THE GROTHENDIECK-RIEMANN-ROCH THEOREM FOR VARIETIES

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ABSTRACT. We give an exposition of the Grothendieck-Riemann-Roch theorem for algebraic varieties. Our proof follows Borel and Serre [3] and Fulton [5] closely, emphasizing geometric considerations and intuition whenever possible.

CONTENTS

1. INTRODUCTION

The familiar Riemann-Roch formula for a non-singular projective algebraic curve (equivalently in the complex case, a Riemann surface) equates algebraic/analytic information, in the form of dimensions of global sections of line bundles, to the purely topological genus:

$$h^0(D) - h^0(K-D) = \deg D - g + 1$$

for a divisor D, where g is the genus of the curve and K the canonical divisor.

This situation of equating a topological invariant with one derived from additional structure is a very familiar one in of geometry, with examples ranging from de Rham cohomology to the Grothendieck trace formula. Usually the idea is that the more tractable algebraic/analytic structure helps one get a handle on the slippery topology, though not always (cf. applications of the Atiyah-Singer index theorem to partial differential equations).

There is one very broad (albeit somewhat ill-defined) class of examples which are grouped with Riemann-Roch, often referred to in general as "Riemann-Roch formulas." It is hard to pin down a definition, but very broadly (and only semi-accurately), such formulas derive their algebraic/analytic invariant from sheaf cohomology, and their topological invariant is the genus of the geometric object under consideration. Often the notions of "sheaf cohomology" and especially "genus" are taken in considerable generality; e.g. the

Riemann-Roch theorem for number fields first formulated (for global fields) in Tate's thesis, though not yet explicitly in the terms above.

Restricting ourselves to the classical case and the algebraic world, generalizations of the theorem for curves arise naturally. How do we handle jumping a dimension? If we reformulate the theorem for curves this way:

$$\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X)$$

it becomes apparent.[1] We can more or less induct: if X becomes a non-singular projective surface rather than a curve, we can use exact sequence associated to the inclusion of a divisor

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i^*\mathcal{O}_D(D) \to 0$$

to write $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \chi(i^*\mathcal{O}_D(D))$ by additivity in short exact sequences. The first term is the arithmetic genus, which is promising. The second term's associated line bundle is tautologically the same as the line bundle on D associated to the self-intersection product $\mathcal{O}_D(D \cdot D)$, which by Riemann-Roch for curves has Euler characteristic $\chi(\mathcal{O}_D) + D \cdot D$.[2] We use here the fact that our last formulation of Riemann-Roch for curves holds even for singular curves.[3]

Using the fact that the degree of the anticanonical class is the topological Euler characteristic of the analytification, we find that the former term is $-K_D/2$, or equivalently $-(K \cdot D + D \cdot D)/2$ by adjunction. So finally we have on surfaces that

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{D \cdot D - K \cdot D}{2}.$$

In principle, this can be continued to higher dimensions, if we were able to continue to cleverly apply adjunction to compute arithmetic genus of the divisor.

There is an equally natural way to generalize in an orthogonal direction: if we take as fundamental not the divisor D but its associated line bundle $L = \mathcal{O}_X(D)$, notice that we can still write down the formulas, since we have an "inverse" in the first (algebraic) Chern class, with $c_1(L) = D$ recovering the divisor. We can then imagine extending the formulas to a general algebraic vector bundle E.[4] For example, the formula for surfaces is

$$\chi(E) = \frac{c_1(E)^2 - 2c_2(E) + c_1(E)c_1(T_X)}{2} + \chi(\mathcal{O}_X)\mathrm{rk}(E)$$

where T_X is the tangent bundle, which clearly generalizes the line bundle case above. To prove it, by the complex splitting principle, we need only show the first term on the RHS above is additive in short exact sequences.

One thing to note is that we can actually write $\chi(\mathcal{O}_X)$ as a polynomial of Chern classes as well, which should come as no surprise given that, as in the cohomological case, every characteristic class can be written in terms of them. Explicitly, for surfaces,

$$\chi(\mathcal{O}_X) = \frac{c_1(T_X)^2 + c_2(T_X)}{12},$$

a result known as Noether's formula. In general, as we move up in dimension of the variety, we should continue to expect $\chi(E)$ to be expressible as a polynomial of Chern classes of E and the tangent bundle.

[1] One should think of the algebraic/holomorphic Euler characteristic $\chi(\mathcal{O}_X)$ as a genus of sorts. Indeed, it is Hirzebruch's preferred definition of "arithmetic genus" for general varieties, over the one corresponding to the familiar geometric genus motivated from topology. This is because it is a genus in the sense of genera theory: a homomorphism from the complex cobordism ring - i.e. it is multiplicative on products and additive on disjoint union. It is thus apparent that it is a genuine topological invariant. Indeed, in the case of curves, we may replace it with $\chi(X)/2$, in terms of the topological Euler characteristic.

[2] The notation $D \cdot D$ to denote an integer is abuse of notation; we are making the natural association of zero-dimensional cycles with their degree - that is, the evaluation of the associated cohomology class on the fundamental class of the analytification. Fulton denotes this homomorphism by an integral sign $\int_X$ for this reason.

[3] Alternately, prove it for divisors represented by smooth curves and extend using the natural homomorphism $\mathrm{Pic}(X) \to K(X)$.

[4] Throughout this paper we will implicitly use the identification of algebraic vector bundles (as geometric objects, i.e. schemes over the base X with appropriate local affine trivializations and transition maps) with locally free sheaves, justified by the equivalence of categories. Hence, for example, the use of the common double meaning of "line bundle" in the algebraic setting.

These lines of generalization find full realization in the Hirzebruch-Riemann-Roch theorem, which says that for a complete nonsingular variety X with a vector bundle E,

$$\chi(E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(X)$$

where the **Chern character** $\mathrm{ch}(-)$ and the **Todd class** $\mathrm{td}(-)$ are formal power series of certain Chern classes (and in practice polynomials, by dimensionality considerations) which satisfy the nice formal properties we desired in the discussion above. This theorem, and in fact even far less general statements like Noether's formula, does not have a short direct proof. Hirzebruch's original proof was for the special case of complex algebraic manifolds, usng analytic techniques and cohomological Chern classes instead of algebraic cycles.[9] The result actually holds in far more generality, in both the algebraic case (which we cover here), where the formulation with algebraic cycles is a strict refinement of the cohomological one, and in the complex one, where it holds for arbitrary complex manifolds by the Atiyah-Singer index theorem.

Grothendieck's insight in the algebraic case was to take a relative point of view, observing that the "degree" on the RHS is a special case of pushforward. If the theorem could be written as a certain commutative diagram claiming naturality of a certain map under pushforward, then one could attack the problem by factoring morphisms into simpler pieces and tackling each individually. The main algebraic innovation which enabled him to do this was the **Grothendieck group**, giving the needed structure to the vector bundles on which the theorem hinged, and marking the nascency of algebraic K-theory.

In this paper, we will give an exposition and proof of the original statement of Grothendieck-Riemann-Roch as given in, e.g., [8].

Theorem 1.1 (Grothendieck-Riemann-Roch for varieties)**.** *If X and Y are nonsingular varieties,*[5] *and $f : X \to Y$ is a proper morphism, then we have the following commutative diagram:*

$$\begin{array}{ccc} K(X) & \xrightarrow{ch(-)\cdot td(X)} & A(X) \otimes \mathbb{Q} \\ f_* \downarrow & & \downarrow f_* \\ K(Y) & \xrightarrow{ch(-)\cdot td(Y)} & A(Y) \otimes \mathbb{Q} \end{array}$$

2. Foundations

To prove theorem 1.1, we must first develop some of the theory behind the undefined objects in its statement. We will work in some more generality than necessary for our particular application, following the philosophy of Lang.

2.1. Algebraic cycles and the Chow groups.

We give a brief review of the intersection theory of cycles in our restricted case.

On any variety X, we consider the additive group of algebraic cycles, modulo rational equivalence, which we will denote by $A(X)$. $A(X)$ is then naturally graded by dimension into what are sometimes referred to as the **Chow groups**, $A_0(X), A_1(X), \ldots, A_k(X)$, where $k = \dim X$. Equivalently, we can grade by codimension as $A^k(X), \ldots, A^1(X), A^0(X)$.[6]

We have the **flat pullback** and **proper pushforward**, which are exactly what they sound like: for $f : X \to Y$ a flat map, we have $f^* : A(Y) \to A(X)$ given by simply taking preimages of the subschemes, and for $f : X \to Y$ proper, we have $f_* : A(X) \to A(Y)$ given by the image of the component subvarieties, with appropriate multiplicities, chosen to be nonzero only when the dimension of the subvariety is not collapsed.

These have nice functorial properties. For details, see [5]. Notice that proper pushforward preserves dimension (the first grading mentioned). Flat pullback, if we add the stipulation of constant relative dimension

[5]In this paper, a variety is over an algebraically closed field.

