2m+2}K(X) = F_{2m+1}K(X)$.

Lemma 3.6. *If $x \in F_{2n}K(X)$, $\psi^k(x) - k^n x \in F_{2n+1}K(X)$.*

Proof: We have an exact sequence that is as follows :

$$K(X^{2n}/X^{2n-1}) \to K(X^{2n}) \to K(X^{2n-1}).$$

Suppose x is in $F_{2n}K(X)$. We have an inclusion map $j : X^{2n} \to X$. Then, we know that $j^* x$ maps into the kernel of i^* in the sequence. By the exactness of the sequence, this implies $j^*(x) = \pi^*(y)$ for some $y \in K(X^{2n}/X^{2n-1})$. But

$$j^*(\psi^k - k^n)(x) = (\psi^k - k^n) j^* x = (\psi^k - k^n)\pi^* y = \pi^*(\psi^k - k^n) y.$$

However, X^{2n}/X^{2n-1} is just a wedge of S^{2n}s. This implies that $\psi^k(y) - k^n y = 0$ for $y \in \tilde{K}(X^{2n}/X^{2n-1})$. Therefore $(\psi^k - k^n)x$ is in the kernel of j, and hence is in $F_{2n+1}K(X)$. $\square$

4 The Hopf invariant one problem

The Hopf invariant was previously defined in terms of ordinary cohomology. We will now give an alternative definition in terms of K theory. We will then use the Adams operations to find obstructions to finding a map of Hopf invariant one.

First, we will define the K theory version of the Hopf invariant. We have a long exact sequence corresponding to the pair (X_f, S^{2n}). A part of this sequence looks like this :

$$\tilde{K}^{-1}(S^{2n}) \to \tilde{K}^0(S^{4n}) \cong \tilde{K}^0(X_f/S^{2n}) \to \tilde{K}^0(X_f) \to \tilde{K}^0(S^{2n}) \to K^1(\tilde{S}^{4n}).$$

Then, as $\tilde{K}(S^{2k+1}) = 0$, for any integer k, we get the following short exact sequence:

$$0 \to \tilde{K}(S^{4n}) \cong \tilde{K}(X_f/S^{2n}) \to \tilde{K}(X_f) \to \tilde{K}(S^{2n}) \to 0.$$

Since, $\tilde{K}(S^{2n}) = \mathbb{Z}$ which is projective, the short exact sequence splits and we get that $\tilde{K}(X_f) = \tilde{K}(S^{2n}) \oplus \tilde{K}(S^{4n})$.

Using the splitting we can conclude that $\tilde{K}(X_f)$ is free on two generators which we call α and β. We can choose these generators such that α is the image of a generator of $\tilde{K}(S^{4n})$ and $i^*\beta$ generates $\tilde{K}(S^{2n})$. Note that while α is well-defined upto a sign, β depends on the choice of splitting. We will now define a K-theoretic version of the Hopf invariant. Consider the element $\beta^2 \in \tilde{K}(X_f)$.We know that $\beta^2 = c\beta + d\alpha$. But we know that

$$0 = (i^*\beta)^2 = i^*(\beta^2) = ci^*(\beta) + di^*(\alpha) = ci^*(\beta).$$

As $i^*(\beta)$ generates $\tilde{K}(S^{2n}) \cong Z$, this implies $c = 0$. Therefore $\beta^2 = d\alpha$ for some d. We will now denote this integer d as $h(f)$.

A priori $h(f)$ may depend on the choice of splitting. We will now show that it is in fact independent of the choice of splitting. Any other choice of β must be of the form $\beta' = k\alpha + \beta$. $\beta'^2 = (k\alpha + \beta)^2 = k^2\alpha^2 + 2k\alpha\beta + \beta^2$. Now $\alpha^2 = 0$, so if we could just show that $\alpha\beta = 0$, we would be done. Now $\alpha\beta = k\alpha + l\beta$. But

$$i^*(\alpha\beta) = i^*(\alpha)i^*(\beta) = 0.$$

This implies $0 = i^*(k\alpha + l\beta) = ki^*(\alpha) + li^*(\beta) = li^*(\beta)$. This implies $l = 0$. Therefore $\alpha\beta = k\alpha$. Multiplying by β on both sides, we get

$$k\alpha\beta = \alpha\beta^2 = \alpha h(f)\alpha = h(f)\alpha^2 = 0.$$

As the group $\tilde{K}(X)$ is torsion free, this implies $\alpha\beta = 0$. Therefore, $h(f)$ is well defined.

