ROMAN DOMINATION

LINFENG XU

Abstract. In his article published in 1999, Ian Stewart discussed a strategy of Emperor Constantine for defending the Roman Empire. Motivated by this article, Cockayne et al.(2004) introduced the notion of Roman domination in graphs. Let $G = (V, E)$ be a graph. A Roman dominating function of G is a function $f : V \to \{0, 1, 2\}$ such that every vertex v for which $f(v) = 0$ has a neighbor u with $f(u) = 2$. The weight of a Roman dominating function f is $w(f) = \sum_{v \in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of all possible Roman dominating functions. This expository paper examines the bounds of Roman domination number and provides examples of calculating this quantity in special graphs such as path, cycles, circulant graphs and regular graphs. In addition, we show the bound for Roman domination number for a connected graph of order n is bounded by $\lfloor 4n/5 \rfloor$ in general. Lastly we look at a local property of Roman domination number, namely the effect of adding an edge to a graph.

Contents

1. Introduction to Roman Domination

About 1700 years ago, the Roman Empire was under attack, and Emperor Constantine had to decide where to station his four field army units to protect eight regions. His trick was to place the army units so that every region was either secured by its own army (one or two units) or was securable by a neighbor with two army units, one of which can be sent to the undefended region directly if a conflict breaks out. Constantine chose to place two army units in Rome and two at his new capital, Constantinople. This meant only Britain could not be reached in one step. As it happened, Constantine's successors lost control of Britain. The causes were surely more complex than anything that can be explained by this simple model. Nevertheless, Stewart [1] is right in arguing that if Constantine had been a better mathematician, the Roman Empire might have lasted a little longer than it did.

Indeed, there are six ways to improve on Constantine's deployment. These results were obtained through a form of zero-one integer programming by ReVelle and Rosing [2]. Besides placing of Roman army units, the same sort of mathematics can also be used for optimizing the location of the declining number of British Fleets at the end of the 19th century or American Military Units during the Cold War [2]. In addition to army placement, the same sort of mathematics is also useful when people want to know the best place in town to put a new hospital, fire station, or fast-food restaurant. Many times such optimization problems can be modeled by Roman domination or its variants.

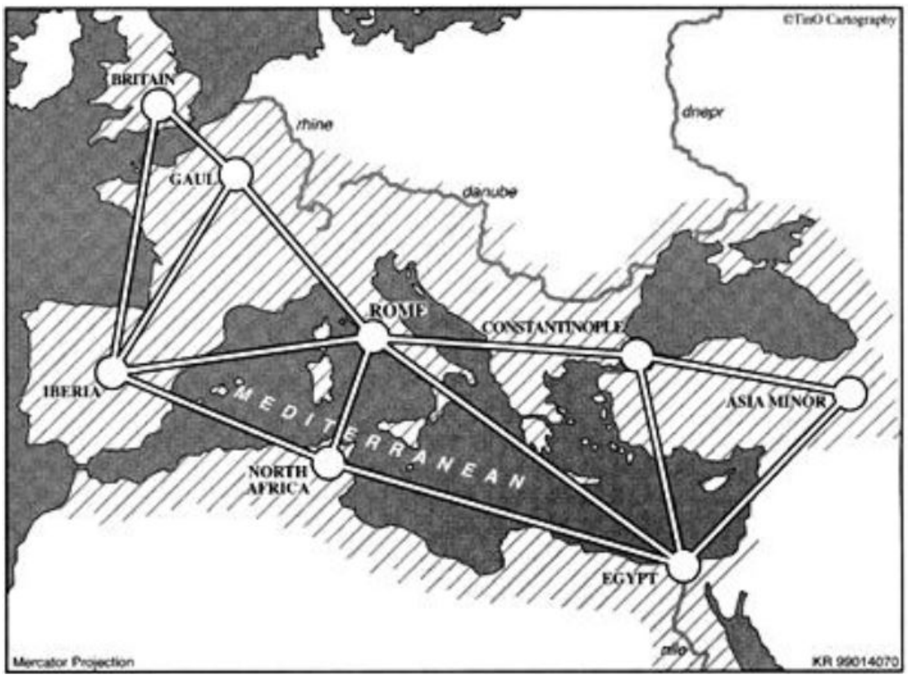

Fig 1.1 Map of Roman Empire. Adapted from http://www.ammc.com.

2. Definitions

Let $G = (V, E)$ be a *graph of order* $|V| = n$. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. The degree of v, denoted by $deg(V)$, is the total number of neighbors of v. In other words $deg(v) = |N(v)|$. For a set $S \subseteq V$, the open neighbourhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood is $N[S] = N(S) \cup S$.

Let $v \in S \subseteq V$. Vertex u is called a *private neighbor of v with respect to S* (denoted by u is an S-pn of v) if u is only adjacent to v but not adjacent to any other vertex in S. In other words, vertex u is a private neighbor of v with respect to S if uv is an edge and u is not in S. More formally speaking, $u \in N[v] - N[S - \{v\}]$. A S-pn of v is *external* if it is a vertex of $V - S$. The set $pn(v, S) = N[v] - N[S - \{v\}]$ of all S-pn's of v is called the *private neighborhood set of v with respect to S*. The set S is said to be *irredundant* if for every $v \in S$, $pn(v, S) \neq \emptyset$. In other words every vertex in S has at least a private neighbor with respect to S.

A subset $S \subseteq V$ is a *dominating set* of G, if for any vertex $u \in V - S$, there exists a vertex $v \in S$ such that $uv \in E$. The *domination number* of G, denoted by $\gamma(G)$, equals the minimum cardinality of a dominating set. A domination set of cardinality $\gamma(G)$ is called a γ-set of G.

A *Roman dominating function (RDF)* on graph $G = (V, E)$ is defined as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of a RDF is the value $f(V) = \sum_{v \in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of all possible RDFs on G. A graph G is a *Roman graph* (or *Roman*) if $\gamma_R(G) = 2\gamma(G)$.

Let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V | f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a $1 - 1$ correspondence between the functions f and the ordered partitions (V_0, V_1, V_2) of V. Thus we will write $f = (V_0, V_1, V_2)$.

The *distance* between two vertices u,v in a graph G, denoted by $d(u, v)$, is defined as the number of edges in a shortest path connecting them. The *diameter* of a graph G, denoted by $D(G)$, is defined as as $D(G) = \max \{d(u, v) | u, v \in V\}$.

The *complement* or *inverse* of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G. We write $H = \overline{G}$.

3. Basic Properties of Roman Domination Number

We start with some of the most basic properties of Roman domination number. As a beginning, we should look at its relationship with the usual domination number.

Proposition 3.1. [3] $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function, and S be a γ-set of G. Then $V_1 \cup V_2$ is a dominating set of G and we can define a RDF of G with $f(v) = 2$ for all $v \in S$ and $f(u) = 0$ for all $u \notin S$. Hence $\gamma(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_R(G)$ by the definition of domination number. But $\gamma_R(G) \leq 2|S| = 2\gamma(G)$ by the definition of a Roman domination number. Hence we have proved the inequality. □

The above inequality characterizes the bounds for Roman domination number for any graph in terms of its domination number in the usual sense. This naturally leads us to the question of when the equality will be achieved.