[6]There are alternate Chow groups given by any (e.g. algebraic, numerical, etc.) equivalence relation on cycles, but rational equivalence is the most common, and gives the finest (in the sense of not coarse) results.

to the morphism, preserves codimension (the second grading). In our restricted context of varieties, this is always true; see Stacks Project section 28.29. As noted earlier, $\int_X$, the degree homomorphism, is just pushforward to the point Spec k, where k is the base field.

In fact, while we need our morphism to be flat to pull back cycles, we can use another moving lemma to obtain a pullback ring homomorphism whenever source and target are regular varieties, by shifting the component cycles suitably. See theorem 5.8 in [4]. This works in the quasiprojective case. Fulton's treatment in [5] allows intersection products by iterating intersections with Cartier pseudo-divisors, which can be pulled back arbitrarily, extending this to all varieties. What is important is that in our context, we in fact have arbitrary pullbacks.

There is a commutative bilinear intersection product on $A(X)$, $\cdot : A^k(X) \otimes A^l(X) \to A^{k+l}(X)$, which takes the classes represented by transversely intersecting (for a suitable notion of transverse) cycles to the class of their actual geometric intersection. In the case where X is regular, this induces a graded ring structure on $A(X)$. The resulting ring is sometimes called the **Chow ring**, and denoted $A^*(X)$ to emphasize the ring structure.[7]

Pullback actually preserves the ring structure as well, as follows from the geometric moving lemma approach (in cases where it applies). That is, we have that $f^* : A^*(Y) \to A^*(X)$ is actually a homomorphism of graded rings. This of course fails for trivial reasons for pushforwards.

The ring $A^*(X)$ is essentially an algebraic version of cohomology. In the complex case, there is in fact a natural homomorphism of graded rings $A^*(X) \to H^{2*}(X^{an})$ on algebraic manifolds given by sending a cycle class represented by a subvariety to the homology class represented by its analytification, and then to its Poincaré dual. In general, singular homology spectacularly fails to be generated by the classes of subvarieties,[8] so this is not surjective. On the other hand, as we mentioned, algebraic cycles are in general a finer theory than cohomology, so this map is not usually injective either.[9]

2.1.1. *Algebraic Chern classes.* In analogy with cohomological Chern classes, we have a finer algebraic theory of Chern classes $c_0(E), c_1(E), c_2(E), \ldots, c_r(E)$ of a rank-r vector bundle $E \to X$, which reside in the gradings $A^0(X), A^1(X), \ldots, A^r(X)$ respectively. Geometrically, they represent the cycle classes of degeneracy loci of tuples of global sections (though this is only actually true for bundles generated by such sections), so we naturally get the equality $c_0(E) = 1 = [X] \in A^0(X)$. More nebulously, we can think of them as recording the "twisting" of the bundle. Indeed, the first Chern class of the line bundle associated to a divisor satisfies $c_1(\mathcal{O}_X(D)) = D$. We also define the total Chern class $c(E) = 1 + c_1(E) + \ldots + c_r(E)$.

Chern classes are natural over pullback, and the total Chern class is multiplicative in short exact sequences, a result known as the Whitney sum formula.[10]

When E is a direct sum of line bundles $\bigoplus L_i$, we obtain $c(E) = \prod(1 + c_1(L_i))$. Again in analogy with the topological theory, we have a splitting principle in which we can obtain a formal factorization (in a ring extension of $A^*(X)$) like this even when E is not such a direct sum, as there is a space whose Chow ring

[7] For those interested in the construction of the intersection product, [4] chapter 5 gives an accessible account in the smooth case using the classical "moving lemma" approach, which moves representatives of the two rational cycle classes until they intersect transversely. For perfect fields, this is general enough, but if the base field is not perfect, regular varieties are not necessarily smooth. [5] gives a more sophisticated approach; a more refined and general "intersection product" is given in chapter 6, and the ring structure for any nonsingular variety is developed in chapter 8 using a Künneth-type exterior product formula together with the pullback of the diagonal map, in strong analogy with the cup product in cohomology.

[8] Iintegral homology is not even in general generated by the classes of topological (i.e. not necessarily complex) submanifolds. Complex submanifolds, which by GAGA are equivalently subvarieties, are a small subset of these, and these classes always fall into the middle term $H^{k,k}$ in the Hodge diamond. Indeed, the statement that this middle term is precisely their image (when taking rational coefficients) is the Hodge conjecture.

[9] That is, the adequate equivalence relation of rational equivalence is in general strictly finer than that of homological equivalence, where we take our Weil cohomology theory to be Betti cohomology (though this conjecturally does not matter). The question of when the two do coincide is unresolved; one case where $A^*(X) \cong H^{2*}(X^{an})$ is for spaces with affine stratifications, e.g. complex projective space.

[10] The above properties (naturality, Whitney sum, and value on line bundles) in fact give an axiomatic characterization of Chern classes.

provides precisely the extension necessary.[11] As a result, we can always treat $c(E)$ as if it factors "linearly," and any formula which holds for a sum of line bundles will hold in general. More specifically, we can treat the total Chern class as if it factors into $(1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_r)$ in some ring extension. The α_i are often called **Chern roots**.

2.2. **Algebraic K-theory and G-theory.** The behavior of the total Chern class given by the Whitney sum formula puts it into a wide class of functions which are additive (or in this case, multiplicative) on short exact sequences. To capture this additivity formally, Grothendieck was led to the beginnings of algebraic K-theory. We give a brief review of the fundamental elements.

Recall that $K^0(X)$ is the free abelian group on algebraic vector bundles over X, modulo short exact sequences. As such, it is the universal structure for additive functions on vector bundles, in the sense that such a function factors through a morphism from the group. Additionally, there is in fact a ring structure on $K(X)$[12] given by tensor product. Locally free sheaves are flat, so this is consistent with the short exact sequences condition.

If we expand from the world of locally free to all coherent sheaves, we get a group $G(X)$ under the same conditions, called the Grothendieck group of the variety. On a regular variety, it is a classic result, the "global version" of Hilbert's syzygy theorem, that every coherent sheaf has a finite projective resolution, so in this case, with some extra work, we actually have $K(X) \cong G(X)$. Tensoring with coherent sheaves is not in general exact, so there is no natural ring structure on $G(X)$,[13] but in the regular case we can transport the structure via isomorphism. The resulting formula for multiplying coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ is

$$[\mathcal{F}\cdot\mathcal{G}] = \sum_{k=0}^{\infty}(-1)^k[\mathrm{Tor}^k_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})],$$

adding on all the derived functors of the tensor product to correct for the lack of exactness.[14]

In our regular setting, we will consider all coherent sheaves for maximal generality, but use $K(X)$ to stand in for $G(X)$ as an abuse of notation so that we can move freely between the two worlds.

Of course $K(X)$ and $G(X)$, being algebraic analogues of cohomology theories, can be pulled back. In G-thery, we use the formula $f^* = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}$ for flat morphisms, but since we have the isomorphism with $K(X)$ and the well-known pullback of vector bundles, we can actually have a pullback homomorphism of rings f^* for arbitrary morphisms - just as with the Chow ring.[15]

As with algebraic cycles, there is also a pushforward theory, but only in G-theory (and hence K-theory in the regular case). Given $f : X \to Y$, we have the direct image sheaf f^*. Under proper maps, the direct image of a coherent sheaf is coherent. This commutes with direct sum (trivially, by definition of direct image), but cannot directly give the pushforward, since it is not exact. We must again use the alternating sum trick of its derived functors, the **higher direct images**, denoted $R^i f_*$, a sort of "global version" of sheaf cohomology.[16] It is a formal verification to see that as usual, the alternating sum will give exactness, but to show we can do this, we need the following result:

[11] Specifically, we construct a flag variety Y so that $Y \to X$ induces an injective pullback $A^*(X) \to A^*(Y)$, such that the total Chern class of E factors in the larger ring. For details and a very nice treatment of algebraic Chern classes in general, see [4] chapter 7.

[12] We will just drop the zero since we have no use for higher algebraic K-theory in this paper.

[13] For this reason, sometimes $G(X)$ is denoted $K_0(X)$, in analogy with how homology has no natural ring structure while cohomology does.

[14] Algebraically, the regularity condition more or less ensures that this sum is finite. This alternating sum of derived functors to get a correction for non-exactness is a common construction, e.g. the Euler characteristic. One might recognize this instance as being very similar to Serre's intersection multiplicity formula, and they are indeed related since once can construct very general intersection products using K-theory; see this exposition on Daniel Dugger's website for a restricted version of this. We see another example of this type of alternating construction below in this same section.

[15] For perfect morphisms f, we can write out this pullback explicitly using an alternating sum of Tor functors for $G(X)$, due to this isomorphism. This provides another example of the utility of the alternating derived functors trick.

[16] Indeed, sheaf cohomology is just the higher direct image sheaves of the map to Spec k. In the general case, the Leray spectral sequence converges to sheaf cohomology on the source.

Theorem 2.1. *Given a proper morphism of regular varieties $f : X \to Y$ and a coherent sheaf $\mathcal{F}$ on X, $R^i f_*(\mathcal{F})$ is coherent.*

Proof. First recall that kernels, cokernels, and extensions of coherent sheaves are also coherent. Hence by the long exact sequence of the derived functor the same applies to the property of having all higher direct images coherent under a given map - i.e., being sandwiched in a long exact sequence by sheaves with the property implies the property.