It turns out that $h(f) = H(f)$. This can be proven using the Chern character, a map which goes from $K(X)$ to $H^*_{even}(X, \mathbb{Q})$. We do not wish to prove this here as it will take us too far afield. A proof can be found in chapter 24 of [8]. Assuming that the two invariants are equal, we can now apply K-theory to study the Hopf invariant. We will use the Adams operations to show that $h(f) \equiv 0 \bmod 2$.

Since α lies in the image of $\tilde{K}(X_f/S^{2n}) = \tilde{K}(S^{4n})$, $\psi^k(\alpha) = k^{2n}\alpha$. Similarly we can prove that $\psi^k(\beta) = k^n\beta + \mu_k\alpha$ for some $\mu_k \in \mathbb{Z}$. We know that $\psi^k \circ \psi^l = \psi^l \circ \psi^k$. Therefore, $\psi^3 \circ \psi^2(\beta) = \psi^2 \circ \psi^2(\beta)$. But we know what ψ^3 does to β and α! We can use our formulas to obtain the equation:

$$3^n 2^n \beta + 2^n \mu_3 \alpha + 3^{2n} \mu_2 \alpha = 2^n 3^n \beta + 3^n \mu_2 \alpha + 2^{2n} \mu_3 \beta$$

.

Rearranging this equation, we get :

$$2^n(1 - 2^n)\mu_3 = 3^n(1 - 3^n)\mu_2$$

(We can ignore the α as α is not a torsion element). This implies

$$2^n | 3^n(1 - 3^n)\mu_2.$$

As 2 and 3 are coprime, this implies $2^n|\mu_2(3^n - 1)$. If μ_2 were odd, this would imply that $2^n|3^n - 1$. But (by an elementary number theoretic argument) this is not true if $n \neq 0, 1, 2, 4$. This forces μ_2 to be even in all other cases. But if μ_2 is even we have :

$$h(f)\alpha \equiv \beta^2 \equiv \psi^2(\beta) \equiv 2^n\beta + \mu_2\alpha \equiv 0(\text{mod } 2).$$

This implies that $h(f) \equiv 0$ mod 2. This proves the result modulo the number theoretic fact, which we now prove.

Lemma 4.1. *If* $2^n|3^n - 1$, *then* $n = 0, 1, 2, 4$.

Proof: We can always write n in the form $2^l m$ with m odd.

Claim : The highest power of 2 dividing $3^n - 1$ is 2^2 if $l = 0$ and 2^{l+2} otherwise.

The claim implies the lemma as if $2^n|3^n-1$ our claim will imply that $n \leq l+2$ and hence, $2^n \leq 4n$. This in turn implies $n \leq 4$. The cases $n = 0, 1, 2, 3, 4$ can be checked by brute force computation.

We will prove the claim by induction. If $l = 0$, then $3^m - 1$ is clearly even. As m is odd and $3^2 \equiv 1($ mod $4)$, $3^m \equiv 3$ (mod 4) and 4 does not divide $3^m - 1$.

If $l = 1$, then $3^{2m} - 1 = (3^m - 1)(3^m + 1)$. The highest power of 2 dividing $3^m - 1$ is 4 due to the previous case. As $3^2 \equiv 1$ (mod 8) and m is odd, $3^m + 1 \equiv 3 + 1 \equiv 4($ mod $8)$. So the highest power dividing $3^{2m} - 1$ is $4 \times 2 = 8$.

For the general induction, note that $3^{2^{l+1}m} - 1 = (3^{2^l m} - 1)(3^{2^l m} + 1)$. The highest power of 2 dividing the first factor is 2^{l+2} by induction. The second factor is congruent to 2 mod 4 by a similar argument to those made above. Hence the highest power dividing $3^{2^{l+1}m} - 1$ is $2 \times 2^{l+2} = 2^{l+3}$. This finishes the proof. □

5 Eigenspaces of ψ^k

Now we would like to move on towards the statement and proof of the mod p Hopf invariant one problem. But first, we would like to look a bit more closely at the the Adams operations ψ^k. We would like to find an eigenspace decomposition of $\tilde{K}(X) \otimes \mathbb{Q}$ with respect to the operator ψ^k.

