ENTROPY AND KINETIC FORMULATIONS OF CONSERVATION LAWS

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Abstract. Entropy and kinetic formulations of conservation laws are introduced, in order to create a well-posed theory. Existence and uniqueness results are proven for both formulations. Convergence of approximations and compactness of solutions are also proven using the kinetic formulation.

Contents

1. Introduction

Conservation laws are prevalent in physics and are some of the most basic examples of nonlinear first-order PDEs. It is known that many conservation laws lack classical solutions for all time, even if the initial data is smooth. A less restrictive notion of solution is thus needed for conservation laws to be well-posed. In this paper, we will explore two alternative equivalent notions of solutions: entropy solutions and solutions to the kinetic formulation, or kinetic solutions. We will prove that both notions lead to conservation laws being well-posed, and using the kinetic formulation we will also derive results concerning the convergence of solutions to approximations of conservation laws, as well as results concerning the compactness of a family of entropy solutions.

We assume the reader is familiar with Lebesgue integration and basic aspects of the theory of distributions.

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2. Kruzkov's entropy formulation

2.1. **Distributional solutions.** It is well known (see [1]) that the problem $u_t + \text{div}_x(A(u)) = 0$ may not have classical solutions for all positive time, even if A and the initial data u^0 are both smooth. In order to provide a well-posed theory of conservation laws, we introduce alternative notions of solutions in order to obtain the existence and uniqueness of solutions.

Definition 2.1. Let $T > 0$, and denote $\pi_T = [0, T] \times \mathbb{R}^d$. Let $A : \mathbb{R} \to \mathbb{R}^d$ be C^1, and let $a = A'$. We say that a bounded locally integrable function $u : \pi_T \to \mathbb{R}$ is a distributional solution to the problem

$$u_t + \text{div}_x(A(u)) = 0, \qquad u(0, x) = u^0(x) \tag{2.1}$$

if it satisfies the above equations in the sense of distributions, i.e. if for all $f \in C_C^\infty([0, T) \times \mathbb{R}^d)$ we have

$$\int_{\pi_T} u f_t + A(u) \cdot \nabla_x f \ \text{d}t \ \text{d}x + \int_{\mathbb{R}^d} u^0(x) f(0, x) \ \text{d}x = 0.$$

For the rest of this section, we will use $C_C^\infty(\pi_T)$ to denote C^∞ functions with compact support strictly inside $(0, T) \times \mathbb{R}^d$; in particular, these functions are zero at times $t = 0$ and $t = T$.

Note that all classical (i.e. C^1) solutions to the PDE are distributional solutions, as can be seen from integrating the equation by parts against any smooth test function. Furthermore, any distributional solution which is C^1 is also a classical solution. In addition, the integral equality in the definition of the distributional solution yields the following condition on the discontinuities of u, known as the Rankine-Hugoniot condition:

Proposition 2.2. *Suppose u is a distributional solution to (2.1) and is C^1 in π_T everywhere except along a finite number of C^1 surfaces, where no two surfaces intersect on a set of positive measure. Let (t, x) be a point belong to exactly one of the above curves. If $\hat{n}$ is the normal vector to the surface at (t, x), then*

$$(u_1 - u_2, A(u_1) - A(u_2)) \cdot \hat{n} = 0$$

where u_1 and u_2 are the limits of u approaching (t, x) from either side of the discontinuity surface. In particular, if $d = 1$, and the discontinuity curve is parametrized by $x = x(t)$, then

$$\dot{x}(t)(u_1 - u_2) = A(u_1) - A(u_2).$$

Conversely, if u is a function which satisfies the PDE in (2.1) at every point in π_T except along a finite number of C^1 surfaces, with no two surfaces intersecting on a set of positive measure, and each non-intersection point of the surfaces satisfies the above relation, then u is a distributional solution to (2.1).

Proof. We prove this in one dimension for simplicity. Let u be a distributional solution, and assume first that we only have one curve of discontinuity C. Let L and R be the regions to the left and right, respectively, of C in the (t, x) plane, and let u_1 and u_2 be the respective limits. Since u is smooth in L and R, we know that $u_t + \text{div}_x(A(u)) = 0$ in the interior of L and R. By the divergence theorem, we have

$$\int_L u f_t + A(u) f_x \ \text{d}t \ \text{d}x = \int_C (u_1 f, A(u_1) f) \cdot \hat{n}_l \ \text{d}s - \int_L f(u_t + \text{div}_x(A(u))) \ \text{d}t \ \text{d}x = \int_C (u_1 f, A(u_1) f) \cdot \hat{n}_1 \ \text{d}s$$

for all $f \in C_C^\infty(\pi_T)$. Similarly, we have

$$\int_R u f_t + A(u) f_x \ \text{d}t \ \text{d}x = \int_C (u_2 f, A(u_2) f) \cdot \hat{n}_2 \ \text{d}s.$$

We thus have

$$0 = \int\limits_L uf_t + A(u)f_x + \int\limits_R uf_t + A(u)f_x = \int\limits_C f(u_1, A(u_1)) \cdot \hat{n}_1 + (u_2, A(u_2)) \cdot \hat{n}_2 \,\mathrm{d}s).$$

Letting $\hat{n} = (\hat{t}, \hat{x})$ with $\hat{x} > 0$, we have $\hat{n}_1 = -\hat{n}_2 = \hat{n}$, and $\dot{x}(t) = -\frac{\hat{t}}{\hat{x}}$. We have

$$\int\limits_C f((u_1 - u_2)\hat{t} + (A(u_1) - A(u_2))\hat{x}) \,\mathrm{d}s = 0.$$

Since this holds for all f, it follows that

$$(u_1 - u_2)\hat{t} + (A(u_1) - A(u_2))\hat{x} = 0 \implies \dot{x}(t)(u_1 - u_2) = -\frac{\hat{t}}{\hat{x}}(u_1 - u_2) = A(u_1) - A(u_2),$$

thus proving the relation in the case of one curve. For the case of multiple curves, we adapt the above proof by testing with f whose support does not intersect any of the other curves.

The converse also follows easily from the above calculations, by noting that every point along a single curve of discontinuity satisfying the above relation will also satisfy the relation $(u_1 - u_2)\hat{t} + (A(u_1) - A(u_2))\hat{x} = 0$. Denoting the curves by $\{C_i\}$, we thus have

$$\sum_i \int\limits_{C_i} f[(u_1 - u_2)\hat{t} + (A(u_1) - A(u_2))\hat{x}] \,\mathrm{d}s = 0.$$

If we denote the regions separated by the curves by $\{R_j\}$, we have

$$\begin{aligned}\int\limits_{\pi_T} uf_t + A(u)f_x \,\mathrm{d}t\,\mathrm{d}x &= \sum_j \int\limits_{R_j} uf_t + A(u)f_x \,\mathrm{d}t\,\mathrm{d}x \\ &= \sum_j \left(\int\limits_{\partial R_j} (uf, A(u)f) \cdot \hat{n} \,\mathrm{d}s - \int\limits_{R_j} f(u_t + \mathrm{div}_x(A(u))) \,\mathrm{d}t\,\mathrm{d}x \right) \\ &= \sum_i \int\limits_{C_i} f[(u_1 - u_2)\hat{t} + (A(u_1) - A(u_2))\hat{x}] \,\mathrm{d}s - 0 = 0,\end{aligned}$$

thus implying that u is a distributional solution. □

While the distributional notion of a solution does not require the solution to be C^1 and hence admits a larger class of possible solutions, it has the disadvantage of admitting *too many* solutions, in that solutions are not generally unique. For example, let $A(u) = \frac{u^2}{2}$ on $(0, T) \times \mathbb{R}$. (The corresponding PDE $u_t + uu_x = 0$ is commonly known as Burgers' equation.) The Rankine-Hugoniot conditions require the curves of discontinuity to satisfy

$$\dot{x}(t)(u_1 - u_2) = A(u_1) - A(u_2) = \frac{(u_1 - u_2)(u_1 + u_2)}{2} \implies \dot{x}(t) = \frac{u_1 + u_2}{2}.$$

For the initial data $u^0 \equiv 0$, it is clear that $u \equiv 0$ is a distributional solution to the problem. However, consider the function

$$u_1(t, x) = \begin{cases} 1 & 0 < x < \frac{t}{2} \\ -1 & -\frac{t}{2} < x < 0 \\ 0 & \text{otherwise} \end{cases}.$$

The initial data for u_1 is also $u^0 \equiv 0$. It has three curves of discontinuity (namely at $x = 0$ and $x = \pm\frac{t}{2}$), and it is easy to check that the Rankine-Hugoniot condition is satisfied along each curve. It follows that u_1 is also a distributional solution to Burgers' equation with $u^0 \equiv 0$, showing that the problem does not admit a unique solution for $u^0 \equiv 0$.

2.2. **Entropy solutions and equivalent notions.** To avoid the issue of non-uniqueness, we introduce a different notion of solution more restrictive than simply satisfying the equation in the sense of distributions.

Definition 2.3. We say that a bounded locally integrable function $u : \pi_T \to \mathbb{R}$ is an entropy solution of the problem

$$u_t + \mathrm{div}_x(A(u)) = 0, \qquad u(0,x) = u^0(x)$$

on π_T if, for all $k \in \mathbb{R}$, we have the inequality

$$\frac{\partial}{\partial t}(|u-k|) + \mathrm{sgn}(u-k)\mathrm{div}_x(A(u)) \leq 0 \tag{2.2}$$

in the sense of distributions, i.e. for any nonnegative $f \in C_C^\infty(\pi_T)$ we have

$$\int\limits_{\pi_T} |u(t,x)-k|f_t + \mathrm{sgn}(u(t,x)-k)(A(u(t,x)) - A(k)) \cdot \nabla_x f \ \mathrm{d}x \ \mathrm{d}t \geq 0, \tag{2.3}$$

and there exists $E \subset [0,T]$ such that $[0,T]\backslash E$ has measure zero, the function $u(t,\cdot)$ is defined almost everywhere in $\mathbb{R}^d$ for all $t \in E$, and for any ball B_R we have

$$\lim_{t\to 0, t\in E} \int\limits_{B_R} |u(t,x) - u^0(x)| \ \mathrm{d}x = 0. \tag{2.4}$$

Remark 2.4. To see that (2.2) and (2.3) are equivalent, integrate (2.2) by parts against any nonnegative $f \in C_C^\infty(\pi_T)$, noting that

$$\begin{aligned} \mathrm{div}_x(\mathrm{sgn}(u-k)(A(u)-A(k))) &= \mathrm{sgn}(u-k)\mathrm{div}_x(A(u)-A(k)) + \nabla_x(\mathrm{sgn}(u-k)) \cdot (A(u)-A(k)) \\ &= \mathrm{sgn}(u-k)\mathrm{div}_x(A(u)) + 2\delta(u-k)\nabla_x u \cdot (A(u)-A(k)) \\ &= \mathrm{sgn}(u-k)\mathrm{div}_x(A(u)), \end{aligned}$$

where δ is the Dirac delta. The last equality following from the fact that $\delta(u-k) = 0$ whenever $A(u) - A(k) \neq 0$.

We first show that all entropy solutions are distributional solutions, showing that the notion of entropy solutions is more strict than that of distributional solutions.

Proposition 2.5. *Let $u \in L^\infty(\pi_T)$ satisfy (2.3). Then u is a distributional solution to the problem (2.1).*

Proof. Let $f \in C_C^\infty(\pi_T)$ and $k \in \mathbb{R}$. Choosing $k > \|u\|_{L^\infty}$, we have $|u-k| = k-u$ and $\mathrm{sgn}(u-k) = -1$, and hence (2.3) becomes

$$\int\limits_{\pi_T} (k-u)f_t - (A(u)-A(k)) \cdot \nabla_x f \ \mathrm{d}x \ \mathrm{d}t \geq 0.$$

Since $f \in C_C^\infty(\pi_T)$, we have $\int_{\pi_T} f_t = 0$ and $\int_{\pi_T} \nabla_x f = 0$. Thus we have

$$\int\limits_{\pi_T} kf_t + A(k) \cdot \nabla_x f \ \mathrm{d}x \ \mathrm{d}t = 0$$

and hence

$$\int\limits_{\pi_T} -uf_t - A(u) \cdot \nabla_x f \ \mathrm{d}x \ \mathrm{d}t \geq 0 \implies \int\limits_{\pi_T} uf_t + A(u) \cdot \nabla_x f \ \mathrm{d}x \ \mathrm{d}t \leq 0.$$

Similarly, choosing $k < -\|u\|_{L^\infty}$ yields

$$\int\limits_{\pi_T} (u-k)f_t + (A(u) - A(k)) \cdot \nabla_x f \, \mathrm{d}x \, \mathrm{d}t \geq 0 \implies \int\limits_{\pi_T} uf_t + A(u) \cdot \nabla_x f \, \mathrm{d}x \, \mathrm{d}t \geq 0.$$

Thus, we have

$$\int\limits_{\pi_T} uf_t + A(u) \cdot \nabla_x f \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all nonnegative $f \in C_C^\infty(\pi_T)$, which is enough to conclude that u satisfies (2.1) in the sense of distributions. ☐

Remark 2.6. If $u \in W^{1,1}(\pi_T)$ and u is a distributional solution, then u is an entropy solution as well, since we may apply the chain rule to obtain

$$\frac{\partial}{\partial t}(|u-k|) + \mathrm{sgn}(u-k)\mathrm{div}_x(A(u)) = \mathrm{sgn}(u-k)(u_t + \mathrm{div}_x(A(u))) = 0.$$

In general, distributional solutions need not be entropy solutions as well. Consider the function u_1 listed above in the Burgers' equation example. It is not an entropy solution, since it is easy to verify that for $f \in C_C^\infty(\pi_T)$ we have

$$\int\limits_{\pi_T} |u_1| f_t + \mathrm{sgn}(u_1)\frac{u_1^2}{2} f_x \, \mathrm{d}t \, \mathrm{d}x = -\int\limits_0^T f(t,0) \, \mathrm{d}t.$$

Hence, if $f \geq 0$ in π_T, and $f\Big|_{x=0}$ is not identically zero, then the right-hand side is negative, showing that u_1 is not an entropy solution. Of course, u_1 is not in $W^{1,1}(\pi_T)$. One way to see this is to note that

$$\int\limits_{\pi_T} u_1 f_t \, \mathrm{d}t \, \mathrm{d}x = -\int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f(2|x|, x) \, \mathrm{d}x$$

which implies that the right-hand side is 0 if $f \in C_C^\infty(\pi_T \backslash \{(t,x) : t = 2|x|\})$. This shows that the distributional time derivative of u_1 is zero almost everywhere outside of $\{(t,x) : t = 2|x|\}$ and hence zero almost everywhere in π_T, implying that it cannot be in $L^1(\pi_T)$ as u_1 is not constant in time.