We use the typical trick to reduce to projective morphisms: Chow's lemma gives us a scheme X' over Y equipped with a projective surjection $h : X' \to X$ over Y which is an isomorphism on an open set U, so that $f' : X' \to Y$ is projective, and everything commutes. Then for $\mathcal{F}$ coherent on X, the kernel and cokernel of $h_* h^* \mathcal{F}$ are supported on $X \setminus U$. All of these are coherent, since the projective pushforward h_* preserves coherence by assumption and h^* does in general. We apply the classic technique of noetherian induction to see that it suffices to prove, by the point about sandwiching above, that $h_* h^* \mathcal{F}$ has all coherent higher direct images, since we can assume coherence under higher direct image of the lower-dimensional kernel and cokernel by inductive hypothesis.

Indeed, we have in general that the Grothendieck spectral sequence $E_2^{i,j} = R^i f_* R^j h_* h^* \mathcal{F}$ converges to $R^{i+j} f_* h_* h^* \mathcal{F} = R^{i+j} (f')_* h^* \mathcal{F}$. The projective case gives us that the latter is coherent. If $j > 0$, $R^i f_* R^j h_* h^* \mathcal{F}$ is coherent by the inductive hypothesis. To see this, observe that higher direct images are local on the target, and that since h is an isomorphism on U, its higher direct images are supported away from it, so are strictly lower dimensional. By sandwiching in the exact sequence in low degrees, the full result follows.

So we have reduced to verifying the projective case. If X is projective over Y, let $i : X \to \mathbb{P}^n_Y$ be the closed immersion and $\pi : \mathbb{P}^n_Y \to Y$ be the projection.[17] The higher direct images of a closed immersion simply are zero, so it suffices to prove that $R^i \pi_*$ preserve coherence.

We can assume Y is affine by localness of coherence. Then by Hilbert's syzygy theorem, every coherent sheaf on $X = \mathbb{P}^n_Y$ has a resolution by powers of the twisting sheaves. By the earlier discussion, it suffices to prove that $R^i \pi_* \mathcal{O}_{\mathbb{P}^n_Y}(k)$ are coherent. Since we can take Y affine, these can be explicitly computed using Čech cohomology to be $\mathcal{O}_Y \otimes H^i(\mathbb{P}^n_Y, \mathcal{O}_{\mathbb{P}^n_Y}(k))$, which is coherent.

This result actually holds in much more generality, over any locally noetherian scheme by basically the same proof; see Stacks Project 55.19, from which this proof was adapted, or EGA III 3.2. □

The higher direct images commute with direct sum because direct image does, so we have the pushforward in K-theory $f_* : K(X) \to K(Y)$ given by

$$[\mathcal{F}] \mapsto \sum_{k=0}^{\infty} (-1)^k [R^k f_*(\mathcal{F})].$$

2.3. **Chern character.** Our motivation for introducing $K(X)$ was to give a structure to capture the behavior of the multiplicative total Chern class $c(E)$ of a vector bundle, and we indeed have that $c : K(X) \to A^*(X)^\times$ is a homomorphism, since existence of stable inverses allows us to extend linearly. This, however, converts an additive theory to a multiplicative one, and essentially forgets the tensor product and ring structure.[18]

To correct this, note that we have the pleasant relation $c_1(E \otimes F) = c_1(E) + c_1(F)$. (Recall the motivation from divisors corresponding to line bundles.) For line bundles, at least, it occurs to us then that if we could map $E \to e^{c_1(E)}$, this would map multiplication to multiplication, undoing the "logarithm"-type

[17] Recall that $\mathbb{P}^n_Y = \mathbb{P}^n_k \times Y$.

[18] It actually is possible to obtain a modification called the *augmented* total Chern class (where the constant 1 term is replaced by the rank of the bundle) which can be made into a ring homomorphism into a new λ-ring structure on $A^*(X)^\times$, where former multiplication becomes addition and a new structure becomes multiplication. This has little relation to the ring homomorphism we define via the Chern character, but Riemann-Roch without denominators (see "Concluding Remarks") can be formulated this way. See [7] for details.

effect of the first Chern class.[19] Moreover, for a direct sum of line bundles $E_1 \oplus \ldots \oplus E_r$, we could map to $e^{c_1(E_1)} + \ldots + e^{c_1(E_r)}$ and have addition and multiplication go where they should. By the splitting principle, we can extend this in general to Chern roots, giving us the Chern character:

$$\mathrm{ch}(E) = e^{\alpha_1} + e^{\alpha_2} + \ldots + e^{\alpha_r}.$$

To make sense of the exponential, of course, we must use its formal power series, with non-integral coefficients, so this takes values in the rationalization $A(X) \otimes \mathbb{Q}$. Because it is symmetrical in the Chern roots, the Chern character can be written in terms of the Chern classes, and indeed we can compute explicitly some terms:

$$\mathrm{ch}(E) = \dim(E) + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \frac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{6} + \ldots$$

Notice that this is a finite series for every actual computation because of the finiteness of the grading. ch is then a genuine ring homomorphism $K(X) \to A(X) \otimes \mathbb{Q}$. We naturally extend its domain to $K(X) \otimes \mathbb{Q}$, in which case it is a remarkable theorem that this is actually an isomorphism.[20] See [5] section 15.2.

We sometimes write $\mathrm{ch}(F)$ instead of $\mathrm{ch}([F])$ for a vector bundle or coherent sheaf F by abuse of notation.

2.4. **Covariance, contravariance, and the Todd class.** It is quite remarkable that K-theory and algebraic cycles are both contravariant and covariant functors on regular varieties, for arbitrary and proper maps respectively. Both of them are in a sense more "naturally" contravariant functors, since after all they are both algebraic analogues of cohomology theories. This is reflected both in how the contravariant theory preserves the ring structure, while the covariant one does not, and in how the covariant theory applies to a more restricted class of morphisms. Hence the pushforward maps are somewhat exceptional objects, and for this reason they are sometimes referred to as "shriek maps" (and, confusingly, denoted $f_!$) or even "Gysin maps" in analogue with the exceptional pushforward of that name in cohomology coming from integration along fibers.[21]

It turns out that the Chern character is actually a natural isomorphism between the contravariant theories. That is, we have the following commutative diagram.

$$\begin{array}{ccc} K(Y) \otimes \mathbb{Q} & \xrightarrow{\mathrm{ch}} & A(Y) \otimes \mathbb{Q} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ K(X) \otimes \mathbb{Q} & \xrightarrow{\mathrm{ch}} & A(X) \otimes \mathbb{Q} \end{array}$$

This is simply a consequence of the functoriality of Chern classes with respect to pullback, so really contains no new information not already practiced in the application of Chern classes. This is not the case for pushforward, so there is no such immediate result.

However, in many applications of intersection theory, we wish to compute with respect to a pushforward of a vector bundle on some parameter space; for example, the classical computation of 27 lines on a cubic entails computing the pushforward of the Grassmanian of lines in $\mathbb{P}^3$. So even though the Chern character does not appear to be a natural transformation of the covariant theories, it would be convenient to have some analogue of the result.

[19] Indeed, one can put some formal weight behind our intuition/motivation for the Chern character using the formal group laws associated to the Chern classes of K-theory and algebraic cycles; the former is the multiplicative group law and the latter the simple additive one, so "logarithm" and "exponential"-type maps go betwen them. If we pass to the topological setting, this is part of a larger framework of Hurewicz-type homomorphisms on generalized cohomology theories given by ring spectra.

[20] Interestingly, this tight comparison of $K(X)$ and $A(X)$ once torsion is killed reflects a parallel in the algebraic/topological dichotomy: just as $A(X)$ is a finer but not as broad version of cohomology, there is $K^{top}(X^{an})$ for complex X, and a natural map $K(X) \to K^{top}(X^{an})$, neither generally injective nor surjective, which is even natural with respect to pullback/pushforward. Geometrically, this is just the fact that not every topological complex vector bundle can be given algebraic (or even holomorphic structure), but those that do often have multiple algebraic structures (think of twisting sheaves on a real circle, all equivalent to the Möbius bundle, as a toy example for intuition). The existence of the parallel isomorphism $K^{top}(X) \otimes \mathbb{Q} \cong H^{2*}(X, \mathbb{Q})$ for complex manifolds is another piece in the interesting dictionary between Riemann-Roch theorems in the algebraic and complex worlds.

[21] Warning: Fulton uses "Gysin map" in quite a different way.

Grothendieck-Riemann-Roch says precisely this analogue: from the diagram in theorem 2.1, ch is not a natural transformation, but $\mathrm{ch}(-) \cdot \mathrm{td}(X)$ is. We drop the rationalization on K-theory because there is no reason for it.[22]

$\mathrm{td}(X)$ is of course the Todd class, and is defined as follows: the Todd class of a vector bundle (or generally an element of $K(X)$) is

$$\prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{-\alpha_i}}$$

in terms of the Chern roots. As with the Chern character, the symmetry allows it to be written as a formal power series in terms of the Chern classes:

$$\mathrm{td}(E) = 1 + \frac{c_1(E)}{2} + \frac{c_1(E)^2 + c_2(E)}{12} + \frac{c_1(E)c_2(E)}{24} + \dots$$

where again in practice this is actually finite due to nilpotence. The Todd class associated to X is defined as $\mathrm{td}(X) = \mathrm{td}(T_X)$; simply the Todd class of its tangent bundle. The **Todd genus** is the degree $\int_X \mathrm{td}(X)$.