We can define the K-theoretic Betti numbers to be

$$B_{2m}(X) = \dim_Q(F_{2m}K(X) \otimes \mathbb{Q}/F_{2m+1}K(X) \otimes \mathbb{Q}).$$

These turn out to be the same as the ordinary Betti numbers, as the associated graded ring of $K(X) \otimes \mathbb{Q}$ is $H^*_{even}(X.\mathbb{Q})$, (this follows from the fact that the Atiyah-Hirzebruch spectral sequence has only zero differentials rationally, see [5]).

We then have the following lemma.

Lemma 5.1. *Let X be a finite dimensional CW complex. Suppose that $B_{2m}(X) = 0$ for $m \neq 0, m_1 < \ldots < m_r$. Let $k_0 \ldots k_r$ be any choice of integers in Z. Then*

$$\prod_{i=0}^{r}(\psi^{k_i} - k_i^{m_i}) = 0$$

in $\tilde{K}(X) \otimes \mathbb{Q}$.

Proof: Note that all terms in the product commute. If $x \in F_{2m_i}K(X) \otimes Q$, $(\psi^{k_i} - {k_i}^{m_i})x \in F_{2m_i+1}K(X) \otimes \mathbb{Q}$ by 3.6. But using the constraints on the Betti numbers, this implies that $(\psi^{k_i} - {k_i}^{m_i})x \in F_{2m_i+1}K(X)$. Repeating this argument, we get $\prod_{i=0}^{r}(\psi^{k_i} - {k_i}^{m_i})(x) \in F_{2m_r+1}K(X)$ which is zero. $\square$

We will use this formula to find an eigenspace decomposition for ψ^k.

Let $V_{i,k} = \mathrm{Im}\prod_{j\neq i}(\psi^k - k^{m_j})$ in $\tilde{K}(X)\otimes\mathbb{Q}$. It is easy to see (by elementary linear algebra) that $V_{i,k} = \ker(\psi^k - k^{m_i})$ and that $\oplus_i V_{i,k} = \tilde{K}(X) \otimes \mathbb{Q}$. Also, using the lemma, we see that $V_{i,k} = \mathrm{Im}\prod_{j\neq i}(\psi^k - k^{m_j}) \subseteq \ker\psi^l - l^{m_i} = V_{i,l}$. Since k and l were arbitrary, we obtain $V_{i,k} = V_{i,l}$ for all choices of k and l. We call this common subspace V_i. We have $\tilde{K}(X) \otimes \mathbb{Q} = \oplus_i V_i$. We have a map $\tilde{K}(X)\otimes\mathbb{Q} \to V_i$, which is the projection map π_i. We have the following explicit formula for π_i :

$$\pi_i(x) = \prod_{j\neq i}\frac{\psi^{k_j} - {k_j}^{m_j}}{{k_j}^{m_i} - {k_j}^{m_j}}$$

where the k_j's are arbitrary choices of integers. This can be easily proven by noting that $\pi_i x = x$ for $x \in V_i$ and $\pi_i x = 0$ for $x \in V_j$ for $j \neq i$. This in turn follows from the fact that the V_i are eigenspaces of ψ^k.

6 Integral elements

We have an obvious map $\tilde{K}(X) \to \tilde{K}(X)\otimes\mathbb{Q}$. We call the elements of $\tilde{K}(X)\otimes\mathbb{Q}$ in the image of this map integral elements. Given any element x in $\tilde{K}(X) \otimes \mathbb{Q}$ there is a minimal positive integer d such that dx is integral. We call this d the denominator of x.

Now we will introduce some notation. Define $d_i(m_1 \ldots m_r)$ to be the gcd of all products of the form $\prod_{j\in\{1,\ldots,r\}\setminus\{i\}}({k_j}^{m_i} - {k_j}^{m_j})$, where the k_j are arbitrary positive integers. Using the formula for $\pi_i(x)$, we get that the denominator d of $\pi_i(x)$ divides $d_i(m_1, \ldots, m_r)$. With this notation we can state the following theorem:

Theorem 6.1. *Let X be a finite dimensional CW complex. Suppose p^{m_i} does not divide $d_i(m_1, \ldots, m_r)$ for each i. Then $\psi^p(x) \in p\tilde{K}(X) + Tors(\tilde{K}(X))$. If $\tilde{K}(X)$ has no p torsion, $\psi^p(x) \in p\tilde{K}(X)$.*