Proposition 3.2. [3] *For any graph G of order n, $\gamma(G) = \gamma_R(G)$ if and only if every vertex in G has degree 0. (Such a graph is denoted as $G = \overline{K_n}$, the complement of a complete graph of order n)*

Proof. When $G = \overline{K_n}$ we have $\gamma(G) = \gamma_R(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_R-function. The equality $\gamma(G) = \gamma_R(G)$ implies that we have equality in

$$\gamma(G) \leq |V_1| + |V_2| = |V_1| + 2|V_2| = \gamma_R(G).$$

Hence $|V_2| = 0$, which implies that $V_0 = \emptyset$. Hence all vertices are assigned with 1 and therefore $\gamma_R(G) = |V_1| = |V| = n$. This implies that $\gamma(G) = n$, which shows that $G = \overline{K_n}$. □

In light of the various problems below, it is necessary to first examine some basic properties which will be used in different proofs. In [3], only proofs for (f) and (g) are given. We present all of (a)-(e) here.

Proposition 3.3. *Let $f = (V_0, V_1, V_2)$ be any γ_R-function. Then*

(a) *$G[V_1]$, the subgraph induced by V_1 has maximum degree 1.*

(b) *No edge of G joins V_1 and V_2.*

(c) *Each vertex of V_0 is adjacent to at most two vertices of V_1.*

(d) *V_2 is a γ-set of $G[V_0 \cup V_2]$.*

(e) *Let $H = G[V_0 \cup V_2]$. Then each vertex $v \in V_2$ has at least two H-pn's (i.e. private neighbors relative to V_2 in the graph H).*

(f) *If v is isolated in $G[V_2]$ and has precisely one external H-pn, say $w \in V_0$, then $N(w) \cup V_1 = \emptyset$.*

(g) *Let k_1 equal the number of non-isolated vertices in $G[V_2]$, let $C = \{v \in V_0 : |N(v) \cap V_2 \geq 2\}$, and let $|C| = c$. Then $n_0 \geq n_2 + k_1 + c$.*

Proof. (a) Suppose $G[V_1]$ has a vertex u with degree more than 1. Let u be adjacent to v and w where $u, v, w \in V_1$. By reassigning $f(u) = 2$ and $f(v) = f(w) = 0$, and keeping all other values of f to be the same, we find a new RDF with smaller weight, a contradiction.

(b) Suppose an edge joins $u \in V_2$ and $v \in V_1$.By reassigning $f(v) = 0$, and keeping all other values of f to be the same, we find a new RDF with smaller weight, a contradiction.

(c) Suppose $u \in V_0$ is adjacent to all $v, w, t \in V_1$. By reassigning $f(u) = 2$ and $f(v) = f(w) = f(t) = 0$, and keeping all other values of f to be the same, we again find a new RDF with smaller weight, a contradiction.

(d) By definition each $u \in V_0$ is adjacent to at least one $v \in V_2$. Hence V_2 will be a dominating set of the subgraph induced by $V_0 \cup V_2$.

(e) Suppose a vertex $v \in V_2$ only has one $H-pn$, denoted by u (here u and v may be the same vertex). If u and v are distinct, then By reassigning $f(v) = 0$, $f(u) = 1$ and keeping all other values of f to be the same, we find a new RDF with smaller weight, a contradiction.If u and v are the same, then By reassigning $f(v) = 1$, and keeping all other values of f to be the same, again we find a contradiction.

(f)[3] Suppose $N(w) \cup V_1 \neq \emptyset$. We form a new function by changing the function values of v and each $y \in N(w) \cup V_1$ to 0, and the value $f(w)$ to 2. This is an RDF with smaller weight than f, which is a contradiction.

(g)[3] Let k_0 equal the number of isolated vertices in $G[V_2]$, so that $k_0 + k_1 = n_2$. By (e), we have $n_0 \geq k_0 + 2k_1 + c = n_2 + k_1 + c$, as required. □

Using the above results, we can further characterize the graphs with $\gamma_R(G) = \gamma(G) + 1$ or $\gamma_R(G) = \gamma(G) + 2$, the details of which are included in [3] as well.

Proposition 3.4. *If G is a connected graph of order n, then $\gamma_R(G) = \gamma(G) + 1$ if and only if there is a vertex $v \in V$ of degree $n - \gamma(G)$.*

Proposition 3.5. *If G is a connected graph of order n, then $\gamma_R(G) = \gamma(G) + 2$ if and only if*

(a) *G does not have a vertex $v \in V$ of degree $n - \gamma(G)$.*

(b) *either G has a vertex of degree $n - \gamma(G) - 1$ or G has two vertices v and w such that $|N[v] \cup N[w]| = n - \gamma(G) + 2$.*

Naturally, we wish to investigate on an even more general case with $\gamma_R(G) = \gamma(G) + k$ where k any given integer. Now for any graph G, by Proposition 3.1, we have $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. It is easy to see if the equality $\gamma_R(G) = \gamma(G) + k$ holds, then $0 \leq k \leq \gamma(G)$. In the following, we give a sufficient and necessary condition for the converse situation.

Proposition 3.6. *(Xing et al.) Let G be a connected graph of order n and the domination number $\gamma(G) \geq 2$. If k is an integer such that $2 \leq k \leq \gamma(G)$, then $\gamma_R(G) = \gamma(G) + k(2 \leq k \leq \gamma(G))$ if and only if*

(a) *for any integer s with $1 \leq s \leq k-1$, G does not have a set U_t of $t(1 \leq t \leq s)$ vertices that $|\cup_{v \in U_t} N[v]| = n - \gamma(G) - s + 2t$.*

(b) *there exists an integer l with $1 \leq l \leq k$, and G has a set W_l of l vertices such that $|\cup_{v \in W_l} N[v]| = n - \gamma(G) - k + 2l$.*

Another natural approach to describe the bound of the Roman domination number is looking at its relationship with the maximum degree of a graph, an important characteristic of any graph. It turns out that the maximum degree of a graph can be used to give a pretty good upper bound of its Roman domination number. In [3] a result is stated with proof omitted. We complete a proof here.

Proposition 3.7. [3] *For any graph G of order n and maximum degree Δ,*

$$\frac{2n}{\Delta + 1} \leq \gamma_R(G)$$

Proof. Let f be a γ_R-function with $f = (V_0, V_1, V_2)$. By definition each $v \in V_0$ will be adjacent to at least one vertex $u \in V_2$, and since the maximum degree for any vertex in G is Δ, and $V_2 \subseteq V$, we must have $|V_0| \leq \Delta|V_2|$. Now we have

$$(\Delta + 1)\gamma_R(G) = (\Delta + 1)(|V_1| + 2|V_2|$$
$$= (\Delta + 1)|V_1| + 2(\Delta + 1)|V_2| \geq (\Delta + 1)|V_1| + 2|V_2| + 2|V_0|$$
$$\geq 2|V_1| + 2|V_2| + 2|V_0| = 2n$$

□

In addition, we notice that the bound of 3.7 is attained for all stars $K_{1,p}$, where $K_{1,p}$ denotes a tree with one internal node and p leaves all adjacent to that internal node. We conclude this section with an upper bound on $\gamma_R(G)$ using a probabilistic method.