The Todd class seems rather odd and arbitrary. It is difficult to give concrete motivation for it, but there are some nice properties which may serve as our best substitute.

The primary light one should view it in is in the sense of a **genus of a multiplicative sequence**, which is a general formal construction from algebraic topology. In the complex case, such a genus is a construction associated to a formal power series $Q(x)$. Then as with the Todd genus, we can write out the class $\prod Q(\alpha_i) = K(c_1, c_2, \dots)$ as a function of the Chern classes (of a bundle) by symmetry. The rational number which is the degree of the tangent bundle is then taken to be the genus. (The "multiplicative sequence" in question is the sequence of homogenous parts of K.) The reason these are useful is that an equivalent abstract formulation is that of a homomorphism from the complex cobordism ring Ω^{MU} into the rationals, meaning such genera are additive on disjoint unions and multiplicative on products of spaces, a property not difficult to verify from the algebraic nonsense. See chapter 7 of these notes by Liviu Nicolaescu for a lucid account of the details. As a simple example, the genus associated to the power series $Q(x) = x$ can be computed to be simply the topological Euler characteristic $\chi(X)$, for which these properties are familiar. This intuition of course all only strictly applies in the complex case, so remember it is simply a motivating framework.

The series for the Todd genus is clearly $\frac{x}{1-e^{-x}}$, which is almost the generating function for the Bernoulli numbers.[23] The nice formal notion of genus above gives us some idea of why it may play a role in our Riemann-Roch formula (from the beginning discussion of such formulas involving topological genera), but why must it be this one?

Again remembering we are motivating from the complex case, it turns out that we can determine such genera purely from their values on complex projective space: the Todd genus is characterized by having value 1 on complex projective spaces.

First, the Todd genus should be 1 for a point, at least. Further, in theorem 2.1, if we take $Y = \mathrm{Spec}\, k$ in general, and $\mathcal{F}$ a coherent sheaf, we find that the statement we recover is, as desired, Hirzebruch-Riemann-Roch:

$$\mathrm{td}(Y)\mathrm{ch}\left(\sum(-1)^i R^i f_*(\mathcal{F})\right) = \mathrm{ch}\left(\sum(-1)^i \underline{H^i(X, \mathcal{F})}\right) = \chi(\mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F})\mathrm{td}(X).$$

In particular, if $k = \mathbb{C}$, $X = \mathbb{P}^k_{\mathbb{C}}$, $\mathcal{F} = \mathcal{O}_X$, we obtain

$$\int_X \mathrm{td}(X) = \chi(\mathcal{O}_X) = 1$$

so this shows we need the Todd genus of complex projective space to be 1. We can even write out the left-hand side as the coefficient of x^n in $Q(x)^{n+1}$, since we have the classical Euler sequence $0 \to \mathcal{O}_X \to$

[22] Though, incredibly, again $\mathrm{ch}(-) \cdot \mathrm{td}(X)$ is also an isomorphism of groups if one passes to the rationalization. See [5] chapter 18, or [8] for a fuller account.

[23] Quote from my advisor: "If you figure out why, you'll win a Fields medal."

$\mathcal{O}(1)^{n+1} \to T_X \to 0$. This condition on $Q(x)$ uniquely determines that the formal series must be $\frac{x}{1-e^{-x}}$, giving us an idea of how this works in general.

There is another, less foundational way to motivate the Todd genus: it is something of a "multiplicative inverse" for the Chern character in the following sense: it is determined by the identity

$$\sum_{i=0}^{r}(-1)^i \mathrm{ch}(\wedge^i E^\wedge) = c_d(E)\cdot \mathrm{td}(E)^{-1}.$$

where d is the dimension. Indeed, if we simply write everything out explicitly, noting that the Chern roots of the ith exterior power are all the i-sums of the Chern roots, this is a short computation.[24]

Notice finally that if in Hirzebruch-Riemann-Roch we take $\mathcal{F} = \mathcal{O}_X$, we get the formula $\int_X \mathrm{td}(X) = \mathcal{O}_X$ in general. Hence the Todd genus actually coincides with Hirzebruch's arithmetic genus, or holomorphic genus, we mentioned earlier. This fits with the classically known fact $\chi(\mathcal{O}_{X\times Y}) = \chi(\mathcal{O}_X)\chi(\mathcal{O}_Y)$.

3. Formulas in algebraic cycles and K-theory

Before proceeding to the proof, we must prove three major formulas which hold in identical form for both $A(X)$ and $K(X)$.

3.1. Excision sequences.

These are straightforward: in analogy with cohomology, there are excision sequences for algebraic cycles and K-theory.

Theorem 3.1. *If X is a regular variety and Y is a closed subvariety, we have the exact sequence of groups*

$$K(Y) \to K(X) \to K(X-Y) \to 0.$$

Proof. The regularity condition is necessary only because this is true on the level of G-theory and coherent sheaves with no additional assumptions. If $i : Y \to X$ and $j : X - Y \to X$ are the inclusion maps, the first map is i_* and the second map j^*.

First we show that the image of the first map is coherent sheaves whose support is Y: indeed, for any coherent $\mathcal{F}$ supported on Y, by the Nullstellensatz, some power of the ideal sheaf $\mathcal{I}$ of Y annihilates it. Then we have the filtration $\mathcal{F} \supseteq \mathcal{I}\mathcal{F} \supseteq \mathcal{I}^2\mathcal{F} \supseteq \ldots \supseteq 0$. The factors in this filtration are naturally $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Y$-modules, so by inductively taking resolutions everything is indeed in the image of i_*. The converse is immediate.

The fact that the second map is surjective is the foundational fact that a coherent sheaf defined on an open set can be extended to a coherent sheaf on the whole variety; we can actually take $j_*\mathcal{F}$ since it is not difficult to check that j^*j_* is actually the identity for open immersions.

To show exactness in the middle, we need only show that a sheaf is zero under j^* iff it is supported on Y. If it is supported on Y, j^* vanishes. Conversely, if there is a point p off of Y so that the stalk there is nonzero, the stalk there in $X - Y$after applying j^* is also clearly nonzero. □

Theorem 3.2. *If X is a variety and Y a closed subvariety, then there is an exact sequence on the dimension-graded Chow group, that is:*

$$A_*(Y) \to A_*(X) \to A_*(X-Y) \to 0.$$

Proof. Taken from [5]. Let Z_k, in analogy with A_k, be the dimension-graded free abelian group of cycles before rational equivalence. Since any subvariety V of $X - Y$ extends to its closure in X, we obtain the exact sequence $Z_kY \to Z_kX \to Z_k(X-Y) \to 0$ on free abelian groups of cycles before taking rational equivalence. Rational functions on subvarieties of $X - Y$ are uniquely identified with rational varieties on the closure of the subvariety, so rational equivalence is also preserved on each part of the grading. Hence we can pass to $A_kY \to A_kX \to A_k(X-Y) \to 0$. □

[24] It is in fact possible to approach the nice interaction of exterior powers with Chern classes as a central part of the theory, giving the K-theory a λ-**ring** structure, though this is more abstract than necessary here. See [8] for such a treatment, and [6] for a highly abstracted account of "Riemann-Roch functors" in general using this approach.

3.2. **Exterior products.** Again in analogy with cohomology, there are Künneth-type exterior products on K-theory and algebraic cycles. In fact, this is a common way to define the intersection product on the Chow ring in more general settings, in analogy with the cup product in cohomology.

Theorem 3.3. *Let X and Y be varieties.*

a) There is a map of rings

$$K(X) \otimes K(Y) \to K(X \times Y),$$

If $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection maps, this map is given by $\alpha \otimes \beta \to \pi_1^*(\alpha) \cdot \pi_2^*(\beta)$.

b) Denote the exterior product from above as $\boxtimes$. Let $f : X' \to X$ and $g : Y' \to Y$ be proper morphisms, so that we have proper $f \times g : X' \times Y' \to X \times Y$. Then

$$(f \times g)_*(\alpha \boxtimes \beta) = f_*(\alpha) \boxtimes g_*(\beta)$$

c) If $Y = \mathbb{P}^n_k$, the map is surjective.