Proof: We can write any $x \in \tilde{K}(X) \otimes \mathbb{Q}$ as $\sum_i \pi_i(x)$. Then

$$\psi^p(x) = \sum_i \psi^p(\pi_i(x)) = \sum_i p^{m_i}\pi_i(x).$$

If x is integral and p^{m_i} does not divide $d_i(m_1, \ldots, m_r)$ then each individual term in the sum is of the form $\frac{py_i}{q_i}$ with y_i integral and $p \not| q_i$. Hence the sum can also be written as $\frac{py}{q}$ with y integral and $p \not| q$. But, if we look at what this means in $\tilde{K}(X)$, we get $\psi^p(x) \in p\tilde{K}(X) + Tors\tilde{K}(X)$. The p-torsion case follows similarly. $\square$

We can in fact, use this result about K-theory to get a result about ordinary cohomology theory, which will give sufficient conditions for the pth power to vanish mod p.

Corollary 6.2. *Assume X and p are as in the previous theorem and that $H^*(X, \mathbb{Z})$ has no p-torsion. Then the pth cup power is zero mod p.*

Proof: We have a spectral sequence $H^*(X, K^*\{pt\}) \Rightarrow F_*K^*(X)$ with all differentials as torsion operators (see [5] for details). We can tensor the spectral sequence with $\mathbb{Z}_{(p)}$ ($\mathbb{Z}$ localized away from the prime ideal (p)). As $H^*(X, K^*\{pt\})$ has no p torsion, the resulting spectral sequence has all differentials zero. Hence we get that

$$H^{2m}(X, \mathbb{Z}_{(p)}) \cong H^{2m}(X) \otimes \mathbb{Z}_{(p)} \cong K_{2m}(X)/K_{2m+1}(X) \otimes \mathbb{Z}_{(p)}.$$

Since X has no p torsion, the map from $H^*(X, \mathbb{Z}_{(p)}) \to H^*(X, \mathbb{F}_p)$ is surjective. Now take any element in $y \in H^*(X, \mathbb{F}_p)$. Using the surjective map given above and the sequence of isomorphisms, we can conclude that it is the image of some element $x \otimes a \in K(X) \otimes \mathbb{Z}_{(p)}$. However $(x \otimes a)^p = x^p \otimes a^p = 0$ as $x^p = 0$ by the theorem. Hence the pth power map is zero. $\square$

But in order to apply these theorems, we have to ensure that the conditions regarding p and m are met. In this regard we have the following lemma:

Lemma 6.3. *Let p be an odd prime and m a positive integer such that m does not divide $p-1$. Then there exists k (a positive integer) such that p^m does not divide $\prod_{j \in \{1,\ldots,p\} \setminus \{i\}} (k^{mi} - k^{mj})$ for all $i \in \{1, \ldots, p\}$.*

Proof: Let p^e be the highest power of p dividing $\prod_{j \in \{1,\ldots,p\} \setminus \{i\}} (k^{mi} - k^{mj})$ (we will choose k later). We wish to show that $e \leq m - 1$. Suppose $\gcd(m, p - 1) = h$. Then $m = ah$ and $p - 1 = bh$. a cannot be 1 as m does not divide $p - 1$. Let p^f be the highest power of p dividing m. $(\mathbb{Z}/p^{f+2}\mathbb{Z})^\times$ is a cyclic group of order $p^{f+1}(p-1)$. We will choose k such that it generates this group (or rather its equivalence class generates the group). k will also automatically generate $(\mathbb{Z}/p^l\mathbb{Z})^\times$ for $l < f + 2$.

We wish to obtain an explicit formula for e. Note that the highest power of p dividing $\prod_{j \in \{1,\ldots,p\} \setminus \{i\}} (k^{mi} - k^{mj})$ is just the product of the highest powers of p dividing $(k^{mi} - k^{mj})$. If $i > j$, $(k^{mi} - k^{mj}) = (k^{m(i-j)} - 1)k^{mj}$. As p and k are coprime, the highest power of p dividing this term is just the highest power of p dividing $k^{m(i-j)} - 1$. But using the fact that k generates $(Z/p^l Z)^\times$ (for $l \leq f + 2$), p^l divides $k^{m(i-j)} - 1$ iff $p^{l-1}(p-1) | m(i-j)$. Using this we can determine what is the highest power of p dividing each term in the product (if $i < j$ we can do something analogous).