Proposition 3.8. [3] *For a graph G on n vertices,*

$$\gamma_R(G) \leq n\frac{2 + \ln(1 + \delta(G))/2)}{1 + \delta(G)}$$

where $\delta(G)$ is the smallest degree over all vertices in G.

Proof. Given a graph of order n we select a subset of its vertices. If each vertex is selected independently, with the same probability p, then the expected size of the selected subset is np. Let V be the set of vertices of the graph and let the selected subset be A. We use B to denote the the set of vertices not dominated by A. Thus $f = (V - (A \cup B), B, A)$ will be an RDF for G. We now compute the expected size of B. The probability that v is in B is equal to the probability that v is not in A and that no vertex in A is the neighbor of v. This probability is $(1-p)^{1+\deg(v)}$. Since $e^{-x} \geq 1-x$ for any $x \geq 0$, and $\deg(v) \geq \delta(G)$, we can conclude that $\Pr(v \in B \leq e^{-p(1+\delta(G))}$. Thus the expected size of B is at most $ne^{-p(1+\delta(G))}$, and the expected weight of f, denoted $E[f(V)]$, is at most $2np + ne^{-p(1+\delta(G))}$.To minimize the upper bound for $E[f(V)]$, differentiate $2np+ne^{-p(1+\delta(G))}$ with respect to p and set it to 0, we get

$$2n - n(1 + \delta(G))e^{-p(1+\delta(G))} = 0$$

Hence when $p = \ln(1 + \delta(G)/2)/(1 + \delta(G))$ the upper bound for $E[f(V)]$ will be minimized and substituting this value in for p gives:

$$E[f(V)] \leq n\frac{2 + \ln(1 + \delta(G)/2)}{1 + \delta(G)}$$

Since the expected weight of $f(V)$ is at most $n(2 + \ln(1 + \delta(G))/2))/(1 + \delta(G))$, there must be some RDF with at most this weight. It turns out that this bound is sharp, being achieved when G is the disjoint union of $n/2$ copies of K_2, where K_2 is the complete graph on two vertices.

□

4. Roman Domination Numbers for Special Graphs

Having investigated the basic properties of the Roman domination number, it will be extremely helpful to look at the special cases of Roman domination to gain a more direct understanding of this idea. To do so we will illustrate the Roman domination number by presenting the value of $\gamma_R(G)$ for several classes of graphs.

The most illustrative case will be paths and cycles. Following conclusions are stated in several papers without proof, which will be completed here.

Proposition 4.1. *For the classes of paths P_n and cycles C_n,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil.$$

Proof. Let f be a γ_R-function with $f = (V_0, V_1, V_2)$. Since in both paths and cycles, the maximum degree of any vertex is 2, we thus have $3|V_2| + |V_1| \geq n$. Hence

$$f(V) = 2|V_2| + |V_1| \geq 2/3(3|V_2| + |V_1|) \geq 2n/3.$$

Therefore $\gamma_R(G) \geq \lceil 2n/3 \rceil$. We can arrive at the same conclusion by applying Proposition 3.7 with $\Delta = 3$. If the total order of the path or cycle is a multiple of 3, the equality is achieved by assigning a value of 2 to every third vertex, leaving all others 0. (In paths we begin at the vertex adjacent to the terminal vertex). If the total order is not a multiple of 3, we assign a value of 1 to additional vertices. □

Some other illustrative examples are also stated in [3] without proof, which are also completed here. A *complete n-partite graph* is a graph whose vertices can be partitioned into n subsets such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph.

Proposition 4.2. *Let $G = K_{m_1,\ldots,m_n}$ be the complete n-partite graph with $m_1 \leq m_2 \leq \cdots \leq m_n$.*

(a) *If $m_1 \geq 3$ then $\gamma_R(G) = 4$*
(b) *If $m_1 = 2$ then $\gamma_R(G) = 3$*
(c) *If $m_1 = 1$ then $\gamma_R(G) = 2$*

Proof. Let S be the corresponding subset of G with $|S| = m_1$. Let f be a γ_R-function with $f = (V_0, V_1, V_2)$.

(a) Suppose $\gamma_R(G) \le 3$.Let $v_1, v_2, v_3 \in S$ with $f(v_1) \ge f(v_2) \ge f(v_3)$. Then $\sum_{w \in N[v_2] \cup N[v_3]} f(w) \ge 2$. Hence $f(v_1) \le 1$. There are two cases to consider. If $f(w) = 2$ for some $w \in N[v_2] \cup N[v_3]$ distinct from v_2 and v_3, then another vertex u in the same subset with w will have $f(u) = 0$ but not be adjacent to any vertex with value 2, a contradiction. Otherwise $f(v_2) \ge 1$ and $f(v_3) \ge 1$. Hence $f(v_1) \ge 1$. Therefore we must have $f(v_1) = f(v_2) = f(v_3) = 1$ and all other vertices 0, yet another contradiction. Hence $\gamma_R(G) \ge 4$. The equality is achieved by assigning one vertex w not in S with $f(w) = 2$, $f(v_1) = 2$ and 0 for the rest.

(b) Suppose $\gamma_R(G) \le 2$.Let $v_1, v_2 \in S$. These two vertices are not adjacent, but each of them is adjacent to all the other vertices. Hence either there exists one vertex w not in S with $f(w) = 2$, or $f(v_1) = f(v_2) = 1$. f is not a RDF in either case, a contradiction. One way $\gamma_R(G) = 3$ is achieved by assigning $f(v_2) = 2$, $f(v_1) = 1$ and 0 for the rest.

(c) Let $v_1 \in S$. With a similar argument, we get $\gamma_R(G) = 2$ is achieved by assigning $f(v_1) = 2$ and 0 for the rest. □

Proposition 4.3. [3] *If G is a graph of order n which contains a vertex of degree $n-1$, then $\gamma(G) = 1$ and $\gamma_R(G) = 2$.*

Proof. Notice that $\gamma_R(G) \ge 2$ and assign 2 to any one vertex of G with degree $n-1$ and 0 to the rest will give us an RDF with weight 2. □

We let $(n/2)K_2$ denote the graph consisting of $n/2$ copies of the complete graph K_2 on two vertices.

Proposition 4.4. [3] *If G a graph of order n and every vertex in G has non-zero degree, then $\gamma_R(G) = n$ if and only if n is even and $G = (n/2)K_2$.*

Proof. [3] If $G = (n/2)K_2$, then each edge contributes at least two to $\gamma_R(G)$, and hence $n \le \gamma_R(G) \le n$.