Proof. a) Since π_1^*, π_2^* are linear, the map given by $(\alpha, \beta) \to \pi_1^*(\alpha)\pi_2^*(\beta)$ is bilinear, so it factors through $K(X) \otimes K(Y)$. We do not have to make any reference to coherent sheaves and are already working at the level of K-elements, so this is a group homomorphism. The ring structure is a formal verification.

b) We can factor the morphism as $(f \times \mathrm{id}_Y) \circ (\mathrm{id}_X \times g)$, and prove it separately for the two. Just working with the former by symmetry, let α be represented by coherent sheaf $\mathcal{F}$ and β by locally free $\mathcal{G}$ - so that we can take the product in K-theory to just be tensor product on both sides. Then we want

$$(f \times \mathrm{id}_Y)_*(\pi_1^*(\mathcal{F}) \otimes \pi_2^*(\mathcal{G})) = \pi_1^*(f_*\mathcal{F}) \otimes \pi_2^*(\mathcal{G})$$

for all i. Indeed, over a basic open set of the form $U \times V$, where $U \subset X', V \subset Y$, the section looks like $\mathcal{F}(f^{-1}(U)) \otimes \mathcal{G}(V)$ on both sides. (In general the sections of the tensor sheaf are not the tensors of the sections, but in this case the section-wise tensor must already be a sheaf, by checking that we can glue basic open sets.)

The same result holds for pullbacks rather than pushforwards, but we do not need it.

c) This is by far the most difficult portion. We follow closely the proof in [3].

Before tackling the main statement, we prove a lemma:

Lemma 3.4. *If X is a variety $K(X \times \mathbb{A}^n_k) = K(X)$.*[25]

Proof. It is clear we need only show it for $n = 1$ by induction.

If $\pi : X \times \mathbb{A}^1_k \to X$, we claim that π^* gives an isomorphism $K(X) \to K(X \times \mathbb{A}^1_k)$. To prove injectivity, let i be the inclusion of X as the zero section in $X \times \mathbb{A}^1_k$. Then πi is the identity, so $(\pi i)^* = i^* \pi^*$ is as well.

To prove surjectivity, let Y be a closed subvariety of X. Then the following diagram is commutative, by theorem 3.1:

$$\begin{array}{ccccccc}
K(Y) & \longrightarrow & K(X) & \longrightarrow & K(X-Y) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K(Y \times \mathbb{A}^1_k) & \longrightarrow & K(X \times \mathbb{A}^1_k) & \longrightarrow & K((X-Y) \times \mathbb{A}^1_k) & \longrightarrow & 0
\end{array}$$

Surjectivity of the outer arrows implies surjectivity of the inner arrow by the four lemma. Hence by noetherian induction, we can excise any closed subvariety from X; thus, we may assume it is irreducible, nonsingular, and affine.

One further result is necessary: the structure sheaves of irreducible closed subvarieties (identified with their pushforwards by inclusion) generates K-theory in this situation. This, too proceeds by noetherian induction: suppose we have a coherent sheaf $\mathcal{F}$ on a variety X. If $\mathcal{F}$ is torsion under the action of $\mathcal{O}_X$, it is supported on a closed subvariety, so it follows by the hypothesis. Otherwise, by definition of coherence, its

[25] Note that when X is a point, this is a generalization of the fact that every finitely generated projective module over a polynomial algebra is stably free, a weaker version of the Quillen-Suslin theorem.

presentation gives it as isomorphic to some power of the total structure sheaf, modulo some sheaf which must then be torsion.[26]

Finally, we conclude the lemma: it suffices to prove that for every subvariety $V \subset X \times \mathbb{A}^1_k$, $\mathcal{O}_V$ (which we identify as a coherent sheaf on the full variety by abuse of notation) lies in the image of π^*. If $\pi(V)$ lies in a closed subvariety of X, this follows by the inductive hypothesis, so $\pi(V)$ is dense in X.

What remains is a familiar algebraic exercise from computing spectra of polynomial rings. Let Spec $A = X$, so that Spec $A[t] = X \times \mathbb{A}^1_k$. If $\mathfrak{p} \subset A[t]$ is the prime ideal corresponding to V, by the above discussion, we have that $\mathfrak{p}$ has trivial intersection with coefficient ring A. If K is the fraction field of A, then the localization of $\mathfrak{p}$ in $K[t]$ is of course a PID generated by, say, irreducible $p(t)$. Then $\mathcal{O}_V = A[t]/\mathfrak{p}$, and say $\mathcal{F} = A[t]/(p(t))$, which is clearly zero in K-theory. $(p(t))$ and $\mathfrak{p}$ agree after localization away from a closed subvariety, hence $\mathcal{O}_V$ and $\mathcal{F}$ are the same in K-theory up to a sheaf supported on that same subvariety, so we are done by the inductive hypothesis. □

Having proved the lemma, We may proceed with the proof of theorem 3.3(c). Once again we have the commutative diagram

$$\begin{array}{ccccccc} K(X) \otimes K(\mathbb{P}^n_r) & \longrightarrow & K(X) \otimes K(H) & \longrightarrow & K(X) \otimes K(\mathbb{P}^n_r - H) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K(X \times \mathbb{P}^n_r) & \longrightarrow & K(X \times H) & \longrightarrow & K(X \times (\mathbb{P}^n_r - H)) & \longrightarrow & 0 \end{array}$$

where H is a hyperplane. The result of removing a hyperplane is affine space, so the third arrow is an isomorphism. Thus once again this follows from the four lemma and induction. Notice this same proof goes through for Y any variety with an affine stratification. □

Theorem 3.5. *Let X and Y be varieties. Then there is a map of rings*

$$A^*(X) \otimes A^*(Y) \to A^*(X \times Y),$$

where the tensor product is considering the Chow groups as $\mathbb{Z}$-modules in the obvious way. At the level of cycles, this is given by the map $[V] \otimes [W] \mapsto [V \times W]$.

b) Denote the exterior product from above as $\boxtimes$. Let $f : X' \to X$ and $g : Y' \to Y$ be proper morphisms, so that we have proper $f \times g : X' \times Y' \to X \times Y$. Then

$$(f \times g)_*(\alpha \boxtimes \beta) = f_*(\alpha) \boxtimes g_*(\beta)$$

Proof. a) Taken from [5]. We have such a map on the level of Z_* before rational equivalence, in the notation of theorem 3.2. To show that it respects rational equivalence, we use part (b): if $\alpha = 0$, we may assume by linearity that $\beta = [W]$, and by (b) that $W = Y$. Then $\alpha \boxtimes \beta$ is clearly just $\pi^*(\alpha)$, where $\pi : X \times Y \to X$ is the projection, so we have the group structure. Again, the ring structure is a simple verification.

Note that the exterior product can equivalently be given $\alpha \boxtimes \beta = \pi_1^*\alpha \cdot \pi_2^*\beta$, as in the K-theory case, by a check of definitions.

b) As in K-theory, it suffices to prove this for $g = \mathrm{id}_Y$. We may assume $\alpha = [V]$ and $\beta = [W]$ by linearity, so that our desired statement reads $(f \times \mathrm{id}_Y)_*([V \times W]) = [f_*(V) \times W]$. This is formal if one works through the algebraic definition of pushforward.

Again, the same result holds for pullbacks, but we have no need for it.

Similarly to theorem 3.3, this map is surjective if one of V and W has an affine stratification (and actually an isomorphism), but this is not important for us. □

We can of course extend the exterior product linearly to the rationalization $A(X) \otimes \mathbb{Q}$, which we will denote $A(X, \mathbb{Q})$ for convenience from this point on.

[26] This is kind of a weaker version of the structure theorem for finitely generated modules over a PID.

Theorem 3.6. *There is a commutative diagram*

$$\begin{array}{ccc} K(X) \otimes K(Y) & \xrightarrow{ch(-)\otimes ch(-)} & A(X,\mathbb{Q}) \otimes A(Y,\mathbb{Q}) \\ {\scriptstyle \cdot\boxtimes\cdot}\downarrow & & \downarrow{\scriptstyle \cdot\boxtimes\cdot} \\ K(X \times Y) & \xrightarrow{ch(-)} & A(X \times Y, \mathbb{Q}) \end{array}$$

Proof. Formal. □

3.3. **Push-pull formulas.** While the pushforward clearly cannot respect the ring structure, we would like to be able to relate it to the tensor product. Simultaneously, we would like a relationship between the functors f^* and f_*, in particular the composition $f_* f^*$. The **push-pull formulas** (sometimes called the **projection formulas**) give an answer to both questions.[27]

Theorem 3.7. *If $f : X \to Y$ is a proper morphism of regular varieties, we have the formula*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

for any $\alpha, \beta \in K(X)$.

