U CHICAGO REU CALCULUS OF VARIATIONS PROBLEM SET 2

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Problem 1: Find the critical points of the functional

$$J(f) = \int_0^1 ((f'(x))^2 + x^2)\,\mathrm{d}x$$

subject to the conditions

$$f(0) = 0, \quad f(1) = 0, \quad \int_0^1 (f(x))^2\,\mathrm{d}x = 2.$$

Problem 2: Consider a particle of mass m moving in $\mathbb{R}^2$ and attracted to the origin with force $F(x,y) = -\dfrac{C}{x^2+y^2}$, where C is a constant.

i) Use the principle of least action to find a functional $J(r,\theta)$ that is minimized by the path $(r(t),\theta(t))$ of the particle in polar coordinates. Find the corresponding equations of motion.
ii) Verify that the functional J is invariant under rotations of the plane, and use Noether's Theorem (in polar coordinates) to find the corresponding conservation law. What geometric fact does this law express?

Problem 3: Consider functions $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}^2$ is a smoothly bounded region. Let

$$J(f) = \int_E \sqrt{1 + \|\nabla f\|^2}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\|\nabla f\|^2 = \left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial x}\right)^2$.

i) Let g be a smooth function defined on ∂E. Show that the Euler-Lagrange equation corresponding to minimizing J over the class of continuously differentiable functions on E such that $f(x) = g(x)$ for $x \in \partial E$ is

$$\frac{d}{dx}\left(\frac{\partial f/\partial x}{\sqrt{1+\|\nabla f\|^2}}\right) + \frac{d}{dy}\left(\frac{\partial f/\partial y}{\sqrt{1+\|\nabla f\|^2}}\right) = 0.$$

You can do this directly, or use the general form of the Euler-Lagrange equation we proved in class. If you prove it directly, you will need a lemma analogous to Lemma 1 from Tuesday's lecture, but that applies for functions defined on subsets of $\mathbb{R}^2$.
ii) If $g(x) = 0$, find the minimizer f of $J(f)$ by pure thought.
iii) A *minimizing sequence* is a sequence f_n of admissible functions (in this case, $f_n \in C^1(E)$ and $f_n = g$ on ∂E) such that $J(f_n) \to \inf J(f)$, the infimum taken over all admissible

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functions. Show by example that if E is the unit circle and $g = 0$, a minimizing sequence f_n does not necessarily converge pointwise to the minimizer f.

(Remark: This issue is a major challenge in "direct methods" of calculus of variations, because one often wants to prove existence of a minimizer by taking the limit of a minimizing sequence. But as this example shows, the minimizing sequence may not converge to a minimizer, and can even fall out of the function space being minimized over. We will explore this and related issues in the second week of lectures.)

Problem 4: A *geodesic* on a surface is the shortest path contained in that surface connecting the two points on that surface. Given two points $(x_0, y_0, z_0), (x_1, y_1, z_1) \in \mathbb{S}^2$ find the geodesic between them. HINT: Set this up as a constrained optimization problem.

Problem 5: Find the shortest curve γ in $\mathbb{R}^2$, connecting $(a, 0)$ and $(-a, 0)$, such that the area between γ and the x-axis is equal to one.