Exercises 6, April 25, 2006

The Normalized Chain Complex

1) Let $A_\bullet$ be a simplicial Abelian group and let $\chi(A_\bullet) = (\hat{A}_*, \hat{d}_*)$ be the (un-normalized) chain complex associated to it. Remember that $\hat{A}_n = A_n$ for $n \geq 0$ and with differential $\hat{d}_n : A_n \to A_{n-1}$ given by

$$\hat{d}_n(x) = \sum_{i=0}^{n} (-1)^i d_i(x)\,.$$

For $n \geq 0$, define

$$N_n = \ker d_0 \cap \ker d_1 \cap \ldots \cap \ker d_{n-1} \subset A_n\,.$$

Use the simplicial identity $d_i d_j = d_{j-1} d_i$ for $i < j$ to show that $N_* \subset \hat{A}_*$ is a sub chain complex. Note also that the differential $N_n \to N_{n-1}$ is given by $d(x) = (-1)^n d_n(x)$.

2) For each $p \geq 0$, let

$$F^p \hat{A}_n = \{x \in A_n \mid d_i(x) = 0,\ 0 \leq i < \min(n,p)\}$$

Show that, for a fixed p, the inclusions $F^{p+1}\hat{A}_n \subset F^p \hat{A}_n$, for $n \geq 0$, give an inclusion of chain complexes $i^p : F^{p+1}\hat{A}_* \subset F^p \hat{A}_*$.

3) Note that for $p \geq n$, we have $F^p \hat{A}_n = N_n$ and that $F^0 \hat{A}_n = \hat{A}_n$. Thus, we have now the following filtration of chain complexes

$$\hat{A}_* \supset F^0 \hat{A}_* \supset F^1 \hat{A}_* \supset \ldots N_*\,.$$

We will now show that every inclusion i^p induce an isomorphism on homology. We do this by constructing a morphism of chain complexes, $f^p : F^p \hat{A}_* \to F^{p+1}\hat{A}_*$ which is an inverse to i^p up to chain homotopy.

Let $x \in F^p \hat{A}_n$. Then define

$$f^p(x) = \begin{cases} x & ; n \leq p \\ x - s_p d_p(x) & ; n > p \end{cases}$$

Check that f^p is a morphism of chain complexes, i.e. check that the following diagram commutes

$$\begin{array}{ccc} F^p \hat{A}_n & \xrightarrow{f^p} & F^{p+1}\hat{A}_n \\ \downarrow{\scriptstyle \hat{d}_n} & & \downarrow{\scriptstyle \hat{d}_n} \\ F^p \hat{A}_{n-1} & \xrightarrow{f^p} & F^{p+1}\hat{A}_{n-1} \end{array}$$

4) Show that the composite $f^p \circ i^p$ is equal to the identity morphism on $F^{p+1}\hat{A}_*$.

5) We will now define a chain homotopy between the identity morphism on $F^p\hat{A}_*$ and the composite $i^p \circ f^p$. Let $x \in F^p\hat{A}_n$, and define $t^p : F^p\hat{A}_n \to F^p\hat{A}_{n+1}$ by the formula

$$t^p(x) = \begin{cases} 0 & ; n < p \\ (-1)^p s_p(x) & ; n \geq p \end{cases}$$

Verify that
$$\hat{d}_{n+1} t^p(x) + t^p \hat{d}_n(x) = x - (i^p \circ f^p)(x)\,.$$

6) Let $f : \hat{A}_* \to N_*$ be the map of chain complexes defined by letting $f_n : \hat{A}_n \to N_n$ be the composite $f^{n-1} \circ f^{n-2} \circ \ldots \circ f^0$. Conclude from the above that the inclusion $i : N_* \subset \hat{A}_*$ induces an isomorphism on homology with inverse $H(f)$.

7) Note that the composite $f \circ i$ is the identity morphism on N_*. Conclude that $\hat{A}_* \cong N_* \oplus \ker f$.

8) Let $D_n = \bigcup\limits_{i=0}^{n-1} s_i(A_{n-1})$. We showed in the previous lecture that $D_* \subset \hat{A}_*$ is a sub chain complex.

   Show by the definition of f that $\ker f \subset D_*$.

9) Let $k \geq 0$ be fixed and let $x^k = \sum_{i=k}^{n-1} s_i(y_i^k)$ where $y_i^k \in A_{n-1}$ for all i. Show by calculation that $f^k(x^k) = \sum_{i=k+1}^{n-1} s_i(y_i^k - s_k d_k y_i^k)$.

   Define inductively $y_i^{k+1} := y_i^k - s_k d_k y_i^k$ and let $x^{k+1} := \sum_{i=k+1}^{n-1} s_i(y_i^{k+1})$.

   Now show that $D_* \subset \ker f$: Let $x = \sum_{i=0}^{n-1} s_i(y_i) \in D_n$, where $y_i \in A_{n-1}$. Then calculate $f(x) := f^{n-1} \circ \ldots f^1 \circ f^0(x)$ using the notation above and verify that the result is zero.

10) Two previous exercises show that we have a splitting of chain complexes $\hat{A}_* \cong N_* \oplus D_*$.

    Conclude from the above that the projection $\hat{A}_* \to \hat{A}_*/D_*$ induce an isomorphism on homology.