U CHICAGO REU CALCULUS OF VARIATIONS PROBLEM SET 3

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Throughout this problem set, E is an open, connected subset of $\mathbb{R}^n$ with ∂E smooth.

Problem 1: Let $1 \leq p < q \leq \infty$.

i) Show that $L^q(E) \subset L^p(E)$.
ii) Show by example that $L^p(\mathbb{R}^n) \not\subset L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n) \not\subset L^p(\mathbb{R}^n)$.

Problem 2: Let $\lambda \in \mathbb{R}$, and define

$$f(x) = \|x\|^\lambda.$$

For what values of n, p, λ is f in $W^{1,p}(B(0,1))$, where $B(0,1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$? For what values of n, p, λ is f in $W^{1,p}(\mathbb{R}^n \setminus B(0,1))$?

Problem 2: Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$. As mentioned in class, V is also a normed vector space with norm $\|v\|_V = \sqrt{\langle v, v \rangle}$. Show that this norm satisfies the parallelogram law: for any $v, w \in V$,

$$2\|v\|^2 + 2\|w\|^2 = \|v + w\|^2 + \|v - w\|^2.$$

With E as in Problem 1, conclude that $L^p(E)$ cannot be given an inner product structure (and therefore is not a Hilbert space) for any $p \neq 2$.

Problem 3: Let H be a Hilbert space. Recall that a sequence u_n in H converges weakly to $u \in H$ if $\langle u_n, v \rangle \to \langle u, v \rangle$ for all $v \in H$.

i) Show that if u_n converges strongly to u, then u_n converges weakly to u.
ii) If $H = \mathbb{R}^n$, show that strong and weak convergence are equivalent.
iii) Show that if u_n converges to u weakly, and $\|u_n\| \to \|u\|$, then u_n converges to u strongly.

Problem 4: Consider the functional $J(u) = \int_E |\nabla u|^2 \, dx$.

i) Show that J is lower-semicontinuous with respect to strong convergence in $H^1(E)$, i.e. $J(u_0) \leq \liminf_{n\to\infty} J(u_n)$ for any sequence u_n converging (in the strong H^1 sense) to u_0.
ii) Show that J is convex, i.e. $J(tu + (1-t)v) \leq tJ(u) + (1-t)J(v)$ for any $u, v \in H^1(E)$ and $t \in [0,1]$.

(It is a theorem that these two properties imply J is lower-semicontinous with respect to weak convergence in $H^1(E)$, which is needed to prove existence of a minimizer.)

Problem 5 (Hard): Recall that $\Delta u = \displaystyle\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.

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i) Let $f \in C^2(E)$ satisfy $\Delta f(x) = 0$ for every $x \in E$ (in other words, f is *harmonic*). Show that f satisfies the following mean value property: for every ball $B(x_0, r) \subset E$,

$$f(x_0) = \frac{1}{\omega_n r^n} \int_{B(x_0,r)} f(x)\,\mathrm{d}x = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} f(x)\,\mathrm{d}\sigma(x),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

(Hint: Show that the quantity

$$\frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B(x_0,\rho)} f(x)\,\mathrm{d}\sigma(x)$$

is constant in ρ for $0 < \rho \le r$.)

ii) Let $u \in L^1(E)$ satisfy

$$\int_E u(x)\Delta\phi(x)\,\mathrm{d}x = 0 \qquad \text{for all } \phi \in C_0^\infty(E).$$

Prove that $u \in C^\infty(E)$.

(Hint: Define

$$u_h(x) = \frac{1}{h^n} \int_E \psi\left(\frac{|x-y|}{h}\right) u(y)\,\mathrm{d}y,$$

where h is a small positive number and ψ is a smooth function on the positive real line such that $\psi(t) \ge 0$ for all t, $\psi(t) = 0$ for $t \ge 1$, and $\int_{B(0,1)} \psi(|x|)\,\mathrm{d}x = 1$. Show that u_h are smooth and harmonic, and that $\|u_h\|_{L^1(E)}$ is bounded uniformly in h. Use i) to estimate the gradient of u_h, and repeat the argument to bound all higher-order derivatives of u_h, such that the bounds are uniform in h. Conclude that u_h converges to a C^∞ function v and argue that $u = v$.)

iii) Using ii), show that if u minimizes the functional J from Problem 4 over the class $\{u \in H^1(E) : u - g \in H_0^1(E)\}$, where $g \in C^\infty(\bar{E})$, then $\Delta u = 0$ in E.

Problem 6: Finish proving the key lemma in our solution to the Sturm-Liouville problem. That is, let $P \in C^1[a,b]$ and $Q \in C[a,b]$ and $f \in C[a,b]$ be such that

$$\int_a^b f(x)(-P(x)h'(x))' + Q(x)f(x)h(x)dx = 0,$$

for all $h \in C^2[a,b]$ with $h(a) = h(b) = h'(a) = h'(b) = 0$. Then, $f \in C^2[a,b]$ and $(-P(x)f'(x))' + Q(x)f(x) = 0$ for all $x \in [a,b]$.

Problem 7: Lets prove the claim from class that the ODE,

$$(-P(x)u'(x))' + Q(x)u(x) = \lambda u(x), \tag{0.1}$$

has at most one solution on $[a,b]$ with $u(a) = u(b) = 0$ and $\int_a^b u^2(x)dx = 1$. Recall that $P \in C^1[a,b]$ satisfies $P(x) > 0$ and that $Q \in C[a,b]$.

i) Let $u, \tilde{u}$ be two solutions to (0.1). Define the *Wronskian* to be $u'\tilde{u} - \tilde{u}'u$. Find an equation for $-P(x)\frac{d}{dx}W(x)$.

ii) Prove that the Wronskian is a constant multiple of the function P. HINT: Use the equation you found in part (i).

iii) Prove that the Wronskian is identically 0.

iv) Conclude that there is a unique solution to (0.1) with zero boundary values and square integral equal to one.

Problem 8: Use the Ritz method to approximate the minimum of the functional

$$J[y] = \int_0^1 [y']^2 - y^2 - 2xy dx, \;\; y(0) = y(1) = 0. \tag{0.2}$$

Can solve the Euler-Lagrange equations to find the actual minimum of (0.2)? Does the minimizing sequence you found using the Ritz method converge to the minimum you found using the E-L equations?

HINT: For the Ritz method, consider choosing the sequence of functions $x(1-x), x^2(1-x), x^3(1-x), \ldots$. For extra challenge, prove that this sequence spans the relevant space.