Assume therefore that $\gamma_R(G) = n$. If G has two incident edges uv and vw, then $f = (V_0, V_1, V_2)$, where $V_0 = \{u, w\}, V_1 = V - \{u, v, w\}$ and $V_2 = \{v\}$, defines a Roman dominating function. Hence $\gamma_R(G) \le |V_1| + 2|V_2| = n - 1$, which is a contradiction. Hence, no two edges of G are incident, and since G is isolate-free, the conclusion will follow. □

Now we shift our focus to another specific type of graph. In [3], the *Cartesian product* of two graphs is mentioned with a simple example of $2 \times n$ grid graph $G_{2,n}$. We now will first restate this simple example and then extend this idea to obtain some new results when it is combined with 3-regular graphs, another interesting class of graphs. For arbitrary graphs G and H, we define the *Cartesian product* of G and H to be the graph $G \times H$ with vertices $\{(u, v) | u \in G, v \in H\}$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if one of the following is true: $u_1 = u_2$ and v_1 is adjacent to v_2 in H; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G. If $G = P_m$ and $H = P_n$, then the Cartesian product $G \times H$ is called the *$m \times n$ grid graph* and is denoted $G_{m,n}$.

The following proposition serves as the starting point of our discussion on this type of graph. In the original paper only one direction is shown, we complete the other.

Proposition 4.5. [3] *For the $2 \times n$ grid graph $G_{2,n}$, $\gamma_R(G_{2,n}) = n + 1$.*

Proof. We first show $\gamma_R(G_{2,n}) \geq n + 1$. Suppose there exists a γ_R-function $f = (V_0, V_1, V_2)$ with $\gamma_R(G_{2,n}) = n$. Now $|V_1| + 4|V_2| \geq 2n$ and $|V_1| + 2|V_2| = n$, hence $|V_1| = 0$ and $4|V_2| = 2n$. This implies that each $u \in V_2$ will be adjacent to 3 vertices in V_0 and each $v \in V_0$ is adjacent to exactly 1 vertex in V_2. Observe the two vertices at the corner and we get a contradiction since they both have to be in V_0 and both adjacent to a vertex in V_2, which is impossible. Now we show by construction that $\gamma_R(G_{2,n}) \leq n + 1$. Let the vertices of $G_{2,n}$ be denoted by $v_{1,1}, \ldots, v_{1,n}, v_{2,1}, \ldots, v_{2,n}$ and define the RDF g as follows: for each i such that $2 + 4i \leq n$, let $g(v_{2,2+4i}) = 2$, and for each j such that $4j \leq n$, let $g(v_{1,4j}) = 2$. Let $g(v_{1,1}) = 1$, and if $n \equiv 1 \pmod 4$, let $g(v_{2,n}) = 1$, and if $n \equiv 3 \pmod 4$, let $g(v_{1,n}) = 1$. For all the remaining vertices u, let $g(u) = 0$. It is easily seen that g is an RDF and that $g(V) = n + 1$. □

On the other hand, a graph is called a *3-regular* graph if the degree of each vertex is 3. The order $V(G)$ of any 3-regular graph G can only be even (since when counting the total degree of all vertices each edge will be counted twice and it also equals $3V(G)$, thus $V(G)$ has to be even). We first present a basic property of 3-regular graphs.

Proposition 4.6. *If G is a 3-regular graph of order n, then $\gamma_R(G) \geq n/2$ when $n \equiv 0 \pmod 4$ and $\gamma_R(G) \geq n/2 + 1$ when $n \equiv 2 \pmod 4$.*

Proof. The conclusion follows immediately from Proposition 3.7. When $n \equiv 0 \pmod 4$, equality can be achieved by letting $|V_1| = 0$ and $|V_2| = n/4$. When $n \equiv 2 \pmod 4$, $|V_1| \geq 1$. Thus $|V_1| + 2|V_2| \geq n/2 + 1$. □

With above propositions we are now able to investigate a Cartesian product of two graphs which turns out to be 3-regular itself. In the following proposition, the graph $C_{n/2} \times P_2$ essentially consists of two cycles with vertices at corresponding positions connected (In other words if we label two cycles respectively, every vertex in one cycle will only be adjacent to the vertex with same label in the other).

Proposition 4.7. *If $G = C_{n/2} \times P_2$, then $\gamma_R(G) = n/2$ when $n \equiv 0 \pmod 8$ and $\gamma_R(G) = n/2 + 1$ when $n \equiv 2, 4, 6 \pmod 8$.*

Proof. Let $V(C_{n/2}) = \{0, 1, 2, ..., n/2 - 1\}$ and $uv \in E(C_{n/2})$ if and only if $u - v \equiv 1$ or $-1 \pmod{n/2}$. Let $V(P_2) = \{1, 2\}$. Consider following cases.

Case 1: $n = 8m$ ($m \geq 1$). Define a γ_R-function $f = (V_0, V_1, V_2)$ for G with

$$V_2 = \{(4k, 1) \mid 0 \leq k \leq m - 1\} \cup \{(4l + 2, 2) \mid 0 \leq l \leq m - 1\},$$

$$V_1 = \emptyset, V_0 = V - V_2.$$

Thus $\gamma_R(G) \leq f(V) = 4m$. By 4.6 $\gamma_R(G) \geq 4m$, thus $\gamma_R(G) = 4m$.

Case 2: $n = 8m + 2$ ($m \geq 1$). Define f with

$$V_2 = \{(4k, 1) \mid 0 \leq k \leq m\} \cup \{(4l + 2, 2) \mid 0 \leq l \leq m - 1\},$$

$$V_1 = \emptyset, V_0 = V - V_2.$$

Thus $\gamma_R(G) \leq f(V) = 4m + 2$. By 4.6 $\gamma_R(G) \geq 4m + 2$, thus $\gamma_R(G) = 4m + 2 = n/2 + 1$.

Case 3: $n = 8m+4$ ($m \geq 1$). By 4.6, $\gamma_R(G) \geq 4m+2$. We use a coloring method to show the equality cannot be achieved.Let X and Y be two partite sets of the bipartite graph G. Color each vertex in X black and each vertex in Y white. Every vertex in one set is adjacent to 3 vertices in the other set. Assume that $\gamma_R(G) = 4m+2$ can be reached with a γ_R-function $f = (V_0, V_1, V_2)$. $2 \mid V_2| + |V_1| = 4m+2$. As $4|V_2| + |V_1| \geq 8m+4$, hence $|V_1| = 0, |V_2| = 2m+1, V_0 = V - V_2$, and for any two distinct $u, v \in V_2$, $N[u] \cap N[v] = \emptyset$.

Assume there are s black vertices and t white vertices in V_2. Hence the union of closed neighbor of V_2 contains $s+3t$ black vertices and $3s+t$ white vertices. Now f is a RDF, thus there are $s+3t$ black vertices and $3s+t$ white vertices in G and both are even. But $s+t = |V_2| = 2m+1$ is odd, leading to a contradiction. Thus $\gamma_R(G) \geq 4m+3$.Define a γ_R-function $f = (V_0, V_1, V_2)$ for G with

$$V_2 = \{(4k, 1) \mid 0 \leq k \leq m\} \cup \{(4l+2, 2) | 0 \leq l \leq m-1\},$$

$$V_1 = \{(4m+1, 2)\}, V_0 = V - V_1 - V_2.$$

Thus $\gamma_R(G) \leq f(V) = 4m+3$ and therefore $\gamma_R(G) = 4m+3 = n/2+1$.