Putting all of this together we get,

$$e = (f+1)([\frac{i-1}{b}] + [\frac{p-i}{b}]).$$

From the equation we learn that

$$e \leq (f+1)\frac{p-1}{b} = h(f+1) \leq hp^f \leq m$$

. Equality cannot hold everywhere as if $f+1 = p^f$, then f is forced to be 0 and $hp^f = h < m$. Therefore $e < m$. □

Now we can prove the following theorem about the cup pth power mod p.

Theorem 6.4. *Let p be an odd prime and m a positive integer not dividing $p-1$. Further assume that:*

- *$H^*(X, Z)$ has no p torsion.*
- *$H^{2k}(X, Q) = 0$ if $m \not| k$.*

Then the cup pth power $H^{2m}(X, \mathbb{F}_p) \to H^{2mp}(X, \mathbb{F}_p)$ is zero.

Proof: First note that we can replace X by X^{2mp+1}. Then we know that all the numbers $B_{2k}(X)$ are zero except possibly when $k \in \{m, 2m, \dots pm\}$. But 6.3 then implies p^{m_i} does not divide $d_i(m_1 \dots m_r)$ and we can then use 6.2 to conclude that the pth power is zero mod p □.

7 The mod p Hopf invariant problem

In this section we will introduce a variation on the original Hopf invariant problem. We begin with a function $f : S^{2mp} \to S^{2m+1}$. We can use the function f to form the pushout $X_f = e^{2mp+1} \cup_f S^{2m+1}$, just as in the definition of the ordinary Hopf invariant.

Now we consider the Steenrod operation $P^m : H^{2m+1}(X_f, \mathbb{F}_p) \to H^{2mp+1}(X, \mathbb{F}_p)$. We will call this map the mod p Hopf invariant. This is reasonable as the ordinary Hopf invariant problem can be thought of as defined by the Steenrod square operation Sq^n. We wish to show that this map is zero.

Theorem 7.1. *The map P^m is zero on $H^{2m+1}(X_f, \mathbb{F}_p)$.*

Proof: Assume m does not divide $p-1$ (that case will be handled later). The plan of action is as follows: We will prove that P^m vanishes in an auxiliary space SY_g, which we will construct. There will be a map $e' : SY_g \to X_f$ which will induce an injective map on cohomology. This will prove the theorem.

Let f be our map from S^{2mp} to S^{2m+1}. We can form the adjoint map g from S^{2mp-1} to ΩS^{2m+1} using the adjunction between loop space and suspension.

We can now form the space $Y_g = e^{2mp} \cup_g \Omega S^{2m+1}$. We know that :

$$H^{2km}(\Omega(S^{2m+1})) = \mathbb{Z} \text{ for } k \in \mathbb{Z}_{\geq 0}$$

and the other cohomology groups are trivial (this is an application of the Serre spectral sequence, see [9] for details). This implies that Y_g satisfies the conditions of the theorem, except for the fact that it is not finite dimensional. However, we know that Y_g has the weak homotopy type of a CW complex. We can therefore approximate Y_g by a CW complex Z. We can choose arbitrarily large l such that the l skeleton of Z satisfies the conditions of 6.2. So the pth cup power $H^{2m}(Z^l) \to H^{2mp}(l)$ is zero. Transferring the result to Y_g, we can conclude that the pth cup power $H^{2m}(Y_g) \to H^{2mp}(Y_g)$ is zero. Hence after suspending, we learn that the map $P^m : H^{2m+1}(SY_g, \mathbb{F}_p) \to H^{2mp+1}(SY_g, \mathbb{F}_p)$ is zero.

So we have constructed our space SY_g. Now we shall construct a map from SY_g to X_f. Let $e : S\Omega S^{2m+1} \to S^{2m+1}$ be the map $e(\omega, t) = \omega(t)$. Note that the composite $e \circ Sg : S^{2mp} \to S\Omega S^{2m+1} \to S^{2m+1}$ is exactly equal to f. Therefore the map $e : S\Omega S^{2m+1} \to S^{2m+1}$ can be extended to a map $e' : SY_g \to X_f$. Since e^* is an isomorphism on H^{2m+1}, e'^* is injective on cohomology, and this proves the theorem if $m \not| p-1$.

If $m|p-1$, then we know that $m < p-1$. In this case, we can use the Adem relations to conclude that $P^m = \frac{1}{m}P^1P^{m-1}$. This map is zero as the intermediate cohomology groups are zero. $\square$.

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