A Primer on Homological Algebra

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1 Modules

For people who have taken the algebra sequence, you can pretty much skip the first section... Before telling you what a module is, you probably should know what a ring is...

Definition 1.1. A ring is a set R with two operations $+$ and $*$ and two identities 0 and 1 such that

1. $(R, +, 0)$ is an abelian group.
2. (Associativity) $(x * y) * z = x * (y * z)$, for all $x, y, z \in R$.
3. (Multiplicative Identity) $x * 1 = 1 * x = x$, for all $x \in R$.
4. (Left Distributivity) $x * (y + z) = x * y + x * z$, for all $x, y, z \in R$.
5. (Right Distributivity) $(x + y) * z = x * z + y * z$, for all $x, y, z \in R$.

A ring is *commutative* if $*$ is commutative. Note that multiplicative inverses do not have to exist!

Example 1.2.
1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ with the standard addition, the standard multiplication, 0, and 1.
2. $\mathbb{Z}/n\mathbb{Z}$ with addition and multiplication modulo n, 0, and 1.
3. $R[x]$, the set of all polynomials with coefficients in R, where R is a ring, with the standard polynomial addition and multiplication.
4. $M_{n\times n}$, the set of all n-by-n matrices, with matrix addition and multiplication, $\mathbf{0}_n$, and $\mathbf{I}_n$.

For convenience, from now on we only consider commutative rings.

Definition 1.3. Assume $(R, +_R, *_R, 0_R, 1_R)$ is a commutative ring. A *R-module* is an abelian group $(M, +_M, 0_M)$ with an operation $\cdot : R \times M \to M$ such that

1. $r \cdot (m +_M n) = (r \cdot m) +_M (r \cdot n)$, for all $r \in R, m, n \in M$.

2. $(r +_R s) \cdot m = (r \cdot m) +_M (s \cdot m)$, for all $r, s \in R, m \in M$.

3. $(r *_R s) \cdot m = r \cdot (s \cdot m)$, for all $r, s \in R, m \in M$.

4. $1_R \cdot m = m$

Remark 1.4. When R is a field, an R-module is exactly a R-*vector space*.

Exercise 1.5. Show that every abelian group can be regarded as a $\mathbb{Z}$-module.

Exercise 1.6. Define what a homomorphism between two rings means. Define what a homomorphism between two R-modules means.

Definition 1.7. Let M, N be two R-modules, and $\varphi : M \to N$ be a homomorphism. The *kernel* of φ, denoted $\ker \varphi$, is defined as

$$\ker \varphi = \varphi^{-1}(\{0_N\})$$

Exercise 1.8. Show that φ is injective if and only if $\ker \varphi = \{0_M\}$.

Remark 1.9. From simplicity, we use 0 to denote the trivial subgroup of every group, i.e., the subgroup containing only the identity element.

Definition 1.10. We say a homomorphism $\varphi : M \to N$ is *trivial* if it maps everything to 0_N.

2 Exact Sequences

From now on R will be a commutative ring.

Definition 2.1. An exact sequence of R-modules consists of a sequence of R-modules $\{M_i\}$ and homomorphisms $\{\varphi_i\}$ looking like

$$\cdots \xrightarrow{\varphi_{-3}} M_{-2} \xrightarrow{\varphi_{-2}} M_{-1} \xrightarrow{\varphi_{-1}} M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \cdots$$

such that

$$\ker \varphi_i = \operatorname{im} \varphi_{i-1}, \forall i.$$

Example 2.2. 1.

$$0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} 0$$

Clearly both φ_0 and φ_1 are trivial maps, so $\ker \varphi_1 = M_1, \operatorname{im} \varphi_0 = 0$. Because the sequence is exact, M_1 must equal to 0.

2.

$$0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} 0$$

We know that

$$\operatorname{im}\varphi_1 = \ker\varphi_2 = M_2$$
$$\ker\varphi_1 = \operatorname{im}\varphi_0 = 0,$$

so φ_1 is both surjective and injective. Hence it is an isomorphism, i.e., $M_1 \cong M_2$.

3.

$$0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} 0$$

This is called a *short exact sequence*. Similar to 2, we know that φ_2 is surjective and φ_1 is injective. In fact, short exact sequences contains more information, but we need the following theorem first.

Definition 2.3. Let $N \subseteq M$ be a submodule. The *quotient* of M by N, denoted M/N, is defined as $M/\sim$, where $\sim$ is the equivalence relation defined as $m \sim n$ if $m - n \in N$. (Check that it is indeed an equivalence relation.)

