REU Algebraic Topology Assignment 2

Problem 1.
1. Prove that $D^n/\partial D^n$ is homeomorphic to S^n.
2. Prove that $(A/\partial A)\wedge(B/\partial B)$ is homeomorphic to $(A\times B)/\partial(A\times B)$. Use this to prove that $S^n \wedge S^m$ is homeomorphic to S^{n+m}.

Problem 2. Find a general formula for $\mathbb{Z}/m\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$.

Problem 3. Let $f, g \in k\,[x]$. Compute $k[x]/(f)\otimes k[x]/(g)$.

Problem 4. Prove that there is a bijective correspondence between bilinear maps from $M \times N$ to P and homomorphisms from $M \otimes N$ to P.

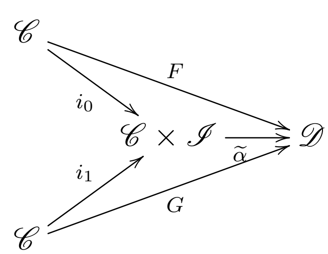
Problem 5. Prove the following properties for R-modules.

1. $R\otimes_R M \cong M \cong M \otimes_R R$.
2. $(M\otimes_R N)\otimes_R P \cong M\otimes_R (N\otimes_R P)$.
3. $M\otimes_R N \cong N\otimes_R M$.

Problem 6. Let G be a finite abelian group. Prove that $\mathbb{Q}\otimes_{\mathbb{Z}} G = 0$. Moreover, prove that if G is an abelian group, then $\mathbb{Q}\otimes_{\mathbb{Z}} G$ is a rational vector space and that a rational vector space is a free abelian group.

Problem 7. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Suppose f is an isomorphism in $\mathscr{C}$. Prove that Ff is an isomorphism in $\mathscr{D}$.

Problem 8. Define $\mathscr{I}$ to be the category $\underline{\mathbf{1}} = \{0 \to 1\}$. Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. Show that defining a natural transformation $\alpha : F \Rightarrow G$ is equivalent to defining a functor $\widetilde{\alpha} : \mathscr{C}\times\mathscr{I} \to \mathscr{D}$ such that the following diagram commutes.



Note that i_0, i_1 sends an object C in $\mathscr{C}$ to $(C,0)\,,(C,1)$ in $\mathscr{C}\times\mathscr{I}$, respectively.

Problem 9. Fix a set X. Define a functor $X_* : \mathbf{Set} \to \mathbf{Set}$ as

$$X_*(A) = \mathbf{Set}\,(X, A)\,.$$

Note that given a function $f : A \to B$, there is a natural function from $\mathbf{Set}(X, A)$ to $\mathbf{Set}(X, B)$ defined by the evident composition. Prove that X_* is a **covariant functor**.

Problem 10. Fix a set X. Define a functor $X^* : \mathbf{Set} \to \mathbf{Set}$ as

$$X^*(A) = \mathbf{Set}\,(A, X)\,.$$

Note that given a function $f : A \to B$, there is a natural function from $\mathbf{Set}(B, X)$ to $\mathbf{Set}(A, X)$ defined by the evident composition. Prove that X^* is a **contravariant functor**. Also prove that $X^*(A)$ is naturally isomorphic to the cartesian product A^X of copies of A indexed by the elements of X.