Finally, we prove a property of entropy solutions which allows an alternative characterization of entropy solutions that, while appearing more restrictive at first, is logically equivalent.

Proposition 2.7. *Let u be an entropy solution to (2.1). Then the inequality*

$$\frac{\partial}{\partial t}(S(u)) + div_x(\eta(u)) \leq 0$$

holds for all convex S, where η satisfies $\eta' = S'A'$.

Note that the definition is equivalent to the above inequality holding for all S of the form $S(u) = |u-k|$, $k \in \mathbb{R}$. Hence, entropy solutions can be defined by the solutions satisfying the above inequality for *all* convex S, and not just those of the form $S(u) = |u-k|$.

Proof. Notice that $F(k) = |k|/2$ is the fundamental solution to the Laplace equation in 1 dimension. This implies that $S''(u) = (S'' * |\cdot|/2)''(u)$, and hence

$$S(u) = \int\limits_{\mathbb{R}} \frac{S''(k)}{2}|u-k| \, \mathrm{d}k + au + b$$

for some constants a and b in the sense of distributions. From Proposition 2.5, we know that u is a distributional solution, and so if S is linear (say $S(u) = au + b$), then

$$\frac{\partial}{\partial t}(S(u)) + \mathrm{div}_x(\eta(u)) = a(u_t + \mathrm{div}_x(A(u))) = 0.$$

Hence, assume that $S(u) = \int_{\mathbb{R}} \frac{S''(k)}{2}|u-k| \, \mathrm{d}k$. Then $\eta'(u) = S'(u)A'(u) = \int_{\mathbb{R}} \frac{S''(k)}{2}\mathrm{sgn}(u-k)A'(u) \, \mathrm{d}k$, and hence $\eta(u) = \int_{\mathbb{R}} \frac{S''(k)}{2}\mathrm{sgn}(u-k)(A(u)-A(k)) \, \mathrm{d}k$, up to an additive constant. For any $f \in C_C^\infty(\pi_T)$ we have

$$\begin{aligned}\int_{\pi_T} S(u)f_t + \eta(u)\cdot\nabla_x f \, \mathrm{d}t \, \mathrm{d}x &= \int_{\mathbb{R}}\int_{\pi_T} \frac{S''(k)}{2}|u-k|f_t + \frac{S''(k)}{2}\mathrm{sgn}(u-k)(A(u)-A(k))\cdot\nabla_x f \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}k \\ &= \int_{\mathbb{R}} \frac{S''(k)}{2}\int_{\pi_T} |u-k|f_t + \mathrm{sgn}(u-k)(A(u)-A(k))\cdot\nabla_x f \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}k \geq 0,\end{aligned}$$

since S convex implies $S'' \geq 0$. □

2.3. **Uniqueness.** In defining the notion of entropy solutions, we aimed to create a notion of solution where existence held in order to fix the main drawback to the notion of distributional solutions. We will now show that entropy solutions are indeed unique.

Theorem 2.8. *Let u, v be entropy solutions with corresponding initial data u^0 and v^0. Let M satisfy $\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq M$, and let $N = \max_{|u|\leq M} |a(u)|$. Then, for any $R > 0$, we have*

$$\int_{S_\tau} |u(\tau,x) - v(\tau,x)| \, dx \leq \int_{B_R} |u^0(x) - v^0(x)| \, dx$$

for almost every $0 < \tau < T_0 = \min(T, R/N)$, where

$$S_\tau = \{x : |x| \leq R - N\tau\}$$

is the cross-section of the plane $t = \tau$ of the cone

$$C = \{(t,x) : |x| \leq R - Nt, 0 \leq t \leq T_0\}.$$

Note that if we take $R \to \infty$, we obtain the L^1 contraction property

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|u^0 - v^0\|_{L^1(\mathbb{R}^d)}.$$

The proof will follows Kruzkov's "doubling variables" proof in [3]. We proceed by proving two lemmas:

Lemma 2.9. *Let $g \in C_C^\infty(\pi_T \times \pi_T)$ be nonnegative. Then*

$$\begin{aligned}\int_{\pi_T\times\pi_T} &|u(t,x) - v(\tau,y)|(g_t + g_\tau) + \\ &sgn(u(t,x) - v(\tau,y))(A(u(t,x)) - A(v(\tau,y)))\cdot(\nabla_x g + \nabla_y g) \; dt \; dx \; d\tau \; dy \geq 0.\end{aligned} \tag{2.5}$$

Proof. We wish to apply the inequality (2.3) by by integrating over the variables t, x, τ, y, integrating over $\mathrm{d}t \, \mathrm{d}x$ and using $v(\tau, y)$ as a constant, and then integrating over $\mathrm{d}\tau \, \mathrm{d}y$, treating $u(t,x)$ as a constant.

We thus have

$$\int_{\pi_T}\left(\int_{\pi_T} |u(t,x) - v(\tau,y)|g_t + \mathrm{sgn}(u(t,x) - v(\tau,y))(A(u(t,x)) - A(v(\tau,y)))\cdot\nabla_x g \, \mathrm{d}t \, \mathrm{d}x\right) \mathrm{d}\tau \, \mathrm{d}y \geq 0$$

and

$$\int_{\pi_T}\left(\int_{\pi_T}|v(\tau,y)-u(t,x)|g_\tau+\operatorname{sgn}(v(\tau,y)-u(t,x))(A(v(\tau,y))-A(u(t,x)))\cdot\nabla_y g\ \mathrm{d}\tau\ \mathrm{d}y\right)\ \mathrm{d}t\ \mathrm{d}x\geq 0$$

Adding the two inequalities yields the desired lemma. □

Lemma 2.10. *Let $f\in C_C^\infty(\pi_T)$ be nonnegative. Then*

$$\int_{\pi_T}|u-v|f_t+sgn(u-v)(A(u)-A(v))\cdot\nabla f\ dt\ dx\geq 0.$$

Proof. Let supp $f\subset K$ for some compact K. We apply Lemma 2.9 with

$$g(t,x,\tau,y)=f\left(\frac{t+\tau}{2},\frac{x+y}{2}\right)\rho_\epsilon\left(\frac{t-\tau}{2},\frac{x-y}{2}\right)$$

where ρ_ϵ is the standard mollifier on $\mathbb{R}^{d+1}$, supp $\rho_\epsilon\subset B_\epsilon$, and ϵ is chosen small enough for g to be well-defined. For notational purposes, let $\bar t=\frac{t+\tau}{2}$, $\bar x=\frac{x+y}{2}$, $\Delta t=\frac{t-\tau}{2}$, and $\Delta x=\frac{x-y}{2}$. We have

$$(g_t+g_\tau)(t,x,\tau,y)=f_t\left(\bar t,\bar x\right)\rho_\epsilon\left(\Delta t,\Delta x\right)$$

and

$$(\nabla_x g+\nabla_y g)(t,x,\tau,y)=\left(\nabla f\left(\bar t,\bar x\right)\right)\rho_\epsilon\left(\Delta t,\Delta x\right)$$

I now claim that

(2.6)

$$\int_{\pi_T\times\pi_T}|u(t,x)-v(\tau,y)|f_t\left(\bar t,\bar x\right)\rho_\epsilon\left(\Delta t,\Delta x\right)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y\xrightarrow{\epsilon\to 0}2^{d+1}\int_{\pi_T}|u(t,x)-v(t,x)|f_t(t,x)\ \mathrm{d}t\ \mathrm{d}x$$

and

$$\int_{\pi_T\times\pi_T}\operatorname{sgn}(u(t,x)-v(\tau,y))(A(u(t,x))-A(u(\tau,y)))\cdot\left(\nabla f\left(\bar t,\bar x\right)\rho_\epsilon\left(\Delta t,\Delta x\right)\right)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y$$

$$\xrightarrow{\epsilon\to 0}2^{d+1}\int_{\pi_T}\operatorname{sgn}(u(t,x)-v(t,x))(A(u(t,x))-v(t,x))\cdot\nabla f(t,x)\ \mathrm{d}t\ \mathrm{d}x. \tag{2.7}$$

For simplicity, we prove (2.6). We re-write the integral on the left hand side as

$$\int_{\pi_T\times\pi_T}\left[|u(t,x)-v(\tau,y)|f_t\left(\bar t,\bar x\right)-|u(t,x)-v(t,x)|f_t(t,x)\right]\rho_\epsilon\left(\Delta t,\Delta x\right)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y$$

$$+\int_{\pi_T\times\pi_T}|u(t,x)-v(t,x)|f_t(t,x)\rho_\epsilon\left(\Delta t,\Delta x\right)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y. \tag{2.8}$$

After a change-of-variables, the second integral becomes

$$2^{d+1}\int_{\pi_T\times\pi_T}|u(t,x)-v(t,x)|f_t(t,x)\rho_\epsilon(\tau-t,y-x)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y$$

$$=2^{d+1}\int_{\pi_T}\left(\left(|u-v|f_t\right)*\rho_\epsilon\right)(\tau,y)\ \mathrm{d}\tau\ \mathrm{d}y\xrightarrow{\epsilon\to 0}2^{d+1}\int_{\pi_T}|u(\tau,y)-v(\tau,y)|f_t(\tau,y)\ \mathrm{d}\tau\ \mathrm{d}y.$$

It suffices to show that the first integral in (2.8) vanishes as $\epsilon \to 0$. To do so, we make the following estimate:

$$\begin{aligned}
&||u(t,x) - v(\tau,y)|f_t(\bar{t},\bar{x}) - |u(t,x) - v(t,x)|f_t(t,x)| \\
&\leq ||u(t,x) - v(\tau,y)| - |u(t,x) - v(t,x)||f_t(\bar{t},\bar{x}) + |u(t,x) - v(t,x)|\,|f_t(\bar{t},\bar{x}) - f_t(t,x)| \\
&\leq |v(t,x) - v(\tau,y)|\|f_t\|_{L^\infty} + |u(t,x) - v(t,x)|\|\nabla f_t\|_{L^\infty}\,|(\Delta t, \Delta x)| \\
&\leq \|f_t\|_{L^\infty} + \epsilon\|\nabla f_t\|_{L^\infty}|u(t,x) - v(t,x)|.
\end{aligned}$$

Note that we may assume $|(\Delta t, \Delta x)| < \epsilon$ since supp $\rho_\epsilon \subset B_\epsilon$. Furthermore, under this assumption we have supp $f_t(\bar{t},\bar{x}) \subset K + B_\epsilon = \{x+y | x \in K, |y| \leq \epsilon\}$. Hence, we can take the limits of integration in the integral to be $(K + B_\epsilon) \times (K + B_\epsilon)$. We thus have

$$\left| \int\limits_{\pi_T \times \pi_T} [|u(t,x) - v(\tau,y)|f_t(\bar{t},\bar{x}) - |u(t,x) - v(t,x)|f_t(t,x)]\,\rho_\epsilon(\Delta t, \Delta x)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y \right|$$

(2.9)

$$\leq \int\limits_{(K+B_\epsilon)\times(K+B_\epsilon)} (\|f_t\|_{L^\infty}|v(t,x) - v(\tau,y)| + \epsilon\|\nabla f_t\|_{L^\infty}|u(t,x) - v(t,x)|)\rho_\epsilon(\Delta t, \Delta x)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y.$$