Case 4: $n = 8m+6$ $(m \geq 0)$.Define a f with

$$V_2 = \{(4k, 1), (4K+2, 2) \mid 0 \leq k \leq m\}$$

$$V_1 = \emptyset, V_0 = V - V_2.$$

Now $\gamma_R(G) \leq f(V) = 4m+4$. By 4.6 $\gamma_R(G) \geq 4m+4$,thus $\gamma_R(G) = 4m+4 = n/2+1$. □

The following graphs show the γ_R-functions for the cases when $n = 2, 4, 6, 8$. Vertices assigned with a value 2 are colored black. Vertices assigned with a value 1 are colored grey. The rest assigned with a value 0 are colored white.

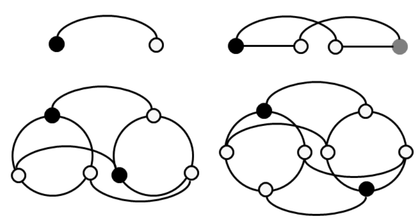

Fig 4.1 Some Cases for Proposition 4.7

In light of this problem, it is interesting to investigate on another class of graph called circulant graphs. A *circulant graph* $C_n\langle a_1, a_2, ..., a_k\rangle$ with n vertices $0, 1, 2, ..., n-1$ refers to a simple graph whose vertex i is adjacent to $i \pm a_1, i \pm a_2, ..., i \pm a_k$ (take the remainder when divided by n), where $a_1, a_2, ...a_k$ are postive integers and $0 < a_i < (n+1)/2, a_i \neq a_j$ for distinct i, j with $1 \leq i, j \leq k$.

The necessary and sufficient condition for a circulant graph $C_n\langle a_1, a_2, ..., a_k\rangle$ to be connected is that the greatest commmon divisor of $(n, a_1, a_2, ...a_k)$ is 1. A 3-regular circulant graph must be $C_n\langle a, n/2\rangle$, where n is an even number with greater than 2 and $1 \leq a \leq n/2-1$. If $G = C_n\langle a, n/2\rangle$ is a connected 3-regular circulant graph, then $\gcd(n, a) = 1$ or 2. If $\gcd(n, a) = 1$, then $C_n\langle a, n/2\rangle$ is isomorphic to $C_n\langle 1, n/2)\rangle$. If $\gcd(n, a) = 2$, then $n/2$ must be odd and $C_n\langle a, n/2\rangle$ is isomorphic to $C_{n/2} \times P_2$, a subject of our discussion above.

Proposition 4.8. *If* $G = C_n\langle a, n/2\rangle$ *is a 3-regular circulant graph and* $gcd(n, a) = 1$*, then* $\gamma_R(G) = n/2$ *when* $n \equiv 4 \pmod 8$ *and* $\gamma_R(G) = n/2+1$ *when* $n \equiv 0, 2, 6 \pmod 8$.

Proof. $G \cong C_n\langle 1, n/2\rangle$. Consider following cases:

Case 1 $n = 8m + 4 (m \geq 0)$. Define a γ_R-function $f = (V_0, V_1, V_2)$ for G with

$$V_2 = \{4k \mid 0 \leq k \leq 2m\},$$

$$V_1 = \emptyset, V_0 = V - V_2.$$

Odd number vertices are neighbors of vertices in V_2. Even number vertices are either in V_2 or equal to $2+4l$ ($l = 0, 1, ..., m-1$). The later ones are also neighbors of vertices in V_2 since $2+4l+n/2 \equiv 2+4l+4m+2 \equiv 4(l+m+1) \equiv 4k \pmod n$.Thus $\gamma_R(G) \leq f(V) = 4m + 2$. By 4.6 $\gamma_R(G) \geq 4m + 2$, thus $\gamma_R(G) = 4m + 2 = n/2$.

Case 2 $n = 8m + 6 (m \geq 0)$. Define a γ_R-function $f = (V_0, V_1, V_2)$ for G with

$$V_2 = \{4k \mid 0 \leq k \leq 2m\} \cap \{n/2 + 2 + 4l \mid 0 \leq l \leq m\},$$

$$V_1 = \emptyset, V_0 = V - V_2.$$

Thus $\gamma_R(G) \leq f(V) = 4m + 4$. By 4.6 $\gamma_R(G) \geq 4m + 4$,thus $\gamma_R(G) = 4m + 4 = n/2 + 1$.

Case 3 $n = 8m (m \geq 0)$. By 4.6 $\gamma_R(G) \geq 4m$. We show by contradiction that equality cannot be achieved. Suppose $\gamma_R(G) = 4m$. $|V_2| = 2m$,$V_1 = \emptyset$ and $V_0 = V - V_2$. Since G is 3-regular and there are $8m$ vertices in G, for any two distinct $u, v \in V_2$, $N[u] \cap N[v] = \emptyset$. Without loss of generality let $0 \in V_2$. Thus $1, 2 \in V_2$. If $3 \in V_2$, consider vertex $n/2 + 1$ in V. At least one of vertices $n/2, n/2 + 1, n/2 + 2$ belongs to V_2. But its closed neighborhood will intersect with $N[0]$ or $N[3]$, a contradiction. Hence $3 \notin V_2$. Then $4 \in V_2$, otherwise 2 and 3 have to be adjacent to vertices $n/2 + 2$ and $n/2 + 3$, both of which should be in V_2, but their closed neighborhoods intersect.

Similarly, vertices $5, 6, 7 \notin V_2$ but $8 \in V_2$. Generally, $4k - 1, 4k - 2, 4k - 3 \notin V_2$ but $4k \in V_2$. Since $n/2 = 4m$, and it intersects with $N[0]$, a contradiction. Hence $\gamma_R(G) > 4m$. Define a γ_R-function $f = (V_0, V_1, V_2)$ for G with

$$V_2 = \{4k \mid 0 \leq k \leq m - 1\} \cap \{n/2 + 2 + 4l \mid 0 \leq l \leq m - 1\},$$

$$V_1 = \{4m - 1\}, V_0 = V - V_2 - V_1.$$

Thus $\gamma_R(G) \leq f(V) = 4m + 1$. Thus $\gamma_R(G) = 4m + 1 = n/2 + 1$.

