Problem 1: Find the critical points of the functional

\[ J(f) = \int_0^1 \left( (f'(x))^2 + x^2 \right) dx \]

subject to the conditions

\[ f(0) = 0, \quad f(1) = 0, \quad \int_0^1 (f(x))^2 dx = 2. \]

Problem 2: Consider a particle of mass \( m \) moving in \( \mathbb{R}^2 \) and attracted to the origin with force \( F(x, y) = -C \frac{x^2 + y^2}{x^2 + y^2} \), where \( C \) is a constant.

i) Use the principle of least action to find a functional \( J(r, \theta) \) that is minimized by the path \( (r(t), \theta(t)) \) of the particle in polar coordinates. Find the corresponding equations of motion.

ii) Verify that the functional \( J \) is invariant under rotations of the plane, and use Noether’s Theorem (in polar coordinates) to find the corresponding conservation law. What geometric fact does this law express?

Problem 3: Consider functions \( f : E \to \mathbb{R} \), where \( E \subset \mathbb{R}^2 \) is a smoothly bounded region. Let

\[ J(f) = \int_E \sqrt{1 + \| \nabla f \|^2} dx dy, \]

where \( \| \nabla f \|^2 = \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \).

i) Let \( g \) be a smooth function defined on \( \partial E \). Show that the Euler-Lagrange equation corresponding to minimizing \( J \) over the class of continuously differentiable functions on \( E \) such that \( f(x) = g(x) \) for \( x \in \partial E \) is

\[ \frac{d}{dx} \left( \frac{\partial f/\partial x}{\sqrt{1 + \| \nabla f \|^2}} \right) + \frac{d}{dy} \left( \frac{\partial f/\partial y}{\sqrt{1 + \| \nabla f \|^2}} \right) = 0. \]

You can do this directly, or use the general form of the Euler-Lagrange equation we proved in class. If you prove it directly, you will need a lemma analogous to Lemma 1 from Tuesday’s lecture, but that applies for functions defined on subsets of \( \mathbb{R}^2 \).

ii) If \( g(x) = 0 \), find the minimizer \( f \) of \( J(f) \) by pure thought.

iii) A minimizing sequence is a sequence \( f_n \) of admissible functions (in this case, \( f_n \in C^1(E) \) and \( f_n = g \) on \( \partial E \)) such that \( J(f_n) \to \inf J(f) \), the infimum taken over all admissible
functions. Show by example that if $E$ is the unit circle and $g = 0$, a minimizing sequence $f_n$ does not necessarily converge pointwise to the minimizer $f$.

(Remark: This issue is a major challenge in “direct methods” of calculus of variations, because one often wants to prove existence of a minimizer by taking the limit of a minimizing sequence. But as this example shows, the minimizing sequence may not converge to a minimizer, and can even fall out of the function space being minimized over. We will explore this and related issues in the second week of lectures.)

**Problem 4:** A *geodesic* on a surface is the shortest path contained in that surface connecting the two points on that surface. Given two points $(x_0, y_0, z_0), (x_1, y_1, z_1) \in \mathbb{S}^2$ find the geodesic between them. HINT: Set this up as a constrained optimization problem.

**Problem 5:** Find the shortest curve $\gamma$ in $\mathbb{R}^2$, connecting $(a, 0)$ and $(-a, 0)$, such that the area between $\gamma$ and the $x$-axis is equal to one.