Since

$$\int\limits_{(K+B_\epsilon)\times(K+B_\epsilon)} |u(t,x) - v(t,x)|\rho_\epsilon(\Delta t, \Delta x)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y$$

$$= 2^{d+1} \int\limits_{(K+B_\epsilon)\times(K+B_\epsilon)} |u(t,x) - v(t,x)|\rho_\epsilon(\tau - t, y - x)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y \xrightarrow{\epsilon \to 0} 2^{d+1} \int\limits_K |u(t,x) - v(t,x)|\ \mathrm{d}t\ \mathrm{d}x$$

it follows that

$$\epsilon\|\nabla f_t\|_{L^\infty} \int\limits_{(K+B_\epsilon)\times(K+B_\epsilon)} |u(t,x) - v(t,x)|\rho_\epsilon(\Delta t, \Delta x)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y \xrightarrow{\epsilon \to 0} 0.$$

Furthermore, we have

$$\int\limits_{(K+B_\epsilon)\times(K+B_\epsilon)} \|f_t\|_{L^\infty}|v(t,x) - v(\tau,y)|\rho_\epsilon(\Delta t, \Delta x)\ \mathrm{d}t\ \mathrm{d}x\ \mathrm{d}\tau\ \mathrm{d}y$$

$$\leq \|f_t\|_{L^\infty}\|\rho\|_{L^\infty} \int\limits_{K+B_\epsilon} \left(\frac{1}{\epsilon^{d+1}} \int\limits_{(t,x)+B_\epsilon} |v(t,x) - v(\tau,y)|\ \mathrm{d}\tau\ \mathrm{d}y \right) \mathrm{d}t\ \mathrm{d}x. \tag{2.10}$$

By the Lebesgue Differentiation Theorem, almost every $(t,x) \in \pi_T$ is a Lebesgue point, i.e.

$$\frac{1}{\epsilon^{d+1}} \int\limits_{(t,x)+B_\epsilon} |v(t,x) - v(\tau,y)|\ \mathrm{d}\tau\ \mathrm{d}y \xrightarrow{\epsilon \to 0} 0$$

for almost every $(t,x) \in \pi_T$. Since

$$\frac{1}{\epsilon^{d+1}} \int\limits_{(t,x)+B_\epsilon} |v(t,x) - v(\tau,y)|\ \mathrm{d}\tau\ \mathrm{d}y \leq \frac{1}{\epsilon^{d+1}} \int\limits_{(t,x)+B_\epsilon} 2M\ \mathrm{d}\tau\ \mathrm{d}y = 2M|B_1|$$

and the outer integral in (2.10) is taken over a bounded set, we can apply the Dominated Convergence Theorem to conclude that the integral in (2.10) vanishes as $\epsilon \to 0$. Hence, the first integral in (2.8) vanishes, so we arrive at (2.6). The statement in (2.7) can be proven similarly.

Hence, applying Lemma 2.9 with our choice of g, and letting $\epsilon \to 0$, we have

$$2^{d+1} \int_{\pi_T} |u(t,x) - v(t,x)| f_t(t,x) + \operatorname{sgn}(u(t,x) - v(t,x))(A(u(t,x)) - A(v(t,x))) \cdot \nabla f(t,x) \, \mathrm{d}t \, \mathrm{d}x \geq 0,$$

which proves the desired lemma. □

With these two lemmas proven, we may now proceed to prove the main theorem.

Proof of Theorem 2.8. Formally, Lemma 2.10 implies that

$$\frac{\partial}{\partial t}(|u - v|) + \operatorname{div}_x(\operatorname{sgn}(u - v)(A(u) - A(v))) \leq 0$$

in the sense of distributions. Furthermore, we have $|A(u) - A(v)| \leq N|u - v|$ by the definition of N, and hence $-N|u - v| \leq \operatorname{sgn}(u - v)(A(u) - A(v)) \cdot \nu$ for any unit vector ν. It follows that

$$\begin{aligned}
\frac{d}{dt}\left(\int_{S_t} |u - v| \, \mathrm{d}x\right) &= \int_{S_t} \frac{\partial}{\partial t}(|u - v|) \, \mathrm{d}x - N \int_{\partial S_t} |u - v| \, \mathrm{d}S \\
&\leq \int_{S_t} \frac{\partial}{\partial t}(|u - v|) \, \mathrm{d}x + \int_{\partial S_t} \operatorname{sgn}(u - v)(A(u) - A(v)) \cdot \hat{n} \, \mathrm{d}S \\
&\leq \int_{S_t} \frac{\partial}{\partial t}(|u - v|) + \operatorname{div}_x(\operatorname{sgn}(u - v)(A(u) - A(v))) \, \mathrm{d}x \leq 0.
\end{aligned}$$

We can formalize the argument as follows: let $\mu(t) = \int_{S_t} |u(t,x) - v(t,x)| \, \mathrm{d}x$, let E_μ be the set of Lebesgue points of μ, and let E_u and E_v be the subsets of $[0,T]$ involved in the definition of entropy solutions. Let $E = E_\mu \cap E_u \cap E_v$. Then $[0,T] \backslash E$ has measure zero. It suffices to show that μ is decreasing on E, and that $\mu(t)$ approaches $\mu(0) = \int_{B_R} |u^0(x) - v^0(x)| \, \mathrm{d}x$ for $t \in E$ approaching 0.

To show that $\mu(t) \to \mu(0)$, note that

$$\int_{S_t} |u(t,x) - v(t,x)| \, \mathrm{d}x \leq \int_{B_R} |u(t,x) - u^0(x)| \, \mathrm{d}x + \int_{B_R} |v(t,x) - v^0(x)| \, \mathrm{d}x + \int_{B_R} |u^0(x) - v^0(x)| \, \mathrm{d}x.$$

For $t \in E$ approaching 0, the first and second terms on the right-hand side vanish, so we obtain the desired result.

We now show that μ is decreasing on E. Let ρ_ϵ be the standard mollifier in $\mathbb{R}$, and let $\chi_\epsilon(x) = \int_{-\infty}^{x} \rho_\epsilon(y) \, \mathrm{d}y$. Note that $\chi_\epsilon(x) = 0$ for $x < -\epsilon$ and $\chi_\epsilon(x) = 1$ for $x > \epsilon$. For $t_1 < t_2 \in E$, let

$$f(t,x) = (\chi_\epsilon(t - t_1) - \chi_\epsilon(t - t_2))(1 - \chi_{\epsilon'}(|x| + Nt - R + \epsilon'))$$

with ϵ' chosen so that $2\epsilon' < R - Nt_2$. Notice that $\operatorname{supp} f \subset [t_1 - \epsilon, t_2 + \epsilon] \times C$, and that $f \geq 0$. Furthermore, f is clearly infinitely differentiable wherever $x \neq 0$, and at $x = 0$, either $t \leq t_2$, in which case $Nt_2 - R + \epsilon' < -\epsilon'$, and hence $\chi_{\epsilon'}(|x| + Nt - R + \epsilon') = 0$ in a neighborhood of (t,x), or $t > t_2$, in which case $f = 0$ in a neighborhood of (t,x). Hence, $f \in C_C^\infty(\pi_T)$, so we may apply Lemma 2.10. We have

$$f_t(t,x) = (\rho_\epsilon(t - t_1) - \rho_\epsilon(t - t_2))(1 - \chi_{\epsilon'}(|x| + Nt - R + \epsilon')) + (\chi_\epsilon(t - t_1) - \chi_\epsilon(t - t_2))(-N\rho_{\epsilon'}(|x| + Nt - R + \epsilon'))$$

and

$$\nabla f(t,x) = (\chi_\epsilon(t - t_1) - \chi_\epsilon(t - t_2))\left(\rho_{\epsilon'}(|x| + Nt - R + \epsilon')\frac{x}{|x|}\right).$$

Hence, we have

$$\begin{aligned}\operatorname{sgn}(u-v)(A(u)-A(v))\cdot\nabla f(t,x) &\le N|u-v|(\chi_\epsilon(t-t_1)-\chi_\epsilon(t-t_2))\rho_{\epsilon'}(|x|+Nt-R+\epsilon') \\ &= |u-v|((\rho_\epsilon(t-t_1)-\rho_\epsilon(t-t_2))(1-\chi_{\epsilon'}(|x|+Nt-R+\epsilon'))-f_t).\end{aligned} \tag{2.11}$$

Applying Lemma 2.10 to f and combining with inequality (2.11), we obtain

$$\int_{\pi_T} |u(t,x)-v(t,x)|(\rho_\epsilon(t-t_1)-\rho_\epsilon(t-t_2))(1-\chi_{\epsilon'}(|x|+Nt-R+\epsilon'))\ \mathrm{d}t\ \mathrm{d}x \ge 0. \tag{2.12}$$

As $\epsilon' \to 0$, we have

$$1-\chi_{\epsilon'}(|x|+Nt-R+\epsilon') \to \begin{cases} 1 & \text{if } |x| < R-Nt \\ 0 & \text{if } |x| > R-Nt \end{cases}$$

and hence we have

$$\int_0^T \mu(t)(\rho_\epsilon(t-t_1)-\rho_\epsilon(t-t_2))\ \mathrm{d}t = \int_0^T\int_{S_t} |u(t,x)-v(t,x)|(\rho_\epsilon(t-t_1)-\rho_\epsilon(t-t_2))\ \mathrm{d}x\ \mathrm{d}t \ge 0. \tag{2.13}$$

Since t_1 is a Lebesgue point of μ, we have

$$\begin{aligned}\left|\left(\int_0^T \mu(t)\rho_\epsilon(t-t_1)\ \mathrm{d}t\right)-\mu(t_1)\right| &= \left|\int_0^T (\mu(t)-\mu(t_1))\rho_\epsilon(t-t_1)\ \mathrm{d}t\right| \\ &\le \int_{t_1-\epsilon}^{t_1+\epsilon} |\mu(t)-\mu(t_1)|\frac{\|\rho\|_{L^\infty}}{\epsilon}\ \mathrm{d}t \\ &= \frac{\|\rho\|_{L^\infty}}{\epsilon}\int_{|t-t_1|<\epsilon} |\mu(t)-\mu(t_1)|\ \mathrm{d}t \xrightarrow{\epsilon\to 0} 0.\end{aligned}$$

A similar result holds for t_2. Hence, the left-hand side of (2.13) converges to $\mu(t_1)-\mu(t_2)$ as $\epsilon \to 0$, so (2.13) implies that $\mu(t_1) \ge \mu(t_2)$ for $t_1 < t_2$, thus proving the theorem. □

2.4. **Existence.** We can also establish a result regarding the existence of entropy solutions. The idea is to consider, for $\epsilon > 0$, the solution to u^ϵ of the problem

$$u_t + \operatorname{div}_x(A(u)) = \epsilon\Delta u, \qquad u|_{t=0} = u^0$$

It is well known[2] that this equation admits a unique classical solution u_ϵ if u^0 is bounded and has sufficient bounded derivatives. Hence, for regular enough u^0, it suffices to show that the family $\{u_\epsilon\}$ is compact, in order to extract a subsequence ϵ_n and a limit u such that $u_{\epsilon_n} \to u$, with u satisfying the desired entropy inequalities, while for $u^0 \in L^\infty$ we can approximate by smooth initial data u^0_h, thus getting a family $\{u_{\epsilon,h}\}$ which converges to some u as $h \to 0$ and $\epsilon \to 0$.

The proof involves finding equicontinuity estimates on u_ϵ and is similar to the corresponding proof for kinetic solutions described later in this paper, so the proof, which can be found in [3], will be omitted. In fact, we will investigate the rate of convergence of the parabolic approximation u_ϵ to the entropy solution u later in this paper.

3. Kinetic Formulation

We now turn our attention to a reformulation of conservation laws which generalizes the notion of entropy solutions. In the kinetic formulation, a function $\chi(\xi, u)$ is introduced to turn the nonlinear conservation law into a linear equation on the nonlinear quantity $\chi(\xi, u)$. This structure provides a method to construct solutions by approximating with solutions of a linear equation, as well as nice estimates on such approximations and compactness results on solutions without requiring compactness of initial data. We will follow the approach of Perthame[5] in introducing the kinetic formulation, existence results, convergence estimates, and compactness results.

We first introduce a simple, yet important function whose properties are critical in forming the kinetic formulation:

Definition 3.1. The function $\chi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\chi(\xi, u) = \begin{cases} 1 & \text{if } 0 < \xi < u \\ -1 & \text{if } u < \xi < 0 \\ 0 & \text{otherwise} \end{cases}$$

We prove a few basic properties:

Proposition 3.2. *We have*

(1) $\int_{\mathbb{R}} S'(\xi)\chi(\xi, u)\ d\xi = S(u) - S(0)$ *for S locally Lipschitz, i.e. $S' \in L^\infty_{loc}$, and in particular* $\int_{\mathbb{R}} \chi(\xi, u)\ d\xi = u$,

(2) $\int_{\mathbb{R}} |\chi(\xi, u) - \chi(\xi, v)|\ d\xi = |u - v|$,

(3) $\frac{\partial}{\partial \xi}(\chi(\xi, u)) = \delta(\xi) - \delta(\xi - u)$, *and*

(4) $\frac{\partial}{\partial u}(\chi(\xi, u)) = \delta(\xi - u)$ *for* $\xi \neq 0$.

The last two statements are made in the sense of distributions.