Case 4 $n = 8m + 2 (m \geq 0)$. Define a γ_R-function $f = (V_0, V_1, V_2)$ for G with

$$V_2 = \{4k \mid 0 \leq k \leq m\} \cap \{n/2 + 2 + 4l \mid 0 \leq l \leq m - 1\}$$

$$V_1 = \emptyset, V_0 = V - V_2.$$

Similar to case 1, we can check f is a γ_R-function. Thus $\gamma_R(G) \leq f(V) = 4m + 2$.By 4.6 $\gamma_R(G) \geq 4m + 2$, thus $\gamma_R(G) = 4m + 2 = n/2 + 1$. □

Proposition 4.9. *If $G = C_n\langle a, n/2\rangle$ is a 3-regular circulant graph and $gcd(n, a) = 2$, then $\gamma_R(G) = n/2 + 1$.*

Proof. The result follows immediately from 4.7. □

Having discussed the special cases of 3-regular graphs above, we are going to look at regular circulant graphs in general. Let G be a r-regular graph with order n ($r \geq 1$), $m = \lfloor \frac{n}{r+1} \rfloor$, $t = n \pmod{r + 1}$, then $n = (r + 1)m + t$, $0 \leq t \leq r$. Let S be an arbitrary dominating set of G, then for each vertex $v \in V(G)$, $N[v] \cap S \neq \emptyset$, and v is being dominated $|N[v] \cap S| \geq 1$ times. We define a function rd counting the times v is dominated as follows:

$$rd(v) = |N[v] \cap S| - 1$$

For a vertex set $V' \subseteq V(G)$, let $rd(V') = \sum_{v\in V'} rd(v)$. Then by Proposition 3.3(d), V_2 is a γ-set of $G[V_0 \cup V_2]$, and this gives us

Lemma 4.10. [4] $rd(V(G[V_0 \cup V_2])) = (r+1)n_2 - (n - n_1)$.

Lemma 4.11. [4] *If $f = (V_0, V_1, V_2)$ is a γ_R-function of G, then*
(a) $n_2 \geq \lceil \frac{n-n_1}{r+1} \rceil$.
(b) $f(V(G)) \geq 2m + \lceil \frac{2r+(r-1)n_1}{r+1} \rceil \geq \frac{2r+(r-1)n_1}{r+1}$.
(c) $f(V(G)) \geq 2m$ *for* $t = 0$.
(d) $f(V(G)) \geq 2m + 2$ *for* $t \geq 1$ *and* $(t, n_1) \neq (1, 1)$.

Proof. (a) By Proposition 3.3(d), V_2 is a γ-set of $G[V_0 \cup V_2]$, hence $(r+1)n_2 \geq n - n_1$. Thus $n_2 \geq \lceil \frac{n-n_1}{r+1} \rceil$.

(b) Since $f(V(G)) = 2n_2 + n_1$, we have

$$(r+1)f(V(G)) = 2(r+1)n_2 + (r+1)n_1$$

$$\geq 2n - 2n_1 + (r+1)n_1 = 2(r+1)m + 2t + (r-1)n_1$$

.

Hence $f(V(G)) \geq 2m + \lceil \frac{2r+(r-1)n_1}{r+1} \rceil \geq 2m + \frac{2r+(r-1)n_1}{r+1} = \frac{2r+(r-1)n_1}{r+1}$.

(c) Suppose $t = 0$, by (b) $f(V(G)) \geq 2m + \lceil \frac{2r+(r-1)n_1}{r+1} \rceil \geq 2m$.

(d) Suppose $t \geq 1$.

Case 1 $n_1 = 0$, then by (a) $n_2 \geq \lceil \frac{n-n_1}{r+1} \rceil = \lceil \frac{(r+1)m+t}{r+1} \rceil = m + 1$. Hence $f(V(G)) = 2n_2 + n_1 = 2n_2 \geq 2m + 2$.

Case 2 $n_1 = 1$ and $t \geq 2$, then by (b) $f(V(G)) \geq 2m + \lceil \frac{2t+(r-1)n_1}{r+1} \rceil \geq 2m + \lceil \frac{4+r-1}{r+1} \rceil = 2m + 2$.

Case 3 $n_1 \geq 2$, then by (b) $f(V(G)) \geq 2m + \lceil \frac{2t+(r-1)n_1}{r+1} \rceil \geq 2m + \lceil \frac{2+2(r-1)}{r+1} \rceil = 2m + 1 + \lceil \frac{r-1}{r+1} \rceil = 2m + 2$. □

The following calculation gives a good example of circulant graphs.

Proposition 4.12. [4] *For 4-regular graph $C_n\langle 1, 3\rangle (n \geq 7)$,*

$$\gamma_R(C_n\langle 1,3\rangle) = \begin{cases} 2m, & \text{if } t = 0 \\ 2m+2, & \text{if } t = 1,2,3 \\ 2m+3, & \text{if } t = 4 \end{cases}$$

Proof. Let

$$S_{1,2} = \begin{cases} \{v_{5i+2} | 0 \leq i \leq m-1\}, & \text{if } t = 0 \\ \{v_{5i+2} | 0 \leq i \leq m\}, & \text{if } t \neq 0 \end{cases}$$

$$S_{1,1} = \begin{cases} \{v_0\}, & \text{if } t = 4 \\ \emptyset, & \text{if } t \neq 4 \end{cases}$$

$$S_{1,0} = N(S_{1,2})$$

Then $N[S_{1,2}] \cup S_{1,1} = V(C_n\langle 1,3\rangle)$, and $f = (V_0, V_1, V_2) = (S_{1,0}, S_{1,1}, S_{1,2})$ is a RDF of $C_n\langle 1,3\rangle$ with

$$f(V(C_n\langle 1,3\rangle)) = \begin{cases} 2m, & \text{if } t = 0 \\ 2m+2, & \text{if } t = 1,2,3 \\ 2m+3, & \text{if } t = 4 \end{cases}$$

Hence we have

$$\gamma_R(C_n\langle 1,3\rangle) \le \begin{cases} 2m, & \text{if } t=0 \\ 2m+2, & \text{if } t=1,2,3 \\ 2m+3, & \text{if } t=4 \end{cases}$$

Now we are going to prove that

$$\gamma_R(C_n\langle 1,3\rangle) \ge \begin{cases} 2m, & \text{if } t=0 \\ 2m+2, & \text{if } t=1,2,3 \\ 2m+3, & \text{if } t=4 \end{cases}$$

Case 1 $t=0$. By Lemma 4.11(c), $\gamma_R(C_n\langle 1,3\rangle) \ge 2m$.

Case 2 $t=1,2,3$ and $(t,n_1) \ne (1,1)$. By Lemma 4.11(d), $\gamma_R(C_n\langle 1,3\rangle) \ge 2m+2$.

Case 3 $(t,n_1)=(1,1)$. By Lemma 4.11(a), $n_2 \ge \lceil \frac{n-n_1}{5} \rceil = \lceil \frac{5m+1-1}{5} \rceil = m$. Hence $\gamma_R(C_n\langle 1,3\rangle) = 2n_2+n_1 \ge 2m+1$. Assume that $\gamma_R(C_n\langle 1,3\rangle) = 2m+1$. Then by Lemma 4.10, $rd(V(G[V_0 \cup V_2])) = (r+1)n_2-(n-n_1) \ge 5m-(5m+1-1) = 0$. Without loss of generality , let $v_{5m} \in V_1$. By Proposition 3.3(b), we have $v_0 \in V_0$. By the definition of RDF, $N(v_0) \cap V_2 \ne \emptyset$, we have $\{v_1, v_3, v_{5m-2}\} \cap V_2 \ne \emptyset$.