Proof. We may assume by linearity that $\alpha = [E]$, $\beta = [F]$, a pair of vector bundles, so that we can take the product on the level of sheaves to be tensor product and pullback to be pullback, instead of having to deal with derived functors. Writing out the definition of pushforward, our formula is

$$\sum(-1)^i [R^i f_*(f^* E \otimes F)] = \sum(-1)^i [E \otimes R^i f_*(F)]$$

Hence it suffices to prove that $R^i f_*(f^* E \otimes F) = E \otimes R^i f_*(F)$ on the level of vector bundles for all i. Indeed, this holds if replace F by any coherent sheaf $\mathcal{F}$. Consider the following diagram of categories, where CohSh is the abelian category of coherent sheaves:

$$\begin{array}{ccc} \mathrm{CohSh}(X) & \xrightarrow{-\otimes f_*(E)} & \mathrm{CohSh}(X) \\ {\scriptstyle f_*(-^\bullet)}\downarrow & & \downarrow{\scriptstyle f_*(-^\bullet)} \\ D_+(\mathrm{CohSh}(Y)) & \longrightarrow & D_+(\mathrm{CohSh}(Y)) \\ {\scriptstyle H^\bullet}\downarrow & & \downarrow{\scriptstyle H^\bullet} \\ \mathrm{CohSh}(Y) & \xrightarrow{-\otimes E} & \mathrm{CohSh}(Y) \end{array}$$

Here D_+ is the bounded-below derived category construction. The top vertical arrows are taking the functor f^* of injective resolutions; it is a standard fact of homological algebra that any two such constructions are homotopy equivalent, so these are well-defined. The middle map consists of applying the exact functor $- \otimes E$ on each degree of the chain complex. The bottom vertical arrows are homology functors. Exact functors commute with homology, so the bottom square commutes. The statement that the top square commutes, term-wise, is just the statement that $f_*(F \otimes f^*(E)) = f_*(\mathcal{F}) \otimes E$. The whole square gives us what we want in general since of course the homology of the f_* functor applied to an acyclic resolution is precisely the higher direct functors, so we have reduced to the case $i = 0$.

When $i = 0$, we have a natural morphism $f^*(f_*(\mathcal{F}) \otimes E) = f^*(f_*(\mathcal{F})) \otimes f^*(E) \to \mathcal{F} \otimes f^*(E)$. Under the hom-set adjunction between f^* and f_*, this induces a morphism $\varphi : f_*(\mathcal{F}) \otimes E \to f_*(\mathcal{F} \otimes f^*(E))$. f_* and f^* commute with restriction, and tensor product does for the locally free (hence flat) sheaf E, so we may work locally on the base of E. Further, we can check that our formula can be broken up under Whitney sums, so we may actually assume $E = \mathcal{O}_Y$. In this case, the fact that φ is an isomorphism is formal, since the morphism from which it is induced as an adjoint is an isomorphism and hom-set adjunction is a natural isomorphism. □

[27]It is incredible in how many different contexts the push-pull formulas arise; seemingly whenever there is an analogue of Grothendieck's six operations, and sometimes when there is not. See this MathOverflow thread.

Theorem 3.8. *If $f : X \to Y$ is a proper morphism of regular quasiprojective varieties, we have the formula*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

for any $\alpha, \beta \in A(X)$.

Proof. Adapted from Stacks Project 42.22.

We may assume $\alpha = [V]$ and $\beta = [W]$ for closed irreducible V, W by linearity. If the relative dimension over the generic fiber of $f(W)$ is positive, then the right hand side is zero. But since the relative dimension over the generic fiber is a lower bound on the relative dimension, this means the left hand side is zero too, so we are done.

Hence we may assume f is generically finite on W. Then we can compute that over each irreducible component of $W \cap f^{-1}(V)$, f is generically finite as well. If p is a generic point of one of these irreducible components, then it is a classical fact that $f|_V$ is quasi-finite on an open set surrounding p, hence finite since it is proper.

Applying our projection formula for K-theory (which holds for general coherent sheaves, recall, by the K/G-theory isomorphism) to the coherent sheaves $\mathcal{O}_V$ and $\mathcal{O}_W$ gives us the formula

$$\sum(-1)^i R^i f_* \left(\sum(-1)^j [\mathrm{Tor}^j(\mathcal{O}_V, f^*\mathcal{O}_W)]\right) = \sum(-1)^i \left[\mathrm{Tor}^i \left(\sum(-1)^j R^j f_*(\mathcal{O}_V), \mathcal{O}_W\right)\right]$$

Here we assume f is flat, so we don't have to write out alternating derived functors for the pullback. But $f|_V$ is finite in a neighborhood of p, so $R^i f_*(\mathcal{F}) = 0$ for $i > 0$ in such a neighborhood for $\mathcal{F}$ coherent supported on V. Hence in a neighborhood of the generic point p,

$$\sum(-1)^j [f_* \mathrm{Tor}^j(\mathcal{O}_V, f^*\mathcal{O}_W)] = \sum(-1)^i [\mathrm{Tor}^i(f_*(\mathcal{O}_V), \mathcal{O}_W)]$$

But this precisely says, by Serre's formula for computing multiplicities through lengths of modules, that the component occurs with equal multiplicity on both sides of the projection formula for cycles. This concludes the flat case.

Finally, in the case where the morphism is non-flat, see Stacks Project 42.27 for how our result implies the general case. It is is the less difficult part of this result, but we omit the discussion here because of the space required to introduce new technology.[28] □

If we write out the formula for Grothendieck-Riemann-Roch,

$$f_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(X)) = \mathrm{ch}(f_*(\mathcal{F})) \cdot \mathrm{td}(Y),$$

notice that if, in the complex case, the map f is a submersion, we have the exact sequence in tangent bundles:

$$0 \to T_{X/Y} \to T_X \to f * T_Y \to 0$$

where $T_{X/Y}$ is the **virtual relative tangent bundle** of the submersion, we can rewrite GRR using the push-pull formula as:

$$\mathrm{ch}(f_*\mathcal{F}) = f_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(T_{X/Y})).$$

This gives some intuition for its relation to our discussion of the Riemann-Roch formulas earlier: here is a formula which on one side computes some invariant in the Chow ring via an alternating sum of derived functors in K-theory, which is like a more refined version of sheaf cohomology. On the other side is a genus.

[28] There is a simple geometric proof for all morphisms in the quasiprojective case. By Chow's moving lemma, all intersections are generically transverse on cycle classes which are represented by closed varieties, in which case this is just the set-theoretic push-pull formula $f(f^{-1}(W) \cap V) = W \cap f(V)$ for $\alpha = [W]$ and $\beta = [V]$. See [4] theorem 5.8. Such geometric ideas can be extended to the full case proven here by Fulton's treatment using refined intersection products, as we briefly mentioned earlier. See [5] section 8.3

4. Main proof

Our main proof follows [5] closely. In particular, we elect for Fulton's treatment of the closed immersion case using deformation to the normal cone rather than Borel/Serre's and SGA's inclusion of a divisor and blow-ups (though the two are essentially equivalent) due to its more geometric nature and concision.

The assumption of quasiprojectivity is necessary to harness the power of the relative point of view, but If X is not quasiprojective, then given $\alpha \in K(X)$, there exists a quasiprojective variety X' and proper $g : X' \to X$, and an element $\alpha' \in K(X')$ with $g^*(\alpha') = \alpha$ - a Chow envelope. See [5] section 18.3 for a reference; the development of the theory in that chapter is not difficult, but omitted here for space reasons.

If the ground field is characteristic zero, we can take X' regular by Hironaka's resolution of singularities. Since the version of the theorem in which the first variety is quasiprojective applies to g and $f \circ g$, the full result follows formally as a result of the natural transformation/relative phrasing. Unfortunately, arbitrary resolution of singularities is an open problem, so we only have the quasiprojective case in positive characteristic using this approach. At this point, we must admit a bit of a fib: we cannot actually prove this case; this requires the treatment of singular GRR given by Fulton in chapter 18, using local Chern characters. We hence only logically have the quasiprojective case in positive characteristic here, since this is the furhest extent our (relatively) straightforward geometric proof goes.

In the case where X is quasiprojective, it is classical that a proper morphism $f : X \to Y$ can be factored into a closed immersion $g : X \to \mathbb{P}^n_Y$ and a projection $\pi : \mathbb{P}^n_Y \to Y$. Again, the relative point of view tells us that from GRR on these two separate cases, the full result follows formally.

4.1. **A projection.**

The projection $\pi : \mathbb{P}^n_Y \to Y$ can be factored as $(p \times \mathrm{id}) : \mathbb{P}^n_k \times Y \to \mathrm{Spec}\, k \times Y$. We may then once again take advantage of the nice formal properties of the relative formulation of GRR to split the proof into the first and second terms as follows.

Notice that $\mathrm{td}(\mathcal{O}_{X\times Y}) = \mathrm{td}(\mathcal{O}_X)\mathrm{td}(\mathcal{O}_Y)$, so the Chern character maps in theorem 3.6 can in fact be augmented by the Todd class. More precisely, if we let $\tau_X(-) = \mathrm{ch}(-) \cdot \mathrm{td}(X)$ (recalling our discussion of genera), so that the statement of GRR becomes $f_*(\tau_X(\mathcal{F})) = \tau_Y(f_*(\mathcal{F}))$, we have commutativity of the top square of

$$\begin{array}{ccc}
K(\mathbb{P}^n_k) \otimes K(Y) & \xrightarrow{\tau_{\mathbb{P}^n_k} \otimes \tau_Y} & A(\mathbb{P}^n_k, \mathbb{Q}) \otimes A(Y, \mathbb{Q}) \\
{\scriptstyle \cdot\boxtimes\cdot}\downarrow & & \downarrow{\scriptstyle \cdot\boxtimes\cdot} \\
K(\mathbb{P}^n_k \times Y) & \xrightarrow{\tau_{\mathbb{P}^n_k \times Y}} & A(\mathbb{P}^n_k \times Y, \mathbb{Q}) \\
{\scriptstyle (p\times\mathrm{id})*}\downarrow & & \downarrow{\scriptstyle (p\times\mathrm{id})_*} \\
K(\mathrm{Spec}\, k \times Y) & \xrightarrow{\tau_{\mathrm{Spec}\, k \times Y}} & A(\mathrm{Spec}\, k \times Y, \mathbb{Q})
\end{array}$$

where we of course substituted in the particular varieties we are working with, and adjoined the desired GRR diagram on the bottom. By theorem 3.4, the left vertical arrow is a surjection. Thus to prove that the bottom square is commutative, our desired statement, it suffices to prove that the outer rectangle is commutative.

