Problem 1. 1. Prove that $D^n/\partial D^n$ is homeomorphic to $S^n$.

2. Prove that $(A/\partial A) \wedge (B/\partial B)$ is homeomorphic to $(A \times B)/\partial(A \times B)$. Use this to prove that $S^n \wedge S^m$ is homeomorphic to $S^{n+m}$.

Problem 2. Find a general formula for $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$.

Problem 3. Let $f, g \in k[x]$. Compute $k[x]/(f) \otimes k[x]/(g)$.

Problem 4. Prove that there is a bijective correspondence between bilinear maps from $M \times N$ to $P$ and homomorphisms from $M \otimes N$ to $P$.

Problem 5. Prove the following properties for $R$-modules.

1. $R \otimes_R M \cong M \cong M \otimes_R R$.

2. $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$.

3. $M \otimes_R N \cong N \otimes_R M$.

Problem 6. Let $G$ be a finite abelian group. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} G = 0$. Moreover, prove that if $G$ is an abelian group, then $\mathbb{Q} \otimes_{\mathbb{Z}} G$ is a rational vector space and that a rational vector space is a free abelian group.

Problem 7. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Suppose $f$ is an isomorphism in $\mathcal{C}$. Prove that $Ff$ is an isomorphism in $\mathcal{D}$.

Problem 8. Define $\mathcal{I}$ to be the category $\mathbf{1} = \{0 \rightarrow 1\}$. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. Show that defining a natural transformation $\alpha : F \Rightarrow G$ is equivalent to defining a functor $\tilde{\alpha} : \mathcal{C} \times \mathcal{I} \rightarrow \mathcal{D}$ such that the following diagram commutes.

\[ \begin{array}{ccc} \mathcal{C} & \xrightarrow{i_0} & \mathcal{C} \\ \downarrow F \downarrow & & \downarrow \tilde{\alpha} \downarrow \mathcal{C} \\ \mathcal{C} \times \mathcal{I} & \xrightarrow{i_1} & \mathcal{D} \end{array} \]

Note that $i_0, i_1$ sends an object $C$ in $\mathcal{C}$ to $(C, 0), (C, 1)$ in $\mathcal{C} \times \mathcal{I}$, respectively.

Problem 9. Fix a set $X$. Define a functor $X_* : \text{Set} \rightarrow \text{Set}$ as

$X_*(A) = \text{Set}(X, A)$.

Note that given a function $f : A \rightarrow B$, there is a natural function from $\text{Set}(X, A)$ to $\text{Set}(X, B)$ defined by the evident composition. Prove that $X_*$ is a covariant functor.
Problem 10. Fix a set $X$. Define a functor $X^* : \text{Set} \to \text{Set}$ as

$$X^*(A) = \text{Set}(A, X).$$

Note that given a function $f : A \to B$, there is a natural function from $\text{Set}(B, X)$ to $\text{Set}(A, X)$ defined by the evident composition. Prove that $X^*$ is a contravariant functor. Also prove that $X^*(A)$ is naturally isomorphic to the cartesian product $A^X$ of copies of $A$ indexed by the elements of $X$.