The proof of these properties is an easy exercise to verify.

We now consider an entropy solution u to $u_t + \text{div}(A(u)) = 0$, and define the distribution $m(t, x, \xi)$ by

$$m(t, x, \xi) = \frac{\partial}{\partial t}\left(\int_0^\xi \chi(\zeta, u(t, x))\ \mathrm{d}\zeta\right) + \text{div}_x\left(\int_0^\xi a(\zeta)\chi(\zeta, u(t, x))\ \mathrm{d}\zeta\right).$$

It turns out that m has some interesting properties. For example, by differentiating both sides in ξ, we have the distributional equation

$$\frac{\partial m}{\partial \xi} = \frac{\partial}{\partial t}(\chi(\xi, u)) + a(\xi) \cdot \nabla_x(\chi(\xi, u)).$$

Furthermore, we can show that m is nonnegative. Multiplying m by $\varphi(\xi)$, where $\varphi \in C^\infty_C(\mathbb{R})$, and integrating by parts yields

$$-\int_{\mathbb{R}} \varphi'(\xi)m(t, x, \xi)\ \mathrm{d}\xi = \frac{\partial}{\partial t}\left(\int_{\mathbb{R}} \varphi(\xi)\chi(\xi, u(t, x))\ \mathrm{d}\xi\right) + \text{div}_x\left(\int_{\mathbb{R}} \varphi(\xi)a(\xi)\chi(\xi, u(t, x))\ \mathrm{d}\xi\right)$$

In particular, if we choose φ approaching S' for S convex, we obtain, using the properties above, that

$$\begin{aligned} -\int_{\mathbb{R}} S''(\xi)m(t, x, \xi)\ \mathrm{d}\xi &= \frac{\partial}{\partial t}\left(\int_{\mathbb{R}} S'(\xi)\chi(\xi, u(t, x))\ \mathrm{d}\xi\right) + \text{div}_x\left(\int_{\mathbb{R}} \eta'(\xi)\chi(\xi, u(t, x))\ \mathrm{d}\xi\right) \\ &= \frac{\partial}{\partial t}(S(u(t, x)) - S(0)) + \text{div}_x(\eta(u(t, x)) - \eta(0)) \\ &= \frac{\partial}{\partial t}(S(u(t, x))) + \text{div}_x(\eta(u(t, x))) \leq 0. \end{aligned}$$

Since this holds for all S convex (i.e. for all $S'' \geq 0$), it follows that m is nonnegative.

Inspired by these results, we define the kinetic formulation of the conservation law as follows:

Definition 3.3. Let $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$. We say that u is a kinetic formulation to the equation

$$u_t + \mathrm{div}_x(A(u)) = 0, \qquad u(0,x) = u_0(x)$$

if there exists a nonnegative bounded measure $m \in C_0(\mathbb{R}, w - M^1(\mathbb{R}^+ \times \mathbb{R}^d))$ such that

$$\frac{\partial}{\partial t}\left(\chi(\xi, u(t,x))\right) + a(\xi) \cdot \nabla_x \left(\chi(\xi, u(t,x))\right) = \frac{\partial m}{\partial \xi}(t,x,\xi), \qquad \chi(\xi, u(0,x)) = \chi(\xi, u^0(x))$$

in the sense of distributions.

Note that this definition does not require a L^∞ bound on u, and as such can be applied to initial data which is not L^∞.

Remark 3.4. If u is regular enough (say $W^{1,1}$), then applying the chain rule just as we did in Proposition 2.5 yields

$$\frac{\partial}{\partial t}(\chi(\xi,u)) + a(\xi) \cdot \nabla_x(\chi(\xi,u)) = \delta(\xi - u)(u_t + a(\xi) \cdot \nabla_x u) = \delta(\xi - u)(u_t + \mathrm{div}_x(A(u))) = 0.$$

Hence, for $u \in W^{1,1}$, the corresponding measure is identically zero.

Remark 3.5. If u is in L^∞, then the corresponding measure m is compactly supported in ξ, with support contained in $|\xi| \leq \|u\|_{L^\infty}$. In particular, the measure associated with entropy solutions are compactly supported in ξ. To see this, note that for $|\xi| > \|u\|_{L^\infty}$ we have

$$\begin{aligned} m(t,x,\xi) &= \frac{\partial}{\partial t}\left(\int_0^\xi \chi(\zeta, u(t,x))\, \mathrm{d}\zeta\right) + \mathrm{div}_x\left(\int_0^\xi a(\zeta)\chi(\zeta, u(t,x))\, \mathrm{d}\zeta\right) \\ &= \frac{\partial}{\partial t}\left(\int_{-\infty}^\infty \chi(\zeta, u(t,x))\, \mathrm{d}\zeta\right) + \mathrm{div}_x\left(\int_{-\infty}^\infty a(\zeta)\chi(\zeta, u(t,x))\, \mathrm{d}\zeta\right) \\ &= \frac{\partial}{\partial t}(u(t,x)) + \mathrm{div}_x(A(u(t,x)) - A(0)) = 0. \end{aligned}$$

The second equality follows from the fact that $\chi(\zeta, u) = 0$ for ζ not between 0 and u, while the last equality follows from the fact that u satisfies the conservation law in the sense of distributions.

4. Existence and Uniqueness

4.1. Statement and Outline.

We have the following existence/uniqueness theorem:

Theorem 4.1. *Let $u^0 \in L^1(\mathbb{R}^d)$, and let A be locally Lipschitz, i.e. $a \in (L^\infty_{loc}(\mathbb{R}))^d$. Then there exists a unique distributional distribution $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ to the kinetic formulation.*

The proof will follow the following steps: We first solve the equation

$$\frac{\partial f_\lambda}{\partial t} + a(\xi) \cdot \nabla_x f_\lambda = \lambda(\chi(\xi, u_\lambda) - f_\lambda), \qquad f_\lambda\Big|_{t=0} = \chi(\xi, u^0)$$

for every $\lambda > 0$ through a fixed point argument, where $u_\lambda = \int f_\lambda \, \mathrm{d}\xi$. For every $\lambda > 0$, we find a (nonnegative bounded) measure m_λ such that

$$\frac{\partial m_\lambda}{\partial \xi} = \lambda(\chi(\xi, u_\lambda) - f_\lambda)$$

and hence

$$\frac{\partial f_\lambda}{\partial t} + a(\xi) \cdot \nabla_x f_\lambda = \frac{\partial m_\lambda}{\partial \xi}.$$

This is similar to the desired equation in the kinetic formulation, except that we do not know if f_λ has the structure of $\chi(\xi, u)$ for some u. Hence, we will show that $\{m_\lambda\}$ is uniformly bounded for all $\lambda > 0$, and that the initial condition $f_\lambda\Big|_{t=0} = \chi(\xi, u^0)$ imposes extra conditions on the sign and bounds of f_λ, in the hope that

$$f_\lambda - \chi(\xi, u_\lambda) = \frac{1}{\lambda}\frac{\partial m_\lambda}{\partial \xi} \xrightarrow{\lambda\to\infty} 0$$

in some way. We then argue that as $\lambda \to \infty$, the sequences $\{u_\lambda\}$, $\{f_\lambda\}$, and $\{m_\lambda\}$ converge to u, f, and m, so that

$$\frac{\partial f}{\partial t} + a(\xi)\cdot\nabla_x f = \frac{\partial m}{\partial \xi}.$$

and that $f_\lambda \to \chi(\xi, u)$, i.e. $f = \chi(\xi, u)$, thus proving the existence of the desired solution.

4.2. **Solving a linear approximation.** We first investigate properties of the solutions of the problem

$$f_t + a(\xi)\cdot\nabla_x f + \lambda f = g, \qquad f\Big|_{t=0} = f^0 \tag{4.1}$$

in $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}_\xi$, where λ is a fixed positive parameter.

Theorem 4.2. *Let $f^0 \in L^1(\mathbb{R}^d \times \mathbb{R}_\xi)$, $g \in L^1((0,T) \times \mathbb{R}^d_x \times \mathbb{R}_\xi)$, and $a \in (L^\infty_{loc}(\mathbb{R}))^d$. Then there exists a distributional solution $f \in C(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}_\xi))$ to (4.1) given by the formula*

$$f(t,x,\xi) = f^0(x - a(\xi)t, \xi)e^{-\lambda t} + \int_0^t e^{-\lambda s} g(t-s, x-a(\xi)s, \xi)\ ds.$$

Furthermore, this solution satisfies the properties

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d\times\mathbb{R}_\xi} f(t,x,\xi)\ dx\ d\xi\right) + \lambda\int_{\mathbb{R}^d\times\mathbb{R}_\xi} f(t,x,\xi)\ dx\ d\xi = \int_{\mathbb{R}^d\times\mathbb{R}_\xi} g(t,x,\xi)\ dx\ d\xi$$

and

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d\times\mathbb{R}_\xi} |f(t,x,\xi)|\ dx\ d\xi\right) + \lambda\int_{\mathbb{R}^d\times\mathbb{R}_\xi} |f(t,x,\xi)|\ dx\ d\xi \le \int_{\mathbb{R}^d\times\mathbb{R}_\xi} |g(t,x,\xi)|\ dx\ d\xi.$$

Proof. For f^0 and g smooth, we have

$$\frac{d}{dt}\left(e^{\lambda t} f(t, x + a(\xi)t, \xi)\right) = e^{\lambda t}(f_t + a(\xi)\cdot\nabla_x f + \lambda f)(t, x+a(\xi)t, \xi) = e^{\lambda t} g(t, x+a(\xi)t, \xi).$$

Integrating from $s = 0$ to $s = t$ gives the desired formula. For f^0 and g in L^1, let $\{f^0_n\}$ and $\{g_n\}$ be sequences of smooth functions such that $f^0_n \to f^0$ in $L^1(\mathbb{R}^{d+1})$ and $g_n \to g$ in $L^1((0,T)\times\mathbb{R}^{d+1})$. Letting $\{f_n\}$ denote the corresponding solutions, we note that the preceding formula gives

$$f_n(t,x,\xi) - f_m(t,x,\xi) = (f^0_n - f^0_m)(x - a(\xi)t, \xi)e^{-\lambda t} + \int_0^t e^{-\lambda s}(g_n - g_m)(t-s, x-a(\xi)s, \xi)\ \mathrm{d}s.$$

Hence, integrating along x and ξ, and making an appropriate change of variables, we obtain

$$\|f_n(t,\cdot) - f_m(t,\cdot)\|_{L^1(\mathbb{R}^{d+1})} \le e^{-\lambda t}\|f^0_n - f^0_m\|_{L^1} + \int_0^t e^{-\lambda s}\|g_n(s,\cdot) - g_m(s,\cdot)\|_{L^1}\ \mathrm{d}s.$$

Thus, $\{f_n\}$ is a Cauchy sequence in $C((0,T);L^1(\mathbb{R}^{d+1}))$, and thus converges to some f satisfying the same distributional formula.

To derive the integral equation, we multiply (4.1) by $\varphi_R(x) = \varphi(x/R)$, where φ is a compactly supported cutoff function, and integrate with respect to x and ξ. As $R \to \infty$, we have $\int f_t \varphi_R \to \frac{d}{dt}\left(\int f\right)$, $\int \lambda f \varphi_R \to \lambda \int f$, $\int g\varphi_R \to \int g$, and

$$\int a(\xi) \cdot \varphi_R \nabla_x f = \int a(\xi) \cdot \left(- \int f \nabla_x \varphi_R \, \mathrm{d}x \right) \, \mathrm{d}\xi = \frac{1}{R} \int a(\xi) \cdot \int f(x) \nabla_x \varphi(\frac{x}{R}) \, \mathrm{d}x \, \mathrm{d}\xi \to 0.$$

This proves the integral equation. Finally, to derive the L^1 inequality, we multiply (4.1) by $\eta_\epsilon(f)\varphi_R(x)$ and integrate, where $\eta_\epsilon(f) = 2 \int\limits_{-\infty}^{f} \rho_\epsilon(y) \, \mathrm{d}y - 1$ and ρ_ϵ is the standard mollifier on $\mathbb{R}$. By similar arguments as above, as $R \to \infty$ we have $\int a(\xi) \cdot \eta_\epsilon(f) \varphi_R \nabla_x f \to 0$, while the φ_R term drops out in the other terms, leading to the equation

$$\int f_t \eta_\epsilon(f) + \lambda \int f \eta_\epsilon(f) = \int g \eta_\epsilon(f).$$

Letting $\epsilon \to 0$ yields

$$\frac{d}{dt}\left(\int |f| \right) + \lambda \int |f| = \int g \mathrm{sgn}(f) \le \int |g|$$

as desired. □

We use the results of this theorem to find solutions to the problem

$$\frac{\partial f_\lambda}{\partial t} + a(\xi) \cdot \nabla_x f_\lambda = \lambda(\chi(\xi, u_\lambda) - f_\lambda), \qquad f_\lambda\Big|_{t=0} = \chi(\xi, u^0) \tag{4.2}$$

To do this, we first fix $v \in C((0,T);L^1(\mathbb{R}^d))$, and consider the solution f to the equation

$$\frac{\partial f}{\partial t} + a(\xi) \cdot \nabla_x f + \lambda f = \lambda \chi(\xi, v), \qquad f\Big|_{t=0} = \chi(\xi, u^0) \tag{4.3}$$