Writing out what this entails on an element $[\mathcal{F}] \otimes [\mathcal{G}]$, we find our desired equation can be written as $\mathrm{ch}(p_*(\mathcal{F})) \cdot \mathrm{ch}(\mathrm{id}_*(\mathcal{G})) \cdot \mathrm{td}(\mathrm{Spec}\, k \times Y) = p_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(Y)) \cdot \mathrm{id}_*(\mathrm{ch}(\mathcal{G}) \cdot \mathrm{td}(\mathbb{P}^n_k))$, or equivalently

$$[\mathrm{ch}(p_*(\mathcal{F}))\mathrm{td}(\mathrm{Spec}\, k)] \cdot [\mathrm{ch}(\mathrm{id}_*(\mathcal{G}))\mathrm{td}(Y)] = [p_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(Y))] \cdot [\mathrm{id}_*(\mathrm{ch}(\mathcal{G}) \cdot \mathrm{td}(\mathbb{P}^n_k))].$$

Hence we see that it suffices to prove GRR separately for the factors p and id.

The statement is easily seen to be vacuous for id. For $p : \mathbb{P}^n_k \to \mathrm{Spec}\, k$, it is Hirzebruch-Riemann-Roch on projective space:

$$\chi(\mathcal{F}) = \int_{\mathbb{P}^n_k} \mathrm{ch}(\mathcal{F})\mathrm{td}(\mathbb{P}^n_k).$$

By additivity on both sides in short exact sequences, it suffices to prove the statement for sheaves generating the K-theory of projective space; by Hilbert's syzygy theorem, we can thus take $\mathcal{F} = \mathcal{O}(k)$, the k-twisted structure sheaf. The left side we can replace with the known Hilbert polynomial, and right side with the explicit calculation of Chern character and Todd class, as we did at the end of section 2.4. Then what is left to prove is

$$\binom{n+k}{k} = [x^n]e^{kx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}$$

for all integers n and k. Indeed we see that the right hand side is equal to the residue $[x^{-1}]\frac{e^{kx}}{(1-e^{-x})^{n+1}}$. Under the change of variables $y = e^{-x}$, the residue $[x^{-1}]$ at 0 becomes the constant term $[y^0]$. Hence, reframing in terms of residues, we need to calculate

$$[y^{-1}]\frac{1}{y^{k+1}(1-y)^{n+1}} = \frac{1}{2\pi i}\int_C \frac{1}{y^{k+1}(1-y)^{n+1}}dy$$

for a small counterclockwise-oriented loop C around the origin. Using the Cauchy integral formula with $f(y) = \frac{1}{(1-y)^{n+1}}$, this is equal to $\frac{1}{k!}f^{(k)}(0)$. But the Taylor series of f is just

$$(1+y+y^2+\dots)^{n+1} = 1 + \frac{n+1}{1!}x + \frac{(n+1)(n+2)}{2!}x^2 + \frac{(n+1)(n+2)(n+3)}{3!}x^3 + \dots$$

from which the value of $f^{(k)}(0)$ is $\frac{(n+k)!}{n!}$, so that our value is indeed $\frac{(n+k)!}{n!k!} = \binom{n+k}{k}$.

4.2. **A closed immersion.** Finally, to show GRR for $g : X \to \mathbb{P}^n_Y$, we show it for a general closed immersion $f : X \to Y$.

4.2.1. *Immersion in the normal cone.* First, we show the result for a particularly nice case. Recall the relation

$$\sum_{i=0}^{r}(-1)^i \mathrm{ch}(\wedge^i E^\wedge) = c_d(E)\cdot \mathrm{td}(E)^{-1}$$

where c_d is the top Chern class. The left-hand side is reminiscient of a resolution by a **Koszul complex**, a standard piece of algebraic machinery which is often used to resolve the structure sheaf of the vanishing of a section of a vector bundle by exterior powers of the vector bundle.[29]

If we could obtain the image of the embedding as the zero locus of a section, we could use a Koszul complex and the above equation to resolve our problem. This naturally leads us to the common construction of embedding of X as the zero section in its normal bundle N, which is equivalently the normal cone.[30] To be more precise, we can take N to be any vector bundle on X, since it is apparent that the normal bundle of a zero section embedding is just the bundle itself. Actually, in order to facillitate the construction of the necessary section, we will embed X in the projectivization $\mathbb{P}(N \oplus 1)$. more precisely, $f : X \to Y = \mathbb{P}(N \oplus 1)$ embeds X in N, and then identifies N in the familiar way with an open set of $\mathbb{P}(N \oplus 1)$.

The reason for this is that the projectivization is equipped with the tautological (or universal) line bundle, which enables us to construct our desired section. Indeed, there is a total line bundle on $\mathbb{P}(N \oplus 1)$ given by the pullback of $N \oplus 1$ itself; i.e. if $\pi : \mathbb{P}(N \oplus 1) \to X$ is the projection, $\pi^*(N \oplus 1)$, in which the tautological bundle can be naturally identified as the subbundle whose fiber over a point q is identified as the line corresponding to q in the copy of $N \oplus 1$ over q. The quotient of this total bundle by the tautological bundle is a vector bundle of the same dimension as N called the **universal quotient bundle**, which we will denote by Q.

The projection of the trivial factor onto Q induces a morphism $s : \mathbb{P}(N \oplus 1) \to Q$ given by sending a point q to, e.g. $(0, 1) \in \pi^* N_q \oplus 1$ in the fiber over it, and then projecting it by the quotient map on vector bundles. Hence s is an (algebraic) section, which vanishes precisely on X: the trivial factor vanishes in projection over a point q precisely when the line corresponding to q is the trivial factor in $N \oplus 1$, i.e. when $q \in X$.

[29] Unfortunately, I am not aware at this time of a good geometric picture for this, and am doubtful whether one can be given, as Koszul complexes exist as algebraic constructions in much greater generality. Stacks Project 17.20 is a reference. We will make use of such complexes without comment.

[30] We mention this only for consistency of nomenclature; it is not particularly important besides the fact that the two are naturally isomorphic on a regular variety. The cone is constructed a different way algebraically, but in nice cases there are natural isomorphisms; see [5] chapter 4 for the cone construction.

Then s determines the Koszul complex

$$0 \to \Lambda^d Q^\wedge \to \ldots \to \Lambda^2 Q^\wedge \to Q^\wedge \to \mathcal{O}_Y \to f_*(\mathcal{O}_X) \to 0$$

where d is the dimension of Q (or N). Tensoring with the vector bundle p^*E, we obtain

$$0 \to \Lambda^d Q^\wedge \otimes p^*(E) \to \ldots \to \Lambda^2 Q^\wedge \otimes p^*(E) \to Q^\wedge \otimes p^*(E) \to p^*(E) \to f_*(E) \to 0$$

where we use the fact that $f_*\mathcal{O}_X \otimes \pi^* E = f_* E$, which is clear by checking sections. Hence we have a resolution of $f_*(E)$, giving us

$$\mathrm{ch}(f_*(E)) = \sum_{i=0}^{d} (-1)^i \mathrm{ch}(\lambda^i Q^\wedge) \cdot \mathrm{ch}(p^*(E)).$$

But we find on the right hand side exactly what we were hoping to work with, and hence

$$\mathrm{ch}(f_*(E)) = c_d(Q) \cdot \mathrm{td}(Q)^{-1} \cdot \mathrm{ch}(p^*(E)) = f_*(f^*(\mathrm{td}(Q)^{-1}) \cdot f^*(\mathrm{ch}(p^*(E)))) = f_*(\mathrm{td}(N)^{-1} \cdot \mathrm{ch}(E)).$$

by the push-pull formula and functoriality. From the exact sequence

$$0 \to T_X \to f^* T_Y \to N \to 0$$

we can rewrite this as

$$\mathrm{ch}(f_*(E)) \cdot \mathrm{td}(Y) = f_*(\mathrm{ch}(E) \cdot \mathrm{td}(X))$$

as desired.

4.2.2. *Deformation to the normal cone.* We conclude by reducing the case in the previous section, via a construction called **deformation to the normal cone**, which, given a closed embedding $f : X \to Y$, allows one to construct a flat family M of copies of Y over $\mathbb{P}^1_k$ so that the copy at ∞ becomes an immersion in the normal cone. We very briefly sketch the construction; consult [5] chapter 5 for details.

