REU Algebraic Topology Assignment

Problem 1. Prove that the following is a homotopy between $p * p^{-1}$ and $id_\star$.

$$H(s,t) = \begin{cases} p(2s-t) & \text{if } t/2 \leq s \leq 1/2 \\ p(2-t-2s) & \text{if } 1/2 \leq s \leq 1-t/2 \\ \star & \text{otherwise .} \end{cases}$$

Problem 2. Prove that the "homotopic" relation between functions from X to Y is an equivalence relation.

Problem 3. Prove that the "homotopy equivalent" relation between two spaces is an equivalence relation.

Problem 4. Prove that $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is contractible.

Problem 5. Prove that X and $X \times [0,1]$ are homotopy equivalent.

Problem 6. Prove the following "compatibility" properties. All the p's and q's are assumed to be loops at $\star \in X$. Let $f, g : X \to Y$ be continuous functions.

1. If $p_1 \simeq p_2$ and $q_1 \simeq q_2$, then $p_1 * q_1 \simeq p_1 * q_2$.
2. If $p \simeq q$, then $f \circ p \simeq f \circ q$.
3. $f \circ (p * q) \simeq (f \circ p) * (f \circ q)$.
4. If $f \simeq g$, then $f \circ p \simeq g \circ p$.

Problem 7. Prove that if $f : (X, \star_X) \to (Y, \star_Y)$ and $g : (Y, \star_Y) \to (X, \star_X)$ give a homotopy equivalence, then $f_* : \pi_1(X, \star_X) \to \pi_1(Y, \star_Y)$ is a group isomorphism.

Problem 8. A *topological group* G is a topological space with a group structure, such that the multiplication $\cdot : G \times G \to G$ and the inverse $(-)^{-1} : G \to G$ are both continuous. Prove that if G is a topological group, then $\pi_1(G, e)$ is abelian.