Let $\Phi : C((0,T), L^1(\mathbb{R}^d)) \to C((0,T), L^1(\mathbb{R}^d))$ be the operator sending v to the integral of the corresponding f, i.e.

$$\Phi(v) : (t,x) \mapsto u(t,x) = \int\limits_{\mathbb{R}} f(t,x,\xi) \, \mathrm{d}\xi.$$

We aim to show that Φ is a strict contraction on the Banach space $C((0,T), L^1(\mathbb{R}^d))$, with the norm

$$\|u\|_{C((0,T);L^1(\mathbb{R}^d))} = \sup_{t \in (0,T)} \|u(t,\cdot)\|_{L^1(\mathbb{R}^d)}$$

in order to apply the Banach fixed point theorem and obtain a solution to (4.2). To do this, let $v_1, v_2 \in C((0,T);L^1(\mathbb{R}^d))$. Letting $f = f_1 - f_2$, where f_1 and f_2 are the solutions to (4.3) corresponding to v_1 and v_2, we see that f is a solution to (4.3) with $g = \chi(\xi, v_1) - \chi(\xi, v_2)$ and $f^0 \equiv 0$. We thus have

$$\begin{aligned} \frac{d}{dt}\left(\int\limits_{\mathbb{R}^d \times \mathbb{R}_\xi} |f(t,x,\xi)| \, \mathrm{d}x \, \mathrm{d}\xi \right) + \lambda \int\limits_{\mathbb{R}^d \times \mathbb{R}_\xi} |f(t,x,\xi)| \, \mathrm{d}x \, \mathrm{d}\xi &\le \lambda \int\limits_{\mathbb{R}^d \times \mathbb{R}_\xi} |\chi(\xi, v_1(t,x)) - \chi(\xi, v_2(t,x))| \, \mathrm{d}x \, \mathrm{d}\xi \\ &= \lambda \int\limits_{\mathbb{R}^d} |v_1(t,x) - v_2(t,x)| \, \mathrm{d}x \\ &\le \lambda \|v_1 - v_2\|_{C((0,T);L^1(\mathbb{R}^d))}. \end{aligned} \tag{4.4}$$

It follows that

$$\frac{d}{dt}\left(e^{\lambda t}\int\limits_{\mathbb{R}^d\times\mathbb{R}_\xi}|f(t,x,\xi)|\ \mathrm{d}x\ \mathrm{d}\xi\right)\leq\frac{d}{dt}\left(e^{\lambda t}\|v_1-v_2\|_{C((0,T);L^1(\mathbb{R}^d))}\right)$$

and, using the fact that $\int|f(0,x,\xi)|\ \mathrm{d}x\ \mathrm{d}\xi=\int|f^0|\ \mathrm{d}x\ \mathrm{d}\xi=0$, we have

$$\int\limits_{\mathbb{R}^d\times\mathbb{R}_\xi}|f(t,x,\xi)|\ \mathrm{d}x\ \mathrm{d}\xi\leq(1-e^{-\lambda t})\|v_1-v_2\|_{C((0,T);L^1(\mathbb{R}^d))}.$$

Hence, we have

$$\begin{aligned}\|\Phi(v_1)-\Phi(v_2)\|_{C((0,T);L^1(\mathbb{R}^d))}&=\sup_{t\in(0,T)}\int\limits_{\mathbb{R}^d}\left|\int\limits_{\mathbb{R}}f_1(t,x,\xi)-f_2(t,x,\xi)\ \mathrm{d}\xi\right|\ \mathrm{d}x\\&\leq\sup_{t\in(0,T)}\int\limits_{\mathbb{R}^d\times\mathbb{R}_\xi}|f(t,x,\xi)|\ \mathrm{d}x\ \mathrm{d}\xi\\&\leq(1-e^{-\lambda T})\|v_1-v_2\|_{C((0,T);L^1(\mathbb{R}^d))}\end{aligned}$$

showing that Φ is a strict contraction on $C((0,T);L^1(\mathbb{R}^d))$. Hence, for all $T>0$, there exists a fixed point of Φ, i.e. there exists $u_\lambda\in C((0,T);L^1(\mathbb{R}^d))$ and $f_\lambda\in C((0,T);L^1(\mathbb{R}^d\times\mathbb{R}_\xi))$ such that

$$\frac{\partial f_\lambda}{\partial t}+a(\xi)\cdot\nabla_x f_\lambda=\lambda(\chi(\xi,u_\lambda)-f_\lambda),\ f_\lambda\Big|_{t=0}=\chi(\xi,u^0)$$

and

$$u_\lambda(t,x)=\int\limits_{\mathbb{R}}f_\lambda(t,x,\xi)\ \mathrm{d}\xi.$$

Moreover, u_λ and f_λ are defined on $(0,T)$ for all $T>0$ and hence can be extended to be defined for all positive time.

4.3. **Properties of approximate solutions and the approximate measure.** We now prove some properties of the solutions f_λ and u_λ obtained above. Since these properties do not depend on the specific value of λ, we drop it in the subscript for ease of notation.

Theorem 4.3. *Let u and f be as above. Then we have the representational formula*

$$f(t,x,\xi)=\chi(\xi,u^0(x-a(\xi)t))e^{-\lambda t}+\lambda\int\limits_0^t e^{-\lambda s}\chi(\xi,u(t-s,x-a(\xi)s))\ ds.$$

Furthermore, we have the following properties:

(1) *The total mass is conserved, i.e.*

$$\frac{d}{dt}\left(\int\limits_{\mathbb{R}^d}u(t,x)\ dx\right)=0.$$

(2) *For any initial data u_1^0 and u_2^0, with corresponding u_1,f_1 and u_2,f_2, we have an L^1 contraction property*

$$\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^1(\mathbb{R}^d)}\leq\|f_1(t,\cdot)-f_2(t,\cdot)\|_{L^1(\mathbb{R}^{d+1})}\leq\|u_1^0-u_2^0\|_{L^1(\mathbb{R}^d)}.$$

Hence, for any initial data u^0, setting $u_1^0 = u^0(\cdot + h)$ and $u_2^0 = u^0$ yields the space-oscillation contraction property

$$\|u(\cdot + h, t) - u(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq \|u^0(\cdot + h) - u^0(\cdot)\|_{L^1(\mathbb{R}^d)}.$$

(3) *$|f(t, x, \xi)| = sgn(\xi) f(t, x, \xi) \leq 1$. In other words, the sign of f matches the sign of ξ, and $|f| \leq 1$.*

(4) *If $u^0 \in L^\infty(\mathbb{R}^d)$, then $\|u(t, \cdot)\|_{L^\infty} \leq \|u^0\|_{L^\infty}$, and $f(t, x, \xi) = 0$ for $|\xi| > \|u^0\|_{L^\infty}$.*

Proof. Since u and f satisfy the equation

$$\frac{\partial f}{\partial t} + a(\xi) \cdot \nabla_x f + \lambda f = g, \qquad f\Big|_{t=0} = \chi(\xi, u^0),$$

with $g = \lambda \chi(\xi, u)$, the representation formula holds, as well as the integral equality

$$\frac{d}{dt}\left(\int\limits_{\mathbb{R}^d \times \mathbb{R}_\xi} f \, \mathrm{d}x \, \mathrm{d}\xi\right) + \lambda \int\limits_{\mathbb{R}^d \times \mathbb{R}_\xi} f \, \mathrm{d}x \, \mathrm{d}\xi = \int\limits_{\mathbb{R}^d \times \mathbb{R}_\xi} \lambda \chi(\xi, u) \, \mathrm{d}x \, \mathrm{d}\xi = \lambda \int\limits_{\mathbb{R}^d} u \, \mathrm{d}x.$$

Since $\int\limits_{\mathbb{R}^d \times \mathbb{R}_\xi} f \, \mathrm{d}x \, \mathrm{d}\xi = \int\limits_{\mathbb{R}^d} u \, \mathrm{d}x$, it follows that

$$\frac{d}{dt}\left(\int\limits_{\mathbb{R}^d} u(t, x) \, \mathrm{d}x\right) + \lambda \int\limits_{\mathbb{R}^d} u(t, x) \, \mathrm{d}x = \lambda \int\limits_{\mathbb{R}^d} u(t, x) \, \mathrm{d}x,$$

thus proving mass conservation. The L^1 contraction property follows similarly, since $f_1 - f_2$ satisfies (4.3) with $g = \chi(\xi, u_1) - \chi(\xi, u_2)$, and hence

$$\begin{aligned} \frac{d}{dt}\left(\int |f_1 - f_2| \, \mathrm{d}x \, \mathrm{d}\xi\right) + \lambda \int |f_1 - f_2| \, \mathrm{d}x \, \mathrm{d}\xi &\leq \lambda \int |\chi(\xi, u_1) - \chi(\xi, u_2)| \, \mathrm{d}x \, \mathrm{d}\xi \\ &= \lambda \int |u_1 - u_2| \, \mathrm{d}x \\ &\leq \lambda \int |f_1 - f_2| \, \mathrm{d}x \, \mathrm{d}\xi \end{aligned}$$

Hence, $\frac{d}{dt}\left(\int |f_1 - f_2| \, \mathrm{d}x \, \mathrm{d}\xi\right) \leq 0$, and since $(f_1 - f_2)\Big|_{t=0} = u_1^0 - u_2^0$, the second inequality in (2) follows. The first inequality follows easily from the fact that $u = \int f \, \mathrm{d}\xi$.

To prove the sign property on f, we note $|\chi| \leq 1$, and the sign of $\chi(\xi, u)$ matches the sign of ξ. Using the representational formula, we see that the sign of f_λ also matches the sign of ξ, while

$$|f_\lambda(t, x, \xi)| \leq e^{-\lambda t} + \lambda \int_0^t e^{-\lambda s} \, \mathrm{d}s = 1$$

as desired.

To show the L^∞ bounds on u, we aim to show that the set

$$\{v \in C((0, T); L^1(\mathbb{R}^d)) : \|v\|_{L^\infty((0,T) \times \mathbb{R}^d)} \leq \|u^0\|_{L^\infty(\mathbb{R}^d)}\},$$

a closed subset of $C((0, T); L^1(\mathbb{R}^d))$, is invariant under Φ, and hence the fixed point (i.e. u) must lie in that subset. For v with $\|v\|_{L^\infty} \leq \|u^0\|_{L^\infty}$, consider the associated solution f_v. We have

$$f_v(t, x, \xi) = \chi(\xi, u^0(x - a(\xi)t))e^{-\lambda t} + \lambda \int_0^t e^{-\lambda s} \chi(\xi, v(t - s, x - a(\xi)s, \xi)) \, \mathrm{d}s.$$

From this formula, we see immediately that $f_v = 0$ whenever $|\xi| \geq \|u^0\|_{L^\infty}$, and, using similar arguments as those used to prove (3), we obtain the sign property $|f_v| = \mathrm{sgn}(\xi) f_v \leq 1$. We can thus write

$$\Phi(v)(t,x) = u_v(t,x) = \int_{\mathbb{R}} f_v(t,x,\xi)\ \mathrm{d}\xi = \int_0^{\|u^0\|_{L^\infty}} f_v(t,x,\xi)\ \mathrm{d}\xi - \int_{-\|u^0\|_{L^\infty}}^0 |f_v(t,x,\xi)|\ \mathrm{d}\xi.$$

Notice that both integrals on the right-hand side are positive, and since $|f_v| \leq 1$, both integrals are bounded by $\|u^0\|_{L^\infty}$. Thus u_v, as a difference of two positive terms, must have absolute value bounded by the maximum of the two terms, and hence $|\Phi(v)(t,x)| \leq \|u^0\|_{L^\infty}$ for all t and x. This shows that $\|u\|_{L^\infty((0,T)\times\mathbb{R}^d)} \leq \|u^0\|_{L^\infty(\mathbb{R}^d)}$. Finally, from the representational formula for f, we clearly see that $f(t,x,\xi) = 0$ for $|\xi| > \|u^0\|_{L^\infty}$.

□

We now show that we can find a function m_λ such that

$$\frac{\partial m_\lambda}{\partial \xi} = \lambda(\chi(\xi, u_\lambda) - f_\lambda) = \frac{\partial f_\lambda}{\partial t} + a(\xi)\cdot\nabla_x f_\lambda.$$

This brings the equation to a form similar to that in the kinetic formulation. We obtain m_λ through the formula

$$m_\lambda(t,x,\xi) = \lambda \int_{-\infty}^{\xi} \chi(\zeta, u_\lambda(t,x)) - f_\lambda(t,x,\zeta)\ \mathrm{d}\zeta.$$

From this, we can prove several properties:

Proposition 4.4. *The function m_λ satisfies the following properties:*

(1) m_λ is nonnegative.