Case 3.1 Suppose $v_1 \in V_2$. Let $v_i \in V_2$ be the vertex dominating v_{5m-2}, then since $rd(V(G[V_0 \cup V_2])) = 0$, we have $v_i \notin \{v_{5m-2}, v_{5m-1}, v_0\}$. By Proposition 3.3(b), we have $v_i \ne v_{5m-3}$. Hence $v_i = v_{5m-5}$. Let $v_j \in V_2$ be the vertex dominating v_{5m-3}, then $v_j \in \{v_{5m-6}, v_{5m-4}, v_{5m-3}, v_{5m-2}\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 0$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 0$.

Case 3.2 Suppose $v_3 \in V_2$. Let $v_i \in V_2$ be the vertex dominating v_1. By Proposition 3.3(b), we have $v_i \ne v_{5m-1}$. So $v_i \in \{v_0, v_1, v_2, v_4\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 0$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 0$.

Case 3.3 Suppose $v_{5m-2} \in V_2$. Let $v_i \in V_2$ be the vertex dominating v_1, then since $rd(V(G[V_0 \cup V_2])) = 0$, we have $v_i \notin \{v_{5m-1}, v_0, v_1\}$. By Proposition 3.3(b), we have $v_i \ne v_2$. Hence $v_i = v_4$. Let $v_j \in V_2$ be the vertex dominating v_2, then $v_j \in \{v_2, v_3, v_4, v_5\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 0$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 0$.

From cases 3.1-3.3, we have $\gamma_R(C_n\langle 1,3\rangle) \ne 2m+1$ for $(t,n_1)=(1,1)$, hence $\gamma_R(C_n\langle 1,3\rangle) \ge 2m+2$.

Case 4 $t=4$. By Lemma 4.11(b),

$$\gamma_R(C_n\langle 1,3\rangle) \ge \frac{2n+(r-1)n_1}{r+1} = \frac{2\times(5m+4)+(4m-1)n_1}{4+1} = 2m+1+\frac{3n_1+3}{5}$$

Hence $\gamma_R(C_n\langle 1,3\rangle) \ge 2m+1+\lceil \frac{3n_1+3}{5} \rceil$.

Case 4.1 Suppose $n_1 \ne 0$. Then $\gamma_R(C_n\langle 1,3\rangle) \ge 2m+1+\lceil \frac{3n_1+3}{5} \rceil \ge 2m+3$.

Case 4.2 Suppose $n_1 = 0$. Then By Lemma 4.11(a), $n_2 \ge \lceil \frac{n-n_1}{r+1} \rceil = \lceil \frac{5m+4}{5} \rceil = m+1$. Assume that $n_2 = m+1$, then by Lemma 4.10, $rd(V(G[V_0 \cup V_2])) = (r+1)n_2-(n-n_1) = 5(m+1)-(5m+4) = 1$. Without loss of generality, we may assume that $rd(v_0) = 1$. Then we have $N[v_0] \cap V_2 = \{v_3, v_{5m+1}\}$. Let $v_i \in V_2$ be the vertex dominating v_1, we have $v_i \in \{v_{5m+2}, v_0, v_1, v_2, v_4\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 1$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$. Hence $n_2 \ne m+1$. $n_2 \ge m+2$, $\gamma_R(C_n\langle 1,3\rangle) = 2n_2+n_1 \ge 2m+4$.

From the above discussion we have

$$\gamma_R(C_n\langle 1,3\rangle) = \begin{cases} 2m, & \text{if } t=0 \\ 2m+2, & \text{if } t=1,2,3 \\ 2m+3, & \text{if } t=4 \end{cases}$$

□

In order to obtain the final conclusion with respect to whether circulant graphs are Roman or not, we need to invoke the following lemma, which is stated in [4] without proof. We complete the proof as follows.

Lemma 4.13. *A graph G is Roman if and only if it has a γ_R-function $f = (V_0, V_1, V_2)$ with $n_1 = |V_1| = 0$.*

Proof. Let G be a graph and $f = (V_0, V_1, V_2)$ be a γ_R-function of G. From Proposition 3.3(d) we know that $V_2 \succ V_0$ and $V_1 \cup V_2 \succ V$, hence

$$\gamma(G) \leq |V_1 \cup V_2| = |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_R(G)$$

. But since G is Roman, we know that

$$2\gamma(G) = 2|V_1| + 2|V_2| = \gamma_R(G) = |V_1| + 2|V_2|$$

. Hence $n_1 = |V_1| = 0$.

Conversely, let $f = (V_0, V_1, V_2)$ be a γ_R-function of G with $n_1 = |V_1| = 0$. Therefore $\gamma_R(G) = 2|V_2|$. Since $V_1 \cup V_2 \succ V$, it follows that V_2 is a dominating set of G. But by Proposition 3.3(d) we know that V_2 is a γ-set of $G[V_0 \cup V_2]$, hence $|V_2| = \gamma(G)$ and $\gamma_R(G) = 2\gamma(G)$, implying G is a Roman graph. □

Proposition 4.14. [4] *The circulant graphs $\gamma_R(C_n\langle 1, 3\rangle)$ are Roman for $n \geq 7$ and $n \not\equiv 4 \pmod 5$.*

Proof. According to the proof of Proposition 4.12 we have $f = (V_0, V_1, V_2) = (S_{1,0}, S_{1,1}, S_{1,2})$ is a γ_R-function with $|V_1| = 0$. By Lemma 4.13, the circulant graphs $\gamma_R(C_n\langle 1, 3\rangle)$ are Roman for $n \geq 7$ and $n \not\equiv 4 \pmod 5$. □

5. A Discussion on the Bounds for Roman Domination Number

Base all the above discussion, in this section we will further discuss the bound for Roman domination number for graphs in general. In the following approach the upper bound of the Roman domination number of a graph is characterized by its order. It is a simple yet insightful way of characterizing the bound. The same conclusion is shown in [5], but in this paper we use a different method to prove it. We first introduce a lemma.

Lemma 5.1. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs with $V = V'$ and $E' \subset E$. Then $\gamma_R(G) \leq \gamma_R(G')$.*

Proof. Observe that any RDF of G' is a RDF of G. □

Proposition 5.2. *For any tree T of order $n \geq 3$, $2 \leq \gamma_R(T) \leq \lfloor 4n/5 \rfloor$.*

Proof. For a star T of order $n \geq 3$, $\gamma_R(T) = 2$. We now show the upper bound by using mathematical induction on $D(T)$, the diameter of the tree.

Case 1 When $D(T) = 2$, it is obvious that $\gamma_R(T) = 2 \leq \lfloor 4n/5 \rfloor$.

Case 2 When $D(T) = 3$, find a path $v_0e_0v_1e_1v_2e_2v_3$ that maximizes $d(v_2)$. If $d(v_1) \geq 2$ and $d(v_2) \geq 2$, removing e_1 results in two isolated trees T_1 and T_2 of diameter 2. $\gamma_R(T) \leq \gamma_R(T_1) + \gamma_R(T_2) \leq \lfloor 4n/5 \rfloor$. Otherwise define a RDF f with $f(v_2) = 2$, $f(v_0) = 1$ and $f = 0$ everywhere else. $\gamma_R(T) \leq f(V) = 3 \leq \lfloor 4n/5 \rfloor$.