We construct M as the blow-up of $Y \times \mathbb{P}^1_k$ along the closed subvariety $X \times \{\infty\}$. It is clear that away from the fiber over ∞, the fiber is still just Y, because of the isomorphism on the open set away from the blow-up. On the fiber over ∞, we are left with the sum of effective Cartier divisors $\mathbb{P}(N \oplus 1) + \tilde{Y}$, since blow-up is the universal way to turn a closed subscheme into a Cartier divisor. Here N is the normal cone to X in Y, and $\tilde{Y}$ is the blowup of Y along X. These two divisors intersect along $\mathbb{P}(N)$, which is embedded respectively the hyperplane at infinity and the exceptional divisor of the blow-up. Notably, $\tilde{Y}$ does not intersect the image of X.

The map back down onto $\mathbb{P}^1_k$ is a blow-down and then a projection, hence we indeed have a flat family. We additionally have the closed immersion $X \times \mathbb{P}^1_k \to M$ still, since the inclusion of the fiber at infinity is just the embedding f' of X in $\mathbb{P}(N \oplus 1)$ in the manner with which we are already familiar.

We apply this now to a general closed immersion $f : X \to Y$ to deduce the final case of GRR. We have the following commutative diagram, reproduced from [5]:

$$\begin{array}{ccccc}
X & \xrightarrow{\bar{f}} & \mathbb{P}(N \oplus 1) + \tilde{Y} & \longrightarrow & \{\infty\} \\
\downarrow{\scriptstyle i_\infty} & & \downarrow{\scriptstyle j_\infty} & & \downarrow \\
X \times \mathbb{P}^1_k & \xrightarrow{F} & M & \longrightarrow & \mathbb{P}^1_k \\
\uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle j_0} & & \uparrow \\
X & \xrightarrow{f} & Y & \longrightarrow & \{0\}
\end{array}$$

where notation is extended from our discussion of the deformation to the normal cone. Let k and l be the restrictions of the inclusion morphism j_∞ to the divisors $\mathbb{P}(N \oplus 1)$ and $\tilde{Y}$ respectively.

As usual, by linearity, we may take a vector bundle E as our representative of the K-theory of X. Let $\tilde{E}$ be its obvious pullback to all of $X \times \mathbb{P}^1_k$. We can find a resolution $G^\bullet$ of $F_*(\tilde{E})$ on M.

The pullback of a flat family to its fibers is exact almost tautologically, so $j_0^* G^\bullet$ and $j_\infty^* G^\bullet$ are resolutions of $j_0^*(F_*(\tilde{E}))$ and $j_\infty^*(F_*(\tilde{E}))$ respectively. But $j_0^*(F_*(\tilde{E})) = f_*(E)$ by diagram chasing (not just functoriality, but the fact that pullbacks of closed immersions are just restrictions), so actually $j_0^* G^\bullet$ resolves $f_*(E)$ on Y. Analagously, $j_\infty^* G^\bullet$ resolves $\overline{f}_*(E)$ on the fiber at infinity. But since $\overline{f}(X)$ does not intersect $\tilde{Y}$, we can say that $k^* G^\bullet$ is already a resolution of the sheaf, while $l^* G^\bullet$ is acyclic, since the pullback of the sheaf supported away from it is just zero.

With all of this set up, we can now compute. Write $\text{ch}(F^\bullet) = \sum(-1)^i \text{ch}(F_i)$ on chain complexes as shorthand.

$$j_{0*}(\text{ch}(f_*(E))) = j_{0*}(\text{ch}(j_0^*(G^\bullet))) = \text{ch}(G^\bullet) \cdot j_{0*}([Y])$$

by the push-pull formula on cycles; using the rational equivalence of fibers in a flat family and working with the divisors in the Chow ring, now, we get that this is

$$= \text{ch}(G^\bullet) \cdot (k_*([\mathbb{P}(N \oplus 1)]) + l_*([\tilde{Y}]) = k_*(\text{ch}(\overline{f}_*(E)))$$

where the last equality follows by applying push-pull again and observations about k_* and l_*. But we already have calculated $\text{ch}(\overline{f}(E))$ since it is an immersion into the normal cone; thus combining the results, we obtain

$$j_{0*}(\text{ch}(f_*(E))) = k_*(\overline{f}_*(\text{td}(N)^{-1} \cdot \text{ch}(E)))$$

in the rational Chow ring of M. Now if we push forward both sides by the blow-down from M to $Y \times \mathbb{P}_k^1$ and then the projection to Y, by functoriality of push-forward in the diagram above this reduces to

$$\text{ch}(f_*(E)) = f_*(\text{td}(N)^{-1} \cdot \text{ch}(E))$$

which reduces as before to

$$\text{ch}(f_*(E)) \cdot \text{td}(Y) = f_*(\text{ch}(E) \cdot \text{td}(X))$$

which is the result. ■

5. Concluding remarks

The theorem we have proven is a very general and powerful way of computing with Chern classes and pushforwards of bundles (among other applications) on varieties. However, it should be noted that in most practical applications, Grothendieck-Riemann-Roch is not the best way to compute, as the morass of definitions and series to sort through often makes such a task arduous and messy. For example, while the 27 lines on a cubic problem mentioned earlier can be computed using GRR (it is the example in [4]), it is almost always done using less general, but much cleaner, techniques or "tricks."

While we have endeavored to make our foundations fairly general in this paper, we clearly have not stated GRR in the greatest generality possible. Some trade-offs had to be made; for example; the ground field can be made arbitrary, but this necessitates separability assumptions in some results (e.g. any theory developed using a moving lemma), and substitution of smoothness assumptions for regularity assumptions almost everywhere.[31]

In other instances, we simply did not adopt the more general viewpoint for the sake of remaining close to geometric roots while not spending too much time developing techniques and theories. For example, we did not actually have the power to prove the non-quasiprojective case in positive characteristic. For this, we need a more general version of GRR for singular varieties, using a construction called the local Chern character. The treatment in [5] chapter 18 extends the results to arbitrary algebraic schemes. Here an algebraic scheme is locally of finite type and separated over a field.

The most general setting known in which the form of GRR stated in this paper holds is mentioned only briefly in chapter 20 of [5]. In the category of schemes over an arbitrary regular scheme, one can define still a relative intersection theory, for locally finite type and separated schemes over it.[32] Restricting to smooth schemes, we again have GRR. The singular formulation with a local Chern character extends as well.

[31]Indeed, we were able to "get away" with regularity in most cases because a regular scheme over a perfect field (hence any algebraically closed field) is smooth.

[32]If the base scheme is one-dimensional, we have even an exterior product on the Chow groups, and hence the Chow ring structure.

In particular, in the case where the base is affine of dimension one and hence a Dedekind domain, we have a number theoretic analogue of GRR (and indeed most of intersection theory, by the previous footnote). This may be extended to the infinite places (in the number field case) by the hermitian techniques of Arakelov geometry to obtain what is known as **arithmetic Grothendieck-Riemann-Roch**.[11] gives an accessible account of the Arakelov version.

There are, of course, countless variants of GRR (or at least related Riemann-Roch type formulas) with statements which appear somewhat different. For example, if one wishes to avoid rationalizing, there is Riemann-Roch "without denominators," treated in section 15.3 of [5]. Equivariant and bivariant formulations also exist; the latter are described in chapter 17 of the same text.

Finally, moving to the analytic world, in [2] Atiyah and Hirzebruch formulated GRR and proved it for a closed immersion for general compact complex manifolds. Indeed, they also showed a close relationship, as is the norm, between the algebraic and analytic worlds, including that the morphism from algebraic K-theory to topological K-theory commutes with pushforward. Hirzebruch-Riemann-Roch was already known, as, e.g., a consequence of the powerful Atiyah-Singer index formula. The general complex analytic case was settled by Nigel O'Brian, Domingo Toledo, and Yue Lin Tong in 1985; see [10]. Less perfect analogues for smooth manifolds without complex structure but a spin-like structure were developed by Atiyah and Hirzebruch in [1]

Even Atiyah-Singer can be seen as a Riemann-Roch type formula, as a vast generalization of the analytic version of Hirzebruch-Riemann-Roch. In fact, SGA 6 includes some musings by Grothendieck on the possibility of a relative version of Atiyah-Singer which would also encapsulate the analytic GRR, but he concluded unhappily: "Sadly, at present, even a heuristic statement which encompasses these two theorems is lacking."[33]

6. Acknowledgements

The writing of this paper is indebted to Professor May (and all the REU lecturers) whose hard work enabled a smooth learning experience for all students involved, including me. I would also like to extend my thanks to the advisors who mentored the undergraduates, especially my own Zhiyuan Ding and Yun Cheng, who were kind enough to politely listen to me stumble over explanations of proofs I did not really understand. Zach Kirsche also deserves a mention, for pointing out several errors in a draft. Professor May additionally provided very helpful comments and revisions.

Finally, I am grateful to Professor Corlette for teaching my first algebraic geometry class, which was a very solid introduction to the subject. Shout out also to Oishee, my favorite TA, who sparked an interest in intersection theory.

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[11] Soulé, Christophe et. al. *Lectures on Arakelov Geometry.* Cambridge: Cambridge University Press, 1992.

[33][8] p. VIII *"Malheureusement, il manque encore à l'heure actuelle un énoncé, même heuristique, qui engloberait ces deux théorèmes..."*