(2) For any convex S with $S'(0) = 0$, we have

$$\int_0^\infty \int_{\mathbb{R}^{d+1}} S''(\xi) m_\lambda(t,x,\xi)\ dx\ d\xi\ dt \leq \int_{\mathbb{R}^d} S(u^0(x)) - S(0)\ dx.$$

In particular, for $S(\xi) = (\xi \mp \xi_0)_\pm$ $(\xi_0 \geq 0)$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} m_\lambda(t,x,\pm\xi_0)\ dx\ dt \leq \int_{\mathbb{R}^d} (u^0(x) \mp \xi_0)_\pm\ dx \leq \mu(\pm\xi_0)$$

where $\mu(\xi) \leq \|u^0\|_{L^1}$ and $\lim_{|\xi|\to\infty} \mu(\xi) = 0$, and if $u^0 \in L^2$, then for $S(\xi) = \frac{\xi^2}{2}$ we have

$$\|m_\lambda\|_{L^1} \leq \frac{1}{2}\|u^0\|_{L^2}$$

.

(3) If $u^0 \in L^\infty$, then $m_\lambda(t,x,\xi) = 0$ for $|\xi| > \|u^0\|_{L^\infty}$.

Proof. Note that if $u_\lambda \geq 0$, then by the sign property of f_λ we have

$$\chi(\zeta, u_\lambda) - f_\lambda(t,x,\zeta) = -f_\lambda(t,x,\zeta) \geq 0$$

for $\zeta \leq 0$, Similarly, for $0 < \zeta < u_\lambda$, the integrand is $1 - f_\lambda(t,x,\zeta) \geq 0$, while for $\zeta > u_\lambda$ the integrand is $-f_\lambda(t,x,\zeta) \leq 0$. Hence, $\chi(\zeta, u_\lambda) - f_\lambda(t,x,\zeta)$ is nonnegative for $\zeta < u_\lambda$ and nonpositive for $\zeta \geq u_\lambda$, assuming $u_\lambda \geq 0$. A similar statement also holds for $u_\lambda < 0$. It follows that $m_\lambda(t,x,\xi)$ is increasing in ξ for $\xi < u_\lambda$ and $\xi > u_\lambda$, and furthermore

$$\lim_{\xi\to\infty} m_\lambda(t,x,\xi) = \int_{-\infty}^{\infty} \chi(\xi, u_\lambda(t,x)) - f_\lambda(t,x,\xi)\ \mathrm{d}\xi = u_\lambda(t,x) - \int_{-\infty}^{\infty} f_\lambda(t,x,\xi)\ \mathrm{d}\xi = 0.$$

Hence, m_λ is nonnegative.

To prove (2), we multiply the equation $\frac{\partial m_\lambda}{\partial \xi} = \frac{\partial f_\lambda}{\partial t} + a(\xi)\cdot\nabla_x f_\lambda$ by $S'(\xi)$ and integrate in x and ξ. By applying the arguments in Theorem 4.2, we may assume that the term $\int S'(\xi)a(\xi)\cdot\nabla_x f$ vanishes. We thus have

$$\int \frac{\partial f_\lambda}{\partial t} S'(\xi)\ \mathrm{d}x\ \mathrm{d}\xi = \int S'(\xi)\frac{\partial m_\lambda}{\partial \xi}\ \mathrm{d}x\ \mathrm{d}\xi = -\int S''(\xi) m_\lambda\ \mathrm{d}x\ \mathrm{d}\xi$$

and hence

$$\int S''(\xi) m_\lambda\ \mathrm{d}x\ \mathrm{d}\xi = -\frac{d}{dt}\left(\int S'(\xi) f_\lambda\ \mathrm{d}x\ \mathrm{d}\xi\right).$$

Integrating both sides with respect to time gives

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^{d+1}} S''(\xi) m_\lambda\ \mathrm{d}x\ \mathrm{d}\xi\ \mathrm{d}t &= \int S'(\xi)\chi(\xi, u^0(x))\ \mathrm{d}x\ \mathrm{d}\xi - \int S'(\xi) f_\lambda(T,x,\xi)\ \mathrm{d}x\ \mathrm{d}\xi \\ &\leq \int S'(\xi)\chi(\xi, u^0(x))\ \mathrm{d}x\ \mathrm{d}\xi \\ &= \int S(u^0(x))\ \mathrm{d}x.\end{aligned}$$

Note that S convex and $S'(0) = 0$ implies that the sign of $S'(\xi)$ matches that of ξ, and hence $S'(\xi) f_\lambda(t,x,\xi) \geq 0$ for all ξ. Letting $T \to \infty$ yields the general estimate. For $S(\xi) = (\xi \mp \xi_0)_\pm$, we have $S''(\xi) = \delta(\xi \mp \xi_0)$, so

$$\int_0^\infty \int_{\mathbb{R}^{d+1}} S''(\xi) m(t,x,\xi)\ \mathrm{d}x\ \mathrm{d}\xi\ \mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^d} m(t,x,\pm\xi_0)\ \mathrm{d}x\ \mathrm{d}t,$$

yielding the first inequality. We note that $\int_{\mathbb{R}^d} (u^0(x) \mp \xi_0)_\pm\ \mathrm{d}x$ is clearly bounded by $\|u^0\|_{L^1}$, while the integral goes to 0 as $\xi_0 \to \infty$ from the Dominated Converge Theorem. Finally, the last inequality follows from noting that $S(\xi) = \frac{\xi^2}{2} \implies S''(\xi) = 1$.

To prove (3), we simply note that $\chi(\xi, u_\lambda) = f_\lambda = 0$ for $|\xi| > \|u^0\|_{L^\infty}$ from Theorem 4.3, so by definition $m_\lambda = 0$ for $|\xi| > \|u^0\|_{L^\infty}$. □

4.4. **Convergence of approximate solutions.** To obtain a kinetic solution from a family of approximate solutions, we seek to show that the family of approximate solutions is compact, in order to extract subsequences $\{f_\lambda\}$, $\{u_\lambda\}$, and $\{m_\lambda\}$ converging to f, u, and m as $\lambda \to \infty$. We also aim to show that $f_\lambda - \chi(\xi, u_\lambda) \to 0$ as $\lambda \to \infty$, which will imply that $f = \chi(\xi, u)$, as desired.

To show compactness, we need uniform boundedness, equicontinuity, and uniform integrability. Uniform boundedness follows from the results of Theorem 4.3, while equicontinuity can be shown using the space-oscillation contraction in Theorem 4.3, combined with the following time continuity estimate:

Proposition 4.5. *There exists a modulus of continuity ω, independent of λ, such that*

$$\|u_\lambda(k,\cdot) - u^0(\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|f_\lambda(k,\cdot) - \chi(\cdot, u^0)\|_{L^1(\mathbb{R}^d\times\mathbb{R}_\xi)} \leq \omega(k).$$

Consequently, setting $u^0 = u(t,x)$ for some $t > 0$ yields the time continuity estimate

$$\|u_\lambda(t+k,\cdot) - u_\lambda(t,\cdot)\|_{L^1} \le \|f_\lambda(t+k,\cdot) - f_\lambda(t,\cdot)\|_{L^1} \le \omega(k).$$

Proof. Set $u^0_\epsilon = u^0 * \rho_\epsilon$, where ρ_ϵ is the standard mollifier in $\mathbb{R}^d$. Then $\|u^0_\epsilon\|_{L^\infty} \le \frac{\|\rho\|_{L^\infty}}{\epsilon}\|u^0\|_{L^1}$. Set $\chi^0_\epsilon = \chi(\xi, u^0_\epsilon)$, and let $N_\epsilon = \sup\limits_{|\xi|\le \|u^0_\epsilon\|_{L^\infty}} |a(\xi)|$. We have

$$\begin{aligned}\frac{\partial}{\partial t}(f-\chi^0_\epsilon) + a(\xi)\cdot\nabla_x(f-\chi^0_\epsilon) + \lambda(f-\chi^0_\epsilon) &= (f_t + a(\xi)\cdot\nabla_x f + \lambda f) - \lambda\chi^0_\epsilon - a(\xi)\cdot\nabla_x\chi^0_\epsilon \\ &= \lambda(\chi(\xi,u) - \chi^0_\epsilon) - a(\xi)\cdot\nabla_x\chi^0_\epsilon.\end{aligned}$$

Hence, $f-\chi^0_\epsilon$ satisfies equation (4.1), with $g = \lambda(\chi(\xi,u)-\chi^0_\epsilon) - a(\xi)\cdot\nabla_x\chi^0_\epsilon$ and initial data $\chi(\xi,u^0)-\chi^0_\epsilon$. We note that

$$\|g\|_{L^1} \le \lambda\|\chi(\xi,u)-\chi(\xi,u^0_\epsilon)\|_{L^1} + \|\nabla_x\chi^0_\epsilon\|_{M^1}\sup_{|\xi|\le\|u^0_\epsilon\|_{L^\infty}}|a(\xi)|$$

and

$$\|\chi(\xi,u)-\chi(\xi,u^0_\epsilon)\|_{L^1} = \|u-u^0_\epsilon\|_{L^1} \le \|f-\chi^0_\epsilon\|_{L^1}.$$

From Theorem 4.2, we have

$$\frac{d}{dt}(\|f(t)-\chi^0_\epsilon\|_{L^1}) + \lambda\|f-\chi^0_\epsilon\|_{L^1} \le \lambda\|f-\chi^0_\epsilon\|_{L^1} + N_\epsilon\|\nabla_x u^0_\epsilon\|_{M^1}.$$

Integrating in time thus gives

$$\|f(k)-\chi(\xi,u^0)\|_{L^1} \le \|\chi(\xi,u^0)-\chi(\xi,u^0_\epsilon)\|_{L^1} + kN_\epsilon\|\nabla_x u^0_\epsilon\|_{L^1} = \|u^0-u^0_\epsilon\|_{L^1} + kN_\epsilon\|\nabla_x u^0_\epsilon\|_{M^1}.$$

Letting $\omega^1(k) = \sup\limits_{|h|\le k}\|u^0(\cdot+h)-u^0\|_{L^1}$, we have

$$\|u^0-u^0_\epsilon\|_{L^1} \le \int\int |u^0(x-y)-u^0(x)|\rho_\epsilon(y)\,\mathrm{d}x\,\mathrm{d}y \le \omega^1(\epsilon).$$

By a similar argument, we obtain

$$\|\nabla_x\chi^0_\epsilon\|_{M^1} = \|\nabla_x u^0_\epsilon\|_{L^1} \le \|\nabla\rho_\epsilon\|_{L^\infty}\omega^1(\epsilon) \le \frac{\|\nabla\rho\|_{L^\infty}}{\epsilon}\omega^1(\epsilon).$$

Hence, we have

$$\|f(k)-\chi(\xi,u^0)\|_{L^1} \le \left(1+\frac{kN_\epsilon}{\epsilon}\right)\omega^1(\epsilon).$$

Setting $\omega(k)$ equal to the infimum of the right-hand side over all $\epsilon > 0$ gives the desired modulus of continuity. □

We now show the uniform integrability of u_λ, which shows the compactness of the family.

Proposition 4.6. *Let $u^0 \in L^\infty$, and let $N = \sup\limits_{|\xi|\le\|u^0\|_{L^\infty}}|a(\xi)|$. Then*

$$\int_{|x|\ge R}|u_\lambda(t,x)|\;dx \le \int_{|x|\ge\frac{R}{2}}|u^0(x)|\;dx + \frac{CNt\|u^0\|_{L^1}}{R}.$$

Note that we only have to consider $u^0 \in L^\infty$ since, given $u^0 \in L^1$, we can always regularize by convolution with some mollifiers such that the resulting convolution approximates u^0.

Proof. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative smooth function satisfying $\varphi(x) = 1$ for $|x| \geq 1$, $\varphi(x) = 0$ for $|x| \leq \frac{1}{2}$, and $\|\varphi\|_{L^\infty} = 1$. Set $\varphi_R(x) = \varphi(x/R)$. We have

$$\begin{aligned}\frac{\partial}{\partial t}(f_\lambda \varphi_R) + a(\xi)\cdot\nabla_x(f_\lambda\varphi_R) + \lambda f_\lambda\varphi_R &= \varphi_R(f_t + a(\xi)\cdot\nabla_x f + \lambda f_\lambda) + f_\lambda a(\xi)\cdot\nabla_x\varphi_R \\ &= \lambda\chi(\xi,u_\lambda)\varphi_R + f_\lambda a(\xi)\cdot\nabla_x\varphi_R.\end{aligned}$$

From Theorem 4.2, we have

$$\frac{d}{dt}\left(\int |f_\lambda|\varphi_R \,\mathrm{d}x\,\mathrm{d}\xi\right) + \lambda\int|f_\lambda|\varphi_R\,\mathrm{d}x\,\mathrm{d}\xi \leq \lambda\int|\chi(\xi,u_\lambda)|\varphi_R\,\mathrm{d}x\,\mathrm{d}\xi + \int|f_\lambda||a(\xi)||\nabla_x\varphi_R|\,\mathrm{d}x\,\mathrm{d}\xi.$$

Since

$$\int|\chi(\xi,u_\lambda)|\,\mathrm{d}x\,\mathrm{d}\xi = \int|u_\lambda|\,\mathrm{d}x \leq \int|f_\lambda|\,\mathrm{d}x\,\mathrm{d}\xi$$

and

$$\int|f_\lambda||a(\xi)||\nabla_x\varphi_R|\,\mathrm{d}x\,\mathrm{d}\xi \leq N\|\nabla\varphi_R\|_{L^\infty}\int|f_\lambda|\,\mathrm{d}x\,\mathrm{d}\xi \leq \frac{N\|\nabla\varphi\|_{L^\infty}}{R}\|u_\lambda(t,\cdot)\|_{L^1} \leq \frac{CN}{R}\|u^0\|_{L^1}$$

we have

$$\frac{d}{dt}\left(\int|f_\lambda|\varphi_R\,\mathrm{d}x\,\mathrm{d}\xi\right) \leq \frac{CN}{R}\|u^0\|_{L^1}$$

and hence

$$\int\limits_{|x|\geq R}|u_\lambda(t,x)|\,\mathrm{d}x \leq \int|f_\lambda|\varphi_R\,\mathrm{d}x\,\mathrm{d}\xi \leq \int|\chi(\xi,u^0)|\varphi_R\,\mathrm{d}x\,\mathrm{d}\xi + \frac{CNt}{R}\|u^0\|_{L^1} \leq \int\limits_{|x|\geq\frac{R}{2}}|u^0|\,\mathrm{d}x + \frac{CNt}{R}\|u^0\|_{L^1},$$

as desired. □

We thus obtain a sequence λ_n and $u \in L^1$ such that $u_{\lambda_n} \to u$ in $L^1([0,T]\times\mathbb{R}^d)$. The next step is to show the convergence of $\{f_{\lambda_n}\}$.

