U CHICAGO REU CALCULUS OF VARIATIONS PROBLEM SET 1

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1. Function Spaces and Functionals

These problems are meant to give you some familiarity with function spaces and functionals. If you have not taken analysis, you may find them difficult but please don't be discouraged. You won't need them for class.

Problem 1: Recall that a set X is *sequentially compact* if for all $\{x_n\} \subset X$ there is a subsequence $\{x_{n_i}\}$ which converges to an element in X.

i) Show that $[0,1]$ is sequentially compact (this is the Heine-Borel theorem).
ii) Show that the set

$$B \equiv \{f \in C[0,1] \mid |f(x)| \leq 1, \forall x \in [0,1]\}$$

is not a sequentially compact subset of $C[0,1]$. HINT: Find a sequence of bounded continuous functions which does not converge to a bounded continuous function.

Problem 2: We say that a functional $\varphi : C^1[0,1] \to V$ (where V is some vector space) is linear if $\varphi(f+g) = \varphi(f)+\varphi(g)$ and $\varphi(\lambda f) = \lambda\varphi(f)$, Verify whether the following functionals are linear:

i) $\varphi(f) = f(1/2)$ (here V is $\mathbb{R}$),
ii) $\varphi(f) = \frac{d}{dx}f$ (here V is $C[0,1]$)
iii) $\varphi(f) = x^2 f(x)$ (here V is $C^1[0,1]$).
iv) $\varphi(f) = f(x)^2$ (here V is $C^1[0,1)$.

Problem 3: We say that a functional J is continuous with respect to $C^1[0,1]$ if for all $f \in C^1[0,1]$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|f-g\|_{C^1} < \delta$ then $|J(f)-J(g)| < \varepsilon$. We can similarly define continuity with respect to the $C[0,1]$ norm.

Show that the arc length functional: $J(f) = \int_0^1 \sqrt{1+[f']^2}dx$, is continuous with respect to the $C^1[0,1]$ norm but not continuous with respect to the $C[0,1]$ norm.

Problem 4: A function $f : \mathbb{R} \to \mathbb{R}$ is lower semi-continuous if for all $x \in \mathbb{R}$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x-y| < \delta$ then $f(x) - f(y) > -\varepsilon$.

i) Give an example of a function which is lower semicontinuous on $\mathbb{R}$ but NOT continuous.
ii) Show that the arc-length function J (defined above) is lower semi-continuous with respect to the $C[0,1]$ norm.

Problem 5: Show that if a functional J has derivative equal to zero at every $\psi \in C^1[0,1]$ then J is constant.

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2. Euler Lagrange Equations

Problem 6: Prove Lemma 3 from class. That if $f \in C[0,1]$ and for every $\alpha(x) \in C^2[0,1]$ with $\alpha'(0) = \alpha'(1) = \alpha(0) = \alpha(1) = 0$ we have

$$\int_0^1 f(x)\alpha''(x)dx = 0,$$

then $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Problem 7: Prove Lemma 4 from class. That if $f, g \in C[0,1]$ are such that for every $\alpha \in C^1[0,1]$ with $\alpha(0) = \alpha(1) = 0$ we have

$$\int_0^1 f(x)\alpha(x) + g(x)\alpha'(x)dx = 0,$$

then g is differentiable and $\frac{d}{dx}g = f$.

Recall from class that if $J[\psi] = \int_0^1 F(x, \psi, \psi')dx$ then the Euler Equation for J is

$$\frac{\partial}{\partial \beta}F + \frac{d}{dx}\frac{\partial}{\partial \gamma}F = 0,$$

where $F = F(\alpha, \beta, \gamma)$.

A notational note: when we write ∂_β or $\frac{\partial}{\partial \alpha}$ or something like that, we are taking the derivative of F assuming that the three variables, α, β, γ are independent. On the other hand, when we write $\frac{d}{dx}$, $\frac{d}{d\psi}$ or $\frac{d}{d\psi'}$ we are taking derivatives assuming interdependence. For example, if $F(x, \psi, \psi') = 1 + [\psi']^2$ then $\partial_\alpha F = 0$ but $\frac{d}{dx}F = 2\psi'\psi''$, since ψ and ψ' depend on x.

Problem 8: Let $\rho(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function.

i) Find the general form of the Euler-Lagrange equations for $J[\psi] = \int_0^1 \rho(x, \psi(x))\sqrt{1 + [\psi']^2}dx$.
ii) What if ρ is independent of the x coordinate? What if it is independent of the y coordinate?
iii) Find minimizers of J such that $\psi(0) = a$ and $\psi(1) = b$ when $\rho(x, y) = y$. These are the curves which generate the surface of minimum area when rotated over the x-axis.
iv) Find minimizers of J such that $\psi(0) = a$ and $\psi(1) = b$ when $\rho(x, y) = \frac{1}{\sqrt{y}}$ (this is supposed to mimic a particle falling under the influence of gravity).
v) Find minimizers of J such that $\psi(0) = a$ and $\psi(1) = b$ (both $a, b > 0$) when $\rho(x, y) = \frac{1}{y^2}$ (these are geodesics in the hyperbolic metric on the upper half plane).

Problem 9: Find the Euler equations for the following functionals, and if you can, solve the Euler equations to find the critical points in $C^1([0,1])$ of the functionals.

i) $\displaystyle\int_0^1 (f^2 + (f')^2 - 2f\sin x)dx.$
ii) $\displaystyle\int_0^1 \frac{(f')^2}{x^3}dx.$

(Note: Since we have not specified any boundary conditions at $x = 0$ and $x = 1$, each answer will involve two integration constants. If we specified $f(0)$ and $f(1)$, the critical points would be unique.)