Theorem 2.4. *(First Isomorphism Theorem) If $\varphi : M \to N$ is a homomorphism, then*

$$\operatorname{im}\varphi \cong M/\ker\varphi.$$

Corollary 2.5. *If*

$$0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} 0$$

is an exact sequence, then

$$M_3 \cong M_2/M_1.$$

(Technically we should write $M_3 \cong M_2/\varphi_1(M_1)$, but $\varphi_1(M_1) \cong M_1$ since φ_1 is injective.)

Remark 2.6. You might think that $M_2 \cong M_1 \oplus M_3$. However, this is not necessarily true. Check that the following is an exact sequence, but $\mathbb{Z}$ and $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ are clearly not isomorphic.

$$n \longmapsto 2n$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$m \longmapsto m + 2\mathbb{Z}$$

However, if R is a field, then $M_2 \cong M_1 \oplus M_3$ is always true.

Theorem 2.7. *(Splitting Lemma) For the exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0$$

, the following statements are equivalent.

1. *(Left Split) There exists a homomorphism $f : M_2 \to M_1$ such that $f \circ \varphi = id_{M_1}$.*

2. *(Right Split) There exists a homomorphism $g : M_3 \to M_2$ such that $\psi \circ g = id_{M_3}$.*

3. *$M_2 \cong M_1 \oplus M_3$.*

We say the exact sequence splits if the above conditions hold.

The following is by far the most important theorem regarding exact sequences!

Theorem 2.8. *(The Five Lemma) Given two exact sequences $A \to B \to C \to D \to E$, $A' \to B' \to C' \to D' \to E'$ and five homomorphisms $\alpha, \beta, \gamma, \delta, \epsilon$ such that the following diagram commutes. (That means all possible compositions of homomorphisms from X to Y must be the same.)*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Then

1. *If β and δ are surjective, and ϵ is injective, then γ is surjective.*

2. *If β and δ are injective, and α is surjective, then γ is injective.*

In particular, if $\alpha, \beta, \delta, \epsilon$ are all isomorphisms, then γ is an isomorphism.

3 Chain Complex and Homology

Definition 3.1. A *chain complex $C_\bullet$* consists of a sequence of R-modules $\{C_i\}$ and *boundary homomorphisms* $\{d_i\}$ looking like

$$\cdots \xrightarrow{d_{-3}} C_{-2} \xrightarrow{d_{-2}} C_{-1} \xrightarrow{d_{-1}} C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} \cdots$$

such that

$$d_i \circ d_{i-1} = 0, \forall i.$$

The condition is called the *boundary condition*.

Exercise 3.2. Show that the boundary condition is equivalent to

$$\operatorname{im} d_{i-1} \subseteq \ker d_i, \forall i.$$

Hence every exact sequence is a chain complex. (We will see that exact sequences are extremely uninteresting chain complexes...)

Definition 3.3. Given a chain complex $C_\bullet$. The homology of this chain complex, denoted $H_\bullet(C_\bullet)$ is defined as

$$H_i(C_\bullet) = \ker d_i / \operatorname{im} d_{i-1}$$

Exercise 3.4. Show that the homology of an exact sequence is all zero. Moreover, show that if a chain complex has zero homology, then it is an exact sequence.

Definition 3.5. Given a homomorphism $\varphi : M \to N$, the *cockerel* of φ is defined as

$$\operatorname{coker} \varphi = N / \operatorname{im} \varphi.$$

Exercise 3.6. Show that $\operatorname{coker} \varphi = 0$ if and only if φ is surjective.

Exercise 3.7. Show that every homomorphism $\varphi : M \to N$ induces the following exact sequence.

$$0 \longrightarrow \ker \varphi \xrightarrow{i} M \xrightarrow{\varphi} N \xrightarrow{q} \operatorname{coker} \varphi \longrightarrow 0$$

Where $i : \ker \varphi \to M$ and $q : N \to \operatorname{coker} \varphi$ are inclusion map and quotient map, respectively.