Proposition 4.7. *$f_{\lambda_n} \to \chi(\xi,u)$ in $L^1([0,T]\times\mathbb{R}^d\times\mathbb{R}_\xi)$ for the sequence $\{\lambda_n\}$ obtained above.*

Proof. Assume first that $u^0 \in L^\infty$. We use the representational formula to have

$$\begin{aligned}f_\lambda(t,x,\xi) - \chi(\xi,u(t,x)) &= \chi(\xi,u^0(x-a(\xi)t))e^{-\lambda t} + \lambda\int_0^t e^{-\lambda s}\chi(\xi,u_\lambda(t-s,x-a(\xi)s))\,\mathrm{d}s \\ &\quad - \chi(\xi,u(t,x))(e^{-\lambda t} + (1-e^{-\lambda t})) \\ &= e^{-\lambda t}(\chi(\xi,u^0(x-a(\xi)t)) - \chi(\xi,u(t,x))) \\ &\quad + \lambda\int_0^t e^{-\lambda s}(\chi(\xi,u_\lambda(t-s,x-a(\xi)s)) - \chi(\xi,u(t,x)))\,\mathrm{d}s.\end{aligned} \tag{4.4}$$

If we integrate the LHS over x and ξ, the first term in the RHS is bounded by $2\|f^0\|_{L^1}e^{-\lambda t}$, while the second term is bounded by

$$\begin{aligned}&\int_0^t \lambda e^{-\lambda s}\int |\chi(\xi,u_\lambda(t-s,x-a(\xi)s))-\chi(\xi,u(t-s,x-a(\xi)s))|\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}s\\+&\int_0^t \lambda e^{-\lambda s}\int |\chi(\xi,u(t-s,x-a(\xi)s))-\chi(\xi,u(t,x-a(\xi)s))|\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}s\\+&\int_0^t \lambda e^{-\lambda s}\int |\chi(\xi,u(t,x-a(\xi)s))-\chi(\xi,u(t,x))|\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}s.\end{aligned}$$

Substituting $\lambda=\lambda_n$ and letting $n\to\infty$, we note that the first term is, after a change of variables, bounded by

$$\int_0^t \lambda_n e^{-\lambda_n s}\int |\chi(\xi,u_{\lambda_n}(t-s,y))-\chi(\xi,u(t-s,y))|\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s=\int_0^t \lambda_n e^{-\lambda_n s}\|u_{\lambda_n}(t-s,\cdot)-u(t-s,\cdot)\|_{L^1}\,\mathrm{d}s,$$

which vanishes as $n\to\infty$ since $u_{\lambda_n}\to u$. The second term also vanishes as $n\to\infty$ by a similar argument. To control the third term, we note that if u is locally Lipschitz and compactly supported, with $N=\sup_{|\xi|\le\|u\|_{L^\infty}}|a(\xi)|$ and $\operatorname{supp} u\subset[0,T]\times K$, then

$$\begin{aligned}\int |\chi(\xi,u(t,x-a(\xi)s))-\chi(\xi,u(t,x))|\,\mathrm{d}x\,\mathrm{d}\xi&\le\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathbb{1}_{\xi\in(u(t,x-a(\xi)s),u(t,x))}\,\mathrm{d}x\,\mathrm{d}\xi\\&\le\int_{\mathbb{R}^d}\sup_{|y|\le Ns}|u(t,x-y)-u(t,x)|\,\mathrm{d}x\\&\le Ns\int_K\|\nabla_x u\|_{L^\infty}\,\mathrm{d}x=Ns|K|\|\nabla_x u\|_{L^\infty}\end{aligned}$$

and hence

$$\int_0^t \lambda e^{-\lambda s}\int |\chi(\xi,u(t,x-a(\xi)s))-\chi(\xi,u(t,x))|\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}s\le N|K|\|\nabla_x u\|_{L^\infty}\int_0^t \lambda_n s e^{-\lambda_n s}\,\mathrm{d}s\xrightarrow{n\to\infty}0.$$

Otherwise, we can regularize u by space convolution and truncation to obtain a locally Lipschitz and compactly supported u_δ such that $\|u(t,\cdot)-u_\delta(t,\cdot)\|_{L^1}\le\delta$ and $\operatorname{supp} u_\delta\subset B_{1/\delta}$ for each t, and the integral with u can thus be controlled by controlling the integral with u_δ. Hence, we see that f_{λ_n} converges to $\chi(\xi,u)$ if u^0 is L^∞.

For u^0 not in L^∞, we can simply regularize by convolution to obtain L^∞ initial data $\{u^0_\delta\}$ such that $\|u^0_\delta-u^0\|_{L^1}\xrightarrow{\delta\to 0}0$, and similarly with u and $\{f_\lambda\}$. The conclusion follows from the contraction $\|f_{\lambda_n,\delta}-f_\lambda\|_{L^1}\le\|u^0_\delta-u^0\|_{L^1}$. □

We conclude that $f_{\lambda_n}\to\chi(\xi,u)$. To conclude the proof of existence, we note that the functions m_λ satisfy the uniform local bound

$$\int_0^\infty\int_{\mathbb{R}^d\times(-R,R)} m_\lambda(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t\le 2R\max_{|\xi|\le R}\mu(\xi)\le 2R\|u^0\|_{L^1}.$$

Hence, we can extract a subsequence from $\{m_{\lambda_n}\}$ which converges weakly to some measure m, thus proving the existence of a solution to the kinetic formulation.

4.5. **Uniqueness.** To finish showing that the kinetic formulation of conservation laws is indeed well-posed, we must show that kinetic solutions are unique and depend continuously (in L^1) on the initial data. Both can be shown by showing the following contraction principle:

Theorem 4.8. *Let u_1 and u_2 be two kinetic solutions with corresponding initial data u_1^0 and u_2^0. Then*
$$\int_{\mathbb{R}^d} |u(t,x) - v(t,x)| \ dx \leq \int_{\mathbb{R}^d} |u_1^0(x) - u_2^0(x)| \ dx.$$

The proof will be sketched below without regard to regularity or rigor. A completely rigorous proof can be found in [5] and involves regularizing the χ functions by convolution in time and space.

Proof. Let m_1 and m_2 be the corresponding measures. Note that, for fixed (t,x), we have
$$\begin{aligned} |u_1 - u_2| = \int |\chi(\xi,u_1) - \chi(\xi,u_2)| \ \mathrm{d}\xi &= \int |\chi(\xi,u_1) - \chi(\xi,u_2)|^2 \ \mathrm{d}\xi \\ &= \int |\chi(\xi,u_1)|^2 + |\chi(\xi,u_2)|^2 - 2\chi(\xi,u_1)\chi(\xi,u_2) \ \mathrm{d}\xi \\ &= \int |\chi(\xi,u_1)| + |\chi(\xi,u_2)| - 2\chi(\xi,u_1)\chi(\xi,u_2) \ \mathrm{d}\xi \end{aligned}$$
since $|\chi|^2 = |\chi|$ and $|\chi(\xi,u_1) - \chi(\xi,u_2)|$ can only take the values 0 or 1. It thus suffices to show that
$$\frac{d}{dt}\left(\int |\chi(\xi,u_1)| + |\chi(\xi,u_2)| - 2\chi(\xi,u_1)\chi(\xi,u_2) \ \mathrm{d}x \ \mathrm{d}\xi\right) \leq 0.$$
If we multiply the equation $\frac{\partial}{\partial t}(\chi(\xi,u)) + a(\xi)\cdot\nabla_x u = \frac{\partial m}{\partial \xi}$ by $\mathrm{sgn}(\xi)$ and integrate, we obtain
$$\frac{d}{dt}\left(\int |\chi(\xi,u)| \ \mathrm{d}x \ \mathrm{d}\xi\right) = \int \mathrm{sgn}(\xi)\frac{\partial m}{\partial \xi} \ \mathrm{d}x \ \mathrm{d}\xi = -2\int m\Big|_{\xi=0} \ \mathrm{d}x.$$
(As before, the term containing $a(\xi)$ drops out after integration). It follows that
$$\frac{d}{dt}\left(\int |\chi(\xi,u_1)| + |\chi(\xi,u_2)| \ \mathrm{d}x \ \mathrm{d}\xi\right) = -2\int (m_1 + m_2)\Big|_{\xi=0} \ \mathrm{d}x. \tag{4.5}$$
Similarly, if we multiply the equation $\frac{\partial}{\partial t}(\chi(\xi,u_1)) + a(\xi)\cdot\nabla_x u_1 = \frac{\partial m_1}{\partial \xi}$ by $\chi(\xi,u_2)$, switch the roles of u_1 and u_2, and add, we obtain
$$\frac{\partial}{\partial t}\left(\chi(\xi,u_1)\chi(\xi,u_2)\right) + a(\xi)\cdot\nabla_x(\chi(\xi,u_1)\chi(\xi,u_2)) = \chi(xi,u_2)\frac{\partial m_1}{\partial \xi} + \chi(\xi,u_1)\frac{\partial m_2}{\partial \xi}$$
and hence integration yields
$$\begin{aligned} \frac{d}{dt}\left(\int 2\chi(\xi,u_1)\chi(\xi,u_2) \ \mathrm{d}x \ \mathrm{d}\xi\right) &= -2\int m_1\frac{\partial}{\partial \xi}\left(\chi(\xi,u_2)\right) + m_2\frac{\partial}{\partial \xi}\left(\chi(\xi,u_1)\right) \ \mathrm{d}x \ \mathrm{d}\xi \\ &= -2\int m_1(\delta(\xi) - \delta(\xi - u_2)) + m_2(\delta(\xi) - \delta(\xi - u_1)) \ \mathrm{d}x \ \mathrm{d}\xi \\ &= -2\int (m_1 + m_2)\delta(\xi) \ \mathrm{d}x \ \mathrm{d}\xi + 2\int m_1\delta(\xi - u_2) + m_2\delta(\xi - u_1) \ \mathrm{d}x \ \mathrm{d}\xi \\ &\geq -2\int (m_1 + m_2)\delta(\xi) \ \mathrm{d}x \ \mathrm{d}\xi = -2\int (m_1 + m_2)\Big|_{\xi=0} \ \mathrm{d}x. \end{aligned}$$
Subtracting the above inequality from equation (4.5) gives the desired result. ◻

5. Convergence in the Diffusion Approximation

We now prove convergence estimates for solutions of the conservation laws approximation $u_t + \mathrm{div}_x(A(u)) = \epsilon\Delta u$, using the kinetic formulation. In this section, let u be a kinetic solution to $u_t + \mathrm{div}_x(A(u)) = 0$ with initial data u^0, and let ω^0 and ω^1 be the time modulus of continuity of u and initial modulus of continuity of u^0, respectively, i.e.

$$\omega^0(k) = \sup_{0\le s\le k} \|u(s,\cdot) - u^0\|_{L^1}$$

and

$$\omega^1(k) = \sup_{|h|\le k} \|u^0(\cdot + h) - u^0\|_{L^1}.$$

We have the following estimate:

Theorem 5.1. *Suppose $v \in L^\infty((0,T); L^1(\mathbb{R}^d))$ satisfies the equation*

$$\frac{\partial}{\partial t}(\chi(\xi, v)) + a(\xi)\cdot\nabla_x(\chi(\xi,v)) = \frac{\partial m}{\partial \xi} + \alpha + \frac{\partial \beta_0}{\partial t} + div_x(\beta) + \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\gamma_{ij}$$

for some nonnegative measure m, with initial data v^0, where α,β_0,β, and γ satisfy

$$\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in M^1((0,T)\times\mathbb{R}^d), \bar{\beta_0} \in L^\infty((0,T); M^1(\mathbb{R}^d))$$

where $\bar{\alpha}(t,x) = \|\alpha(t,x,\cdot)\|_{L^1(\mathbb{R})}$, and similarly for the other terms. Then, for $T > 0$ and any $\epsilon_1, \epsilon_2 > 0$, we have

$$\begin{aligned}\|v(T) - u(T)\|_{L^1} &\le \|v^0 - u^0\|_{L^1} + \omega^1(\epsilon_2) + \omega^0(\epsilon_1) \\ &\quad + \|\bar{\alpha}\|_{M^1} + \frac{C}{\epsilon_2}\|\bar{\beta}\|_{M^1} + \frac{C}{\epsilon_2^2}\|\bar{\gamma}\|_{M^1} + \left(2 + \frac{CT}{\epsilon_1}\right) \sup_{t\in[0,T]} \int \beta_0(\bar{t}, x)\ dx.\end{aligned}$$

The estimate is proven in a similar fashion to that of the uniqueness theorem.