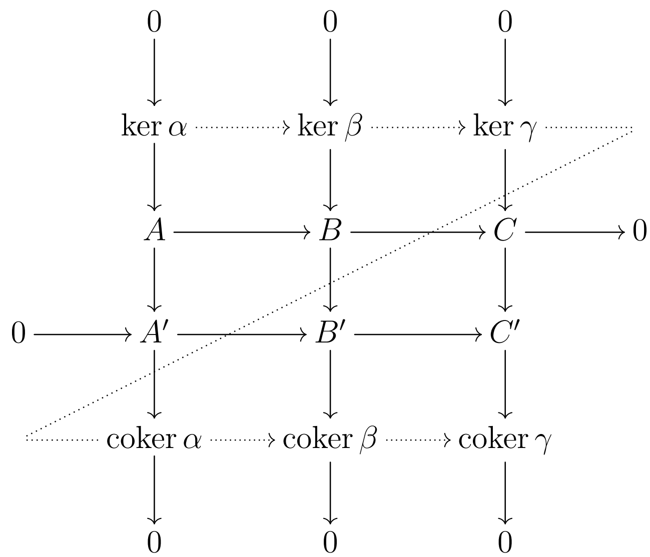
Theorem 3.8. *(The Snake Lemma) Given two exact sequences $A \to B \to C \to 0$, $0 \to A' \to B' \to C'$ and homomorphisms α, β, γ such that the following diagram commutes.*

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

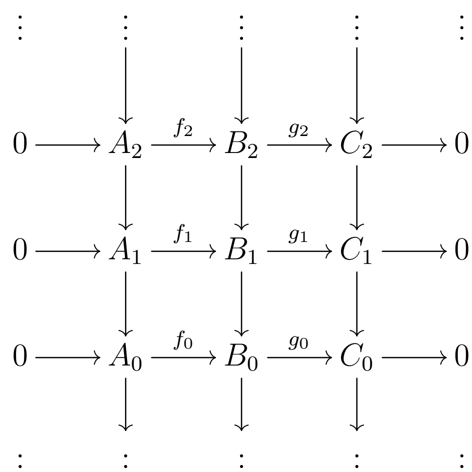
Then there is an exact sequence

$$\ker \alpha \xrightarrow{\tilde{f}} \ker \beta \xrightarrow{\tilde{g}} \ker \gamma \xrightarrow{\delta} \operatorname{coker} \alpha \xrightarrow{\tilde{f'}} \operatorname{coker} \beta \xrightarrow{\tilde{g'}} \operatorname{coker} \gamma$$

, where δ is called the connecting homomorphism. It is called Snake Lemma because the induced exact sequence zig zags through the original diagram.



Definition 3.9. We say that $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of chain complexes if there exist homomorphisms $f_i : A_i \to B_i$ and $g_i : B_i \to C_i$ such that the following diagram commutes and every row is an exact sequence.



Corollary 3.10. *A short exact sequence of chain complexes* $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ *induces a long exact sequence in their homologies.*

$$
\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
 & & \xrightarrow{\delta_3} & & \\
H_2(A_\bullet) & \xrightarrow{f_2} & H_2(B_\bullet) & \xrightarrow{g_2} & H_2(C_\bullet) \\
 & & \xrightarrow{\delta_2} & & \\
H_1(A_\bullet) & \xrightarrow{f_1} & H_1(B_\bullet) & \xrightarrow{g_1} & H_1(C_\bullet) \\
 & & \xrightarrow{\delta_1} & & \\
H_0(A_\bullet) & \xrightarrow{f_0} & H_0(B_\bullet) & \xrightarrow{g_0} & H_0(C_\bullet) \\
 & & \xrightarrow{\delta_0} & & \\
\vdots & & \vdots & & \vdots \quad \vdots
\end{array}
$$

4 Cochain Complex and Cohomology

Cochain complexes and cohomologies are nothing special but chain complexes and homologies with arrows reversed.

Definition 4.1. A *cochain complex* $C^\bullet$ consists of a sequence of R-modules $\{C^i\}$ and *coboundary homomorphisms* $\{d^i\}$ looking like

$$\cdots \xrightarrow{d^3} C^2 \xrightarrow{d^2} C^1 \xrightarrow{d^1} C^0 \xrightarrow{d^0} C^{-1} \xrightarrow{d^{-1}} C^{-2} \xrightarrow{d^{-2}} \cdots$$

such that

$$d^i \circ d^{i+1} = 0, \forall i.$$

The condition is called the *coboundary condition*.

Definition 4.2. Given a cochain complex $C^\bullet$. The cohomology of this cochain complex, denoted $H^\bullet(C^\bullet)$ is defined as

$$H^i(C^\bullet) = \ker d^i / \operatorname{im} d^{i+1}$$