We now wish to consider a solution u_ϵ to the diffusion problem

$$(u_\epsilon)_t + \mathrm{div}_x(A(u_\epsilon)) = \epsilon\Delta u_\epsilon, \qquad u_\epsilon(0,x) = u^0(x)$$

and estimate the rate of convergence of u_ϵ to u as $\epsilon \to 0$. We have

$$\begin{aligned}\frac{\partial}{\partial t}(\chi(\xi, u_\epsilon)) + a(\xi)\cdot\nabla_x(\chi(\xi,u_\epsilon)) &= \delta(\xi - u_\epsilon)((u_\epsilon)_t + a(\xi)\cdot\nabla_x u_\epsilon) \\ &= \delta(\xi - u_\epsilon)((u_\epsilon)_t + \mathrm{div}_x(A(u_\epsilon))) = \delta(\xi - u_\epsilon)\epsilon\Delta u_\epsilon.\end{aligned}$$

We can apply Theorem 5.1 either with $\gamma_{ij} = \epsilon\delta_{ij}\delta(\xi - u_\epsilon)u_\epsilon$ (δ_{ij} being the Kronecker delta) and all other terms equal to zero, or with $\beta = \epsilon\delta(\xi - u_\epsilon)\nabla_x u_\epsilon$ and all other terms zero. These lead to the following results:

Theorem 5.2. *For any $\epsilon' > 0$, we have*

$$\|u_\epsilon(T) - u(T)\|_{L^1} \le \omega^1(\epsilon') + \frac{C}{(\epsilon')^2}(T\epsilon\|u^0\|_{L^1}).$$

If, in addition, $u^0 \in BV$, we have

$$\|u_\epsilon(T) - u(T)\|_{L^1} \le C\|u^0\|_{TV}\sqrt{T\epsilon}.$$

Proof. If we let $\gamma_{ij} = \epsilon\delta_{ij}\delta(\xi - u_\epsilon)u_\epsilon$, then in Theorem 5.1 we can take $\epsilon_1 = 0$ (since all terms associated with ϵ_1 vanish) and $\epsilon_2 = \epsilon'$. We thus have

$$\|u_\epsilon(T) - u(T)\|_{L^1} \le \omega^1(\epsilon') + \frac{C}{(\epsilon')^2}\|\bar{\gamma}\|_{M^1}.$$

Since $\|\gamma_{ij}\|_{M^1} = \epsilon\delta_{ij}\|u_\epsilon\|_{M^1} = \epsilon\delta_{ij}\int_0^T \|u_\epsilon(T)\|_{L^1(\mathbb{R}^d)}\ \mathrm{d}t \leq T\epsilon\delta_{ij}\|u^0\|_{L^1}$, it follows that $\|\bar{\gamma}\|_{M^1} = \sum_{i,j=1}^d \|\gamma_{ij}\|_{M^1} \leq T\epsilon d\|u^0\|_{L^1}$, and the first result follows. If we assume $u^0 \in BV$ and take $\beta = \epsilon\delta(\xi - u_\epsilon)\nabla_x u_\epsilon$, we can again take $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon'$, this time yielding

$$\|u_\epsilon(T) - u(T)\|_{L^1} \leq \omega^1(\epsilon') + \frac{C}{\epsilon'}\|\bar{\beta}\|_{M^1} \leq \epsilon'\|u^0\|_{TV} + \frac{C}{\epsilon'}\|\bar{\beta}\|_{M^1}.$$

Since $\|\beta\|_{M^1} = \epsilon\int \delta(\xi - u_\epsilon)|\nabla_x u_\epsilon|\ \mathrm{d}\xi\ \mathrm{d}x\ \mathrm{d}t = \epsilon\int_0^T\int |\nabla_x u_\epsilon(t,x)|\ \mathrm{d}x\ \mathrm{d}x \leq T\epsilon\|u^0\|_{TV}$, we obtain

$$\|u_\epsilon(T) - u(T)\|_{L^1} \leq \|u^0\|_{TV}\left(\epsilon' + \frac{CT\epsilon}{\epsilon'}\right).$$

Minimizing the RHS with respect to ϵ' yields the desired result. □

6. Compactness and Averaging Lemmas

We conclude by proving a result on the compactness of entropy solutions. To do so, we will use the following averaging compactness theorem:

Theorem 6.1. *Let $R > 0$, and let a satisfy the non-degeneracy condition*

$$|\{\xi : |\xi| < R, a(\xi)\cdot\zeta + \alpha = 0\}| = 0 \qquad \forall\alpha \in \mathbb{R}, \zeta \in S^{d-1}.$$

Consider the transport equation

$$f_t + a(\xi)\cdot\nabla_x f = \sum_{k=0}^m \frac{\partial^k}{\partial\xi^k}\left(div_{t,x,\xi}\, g_k\right),$$

and let $\{f_n\}_n$ and $\{g_{k,n}\}_n$ be sequences of functions satisfying this equation. Let $\psi \in C_C^\infty(\mathbb{R})$, and let $\rho_n(t,x) = \int \psi(\xi) f_n(t,x,\xi)\ d\xi$. If, for some $1 < q < \infty$, the sequence $\{f_n\}$ is bounded in $L^q(\mathbb{R}^+ \times \mathbb{R}^d)$, and $\{g_{k,n}\}$ is relatively compact in $L^q(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^\xi, \mathbb{R}^{d+2})$, then the averages $\{\rho_n\}$ are relatively compact in $L^q(\mathbb{R}^+ \times \mathbb{R}^d)$.

The proof can be found in [4] or [5]. The idea is to take the Fourier transform with respect to time and space of the transport equation to obtain

$$(\tau + a(\xi)\cdot\eta)\hat{f} = \sum_{k=0}^m \frac{\partial^k}{\partial\xi^k}\left(\left(\tau, \eta, \frac{\partial}{\partial\xi}\right)\cdot\widehat{g_k}\right)$$

and hence

$$\hat{f} = \frac{1}{\tau + a(\xi)\cdot\eta}\sum_{k=0}^m \frac{\partial^k}{\partial\xi^k}\left(\left(\tau, \eta, \frac{\partial}{\partial\xi}\right)\cdot\widehat{g_k}\right).$$

where τ and η are the time and space Fourier variables, respectively. It follows that $\hat{f}$ decays quickly (and hence ensures regularity) whenever the quantity $\tau + a(\xi)\cdot\eta$ is far from zero, and the non-degeneracy conditions ensure that this is indeed the case almost everywhere.

We can now prove the following compactness result on entropy solutions:

Theorem 6.2. *Assume a satisfies the non-degeneracy condition in Theorem 6.1. Let $\{u_n\}$ be a sequence of entropy solutions to $u_t + div_x(A(u)) = 0$ with uniform L^1 and L^∞ bounds. Then $\{u_n\}$ is locally relatively compact in $L^p(\mathbb{R}^+ \times \mathbb{R}^d)$ for all $1 \leq p < \infty$.*

Remark 6.3. Notice that no compactness requirement is needed for the initial data, only that the solutions be uniformly bounded. Furthermore, the non-degeneracy condition is needed for the result, to rule out the cases of transport equations of the form $u_t + \mathrm{div}_x(au) = 0$, $a \in \mathbb{R}^d$.

Proof. Let $\chi_n = \chi(\xi, u_n)$. From the kinetic formulation, we have $\frac{\partial \chi_n}{\partial t} + a(\xi) \cdot \nabla_x \chi_n = \frac{\partial m_n}{\partial \xi}$. We first localize the functions χ_n so that they are uniformly supported in t and x for every ξ in order to attain convergence more easily, since we are only seek local compactness results on u_n.

More specifically, let $0 < t_1 < t_2$, let $K \subset \mathbb{R}^d$ be compact, let $\phi_1 \in C_C^\infty(\mathbb{R}^+)$ satisfy $\phi_1(t) = 1$ for $t_1 \leq t \leq t_2$, let $\phi_2 \in C_C^\infty(\mathbb{R}^d)$ satisfy $\phi_2(x) = 1$ for $x \in K$, and set $f_n(t, x, \xi) = \phi_1(t)\phi_2(x)\chi_n$. Then $\{f_n\}$ is uniformly supported in t and in x, and since $\{u_n\}$ has a uniform L^∞ bound, it follows that $\{\chi_n\}$ and hence $\{f_n\}$ are uniformly supported in ξ as well. We have

$$\begin{aligned}\frac{\partial f_n}{\partial t} + a(\xi) \cdot \nabla_x f_n &= \phi_1' \phi_2 \chi_n + \phi_1 \phi_2 \frac{\partial \chi_n}{\partial t} + \phi_1 (a(\xi) \cdot \nabla_x \phi_2) \chi_n + \phi_1 \phi_2 a(\xi) \cdot \chi_n \\ &= \phi_1 \phi_2 \frac{\partial m_n}{\partial \xi} + (\phi_1' \phi_2 + \phi_1 a(\xi) \cdot \nabla_x \phi_2) \chi_n \\ &:= \frac{\partial m_n^1}{\partial \xi} + m_n^2\end{aligned}$$

where $m_n^1 = \phi_1 \phi_2 m_n$ and $m_n^2 = (\phi_1' \phi_2 + \phi_1 a(\xi) \cdot \nabla_x \phi_2)\chi_n$ are uniformly bounded measures with uniform compact support.

We now claim that there exist M_n^1 and M_n^2 in $W^{1,1}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}_\xi, \mathbb{R}^{d+2})$ such that $m_n^i = \mathrm{div}_{t,x,\xi} M_n^i$. Indeed, if we solve the equations $\Delta v_n^i = m_n^i$, and set $M_n^i = \Delta v_n^i$, then $m_n^i = \mathrm{div}_{t,x,\xi} M_n^i$, and the uniform bound on $\{m_n^i\}$ implies that $\{v_n^i\}$ uniformly bounded in $W^{2,1}$, and hence $\{M_n^I\}$ is uniformly bounded in $W^{1,1}$. Since $\{M_n^i\}$ are also compactly supported, we can apply the Rellich-Kondrachov theorem to conclude that $\{M_n^i\}_n$ is compact in L^q for $1 \leq q < \frac{d+2}{d+1}$.

Hence, if we fix some $1 < q < \frac{d+2}{d+1}$, we can apply Theorem 6.1 to conclude that $\{\rho_n\}$ is compact in $L^q(\mathbb{R}^{d+1})$, where $\rho_n = \int \psi(\xi) f_n(t, x, \xi) \, \mathrm{d}\xi$, for any $\psi \in C_C^\infty(\mathbb{R})$. Since $f_n(t, x, \xi) = 0$ for $|\xi| > \sup \|u_n\|_{L^\infty}$, we can choose ψ so that $\psi(\xi) = 1$ for $|\xi| \leq \sup \|u_n\|_{L^\infty}$, in which case

$$\rho_n(t, x) = \int \phi_1(t)\phi_2(x)\chi(\xi, u_n) \, \mathrm{d}\xi = \phi_1(t)\phi_2(x)u_n(t, x).$$

It follows that $\{\phi_1 \phi_2 u_n\}$ is compact in $L^q(\mathbb{R}^{d+1})$, and in particular $\{u_n\}$ is compact in $L^q(K')$, where $K' = [t_1, t_2] \times K$. Hence, there exists $u \in L^q(K')$ such that $u_n \to u$, up to subsequence. Since $\{u_n\}$ is uniformly bounded in $L^\infty(K')$, it follows that $u \in L^\infty(K')$. Note that we have the continuous injection $L^p(K') \hookrightarrow L^q(K') \cap L^\infty(K')$ for all $q < p < \infty$, and since K' has finite measure, we also have the continuous injection $L^p(K') \hookrightarrow L^q(K')$ for all $1 \leq p < q$. It follows that $u \in L^p(K')$ and $u_n \to u$ in $L^p(K')$ for all $1 \leq p < \infty$. This shows that $\{u_n\}$ is relatively compact in $L^p(K')$ for all $1 \leq p < \infty$. Since every compact $K' \in \mathbb{R}^+ \times \mathbb{R}^d$ is contained in $[t_1, t_2] \times K$ for some $0 < t_1 <_2$ and $K \subset \mathbb{R}^d$ compact, it follows that $\{u_n\}$ is locally relatively compact in L^p for all $1 \leq p < \infty$. □

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