Case 3 When $D(T) = 4$, find a path $v_0e_0v_1e_1v_2e_2v_3e_3v_4$ that maximizes $d(v_3)$. If $d(v_3) > 2$, remove it together with all its neighboring end vertices as a tree of diameter 2. Repeat until the tree decreases in diameter to previous cases, or becomes a tree T' where $d(v_1) = d(v_3) = 2$ and $d(v_2) \geq 2$. For the last case, define a RDF f with $f(v_2) = 2$, $f(v) = 1$ for all end vertices $v \in T'$, and 0 everywhere else. $\gamma_R(T') \leq \frac{2+d(v_2)}{2d(v_2)+1} n' \leq \lfloor 4n'/5 \rfloor$, where n' is the order of T'.

Inductive hypothesis If $\gamma_R(T) \leq \lfloor 4n/5 \rfloor$ for any tree T where $k-3 \leq D(T) \leq k-1$, then for any tree T of $D(T) = k$, $\gamma_R(T) \leq \lfloor 4n/5 \rfloor$. To show this,

Step 1 Given a tree T with $D(T) = k$, find its longest path $v_0 e_0 v_1 e_1 v_2 ... e_{k-1} v_k$.

Step 2 Remove edge e_{k-3}, resulting in two disjoint trees T_{b1} and T_1. T_{b1} contains path $v_{k-2} e_{k-2} v_{k-1} e_{k-1} v_k$. Since the longest path is chosen, $2 \leq D(T_{b1}) \leq 4$. Let $v(T_{b1})$ denote the order of T_{b1}, then $\gamma_R(T_{b1}) \leq \lfloor \frac{4v(T_{b1})}{5} \rfloor$. T_1 contains path $v_0 e_0 v_1 e_1 v_2 ... e_{k-4} v_{k-3}$. Obviously $D(T_1) \geq k-3$. Thus $k-3 \leq D(T_1) \leq D(T) = k$. If $D(T_1) = k$, note that there are fewer paths of length k in T_1 than in T.

Step 3 If $D(T_1) = k$, repeat Step 1 and 2. At the i-th repetition we divide T_{i-1} into T_{bi} and T_i. Since the number of path of length k is finite and it decreases each time Step 1 and 2 are applied, after some m times of repeats, $D(T_m) < k$. Hence $k-3 \leq D(T_m) \leq k-1$. By inductive hypothesis, $\gamma_R(T_m) \leq \lfloor \frac{4v(T_m)}{5} \rfloor$. Thus,

$$\gamma_R(T) \leq \gamma_R(\sum_{i=1}^{m} T_{bi} + T_m) = \sum_{i=1}^{m} \gamma_R(T_{bi}) + \gamma_R(T_m) \leq \sum_{i=1}^{m} \lfloor \frac{4v(T_{bi})}{5} \rfloor + \lfloor \frac{4v(T_m)}{5} \rfloor$$
$$\leq \lfloor \frac{4 \sum v(T_{bi}) + 4v(T_m)}{5} \rfloor = \lfloor \frac{4v(T)}{5} \rfloor$$

□

Remark 5.3. This upper bound is achievable by structures where successive paths of length 5 are only connected at their central vertex. If n is not a multiple of 5, we just eliminate terminal vertices of the last path accordingly.

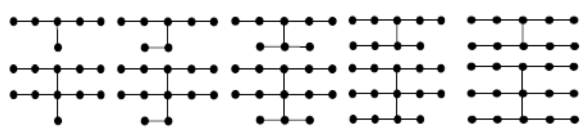

Fig 5.1 Cases when upper bound is achieved

Remark 5.4. Given a tree T of order $n \geq 3$, $\gamma_R(G) = 4n/5$ if and only if T has a structure where successive paths of length 5 are only connected at their central vertex.

Proof. Sufficiency is shown by the proposition. For necessity, given $2 \leq D(T) \leq 4$,only $\gamma_R(P_5) = 4n/5$. Thus only when $T_{bi} = P_5$ for all $1 \leq i \leq m$ will we have $\gamma_R(G) = 4n/5$. □

Corollary 5.5. *For any connected graph G of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \lfloor \frac{4n}{5} \rfloor$.*

Proof. It follows immediately from 5.1 and 5.2. □

6. Impact of edges on Roman domination number

The previous sections all focus on the relationship between the Roman domination number and the graph as a whole. To have a more thorough understanding of the whole set-up, we conclude our discussion with a local property. It is a natural question to ask how Roman domination number of a particular graph will change when some edges are added or removed. There have been some discussion in [6] that removing any edge from a graph G can increase by at most one the Roman domination number of G. We now show another approach. Let G be a graph and u, v be two non-adjacent vertices in G. Let H be the graph where an additional edge between u and v is added to G. The *Roman dominating index* of the edge connecting u and v, denoted by $R(u,v)$, is defined by $R(u,v) = \gamma_R(G) - \gamma_R(H)$. By Lemma 5.1 we have $R(u,v) \geq 0$. In what follows, we will show that this quantity is always bounded above by 1.

Proposition 6.1. *Let G be a graph and u, v be two non-adjacent vertices in G. $0 \leq R(u,v) \leq 1$.*

Proof. We show the proof for the upper bound. Let H be the graph where an additional edge between u and v is added to G and let h be a γ_R-function of H. There are 2 cases to consider.

Case 1: $h(u) = 0,\ h(v) = 2$ or $h(v) = 0,\ h(u) = 2$. Without loss of generality we only consider the case where $h(u) = 0,\ h(v) = 2$. We define a RDF $f : V \to \{0,1,2\}$ on G with $f(u) = 1$ and $f(w) = h(w)$ for all other $w \in V, w \neq u$. Now $f(V) = \gamma_R(H) + 1$. Hence $\gamma_R(G) \leq f(V) = \gamma_R(H) + 1$, which implies $R(u,v) = \gamma_R(G) - \gamma_R(H) \leq 1$.

Case 2: $h(u) \leq 1,\ h(v) \leq 1$. We define a RDF $f : V \to \{0,1,2\}$ on G with $f = h$. Since neither of $h(u)$ or $h(v)$ is 2, each of them must be adjacent to another vertex with value 2 in h, hence after removing the edge between u and v the original function for H is still a RDF for G. Now $f(V) = \gamma_R(H)$. Hence $\gamma_R(G) \leq f(V) = \gamma_R(H)$, which combining with Lemma 5.1 implies $R(u,v) = \gamma_R(G) - \gamma_R(H) = 0$. □

Proposition 6.2. *Let G be a graph and u, v be two non-adjacent vertices in G. $R(u,v) = 1$ if and only if there exists a γ_R-function f of G such that $f(u) = 1,\ f(v) = 2$ or $f(u) = 2,\ f(v) = 1$.*

Proof. Without loss of generality, we first assume that $f(u) = 1,\ f(v) = 2$. Let H be the graph where an additional edge between u and v is added to G and define h to be a RDF of H with $h(u) = 0$ and $h(w) = f(w)$ for all other $w \in V, w \neq u$. Here h is RDF since now u is adjacent to v in H. Now $h(V) = \gamma_R(G) - 1$. Hence $R(u,v) = \gamma_R(G) - \gamma_R(H) \geq \gamma_R(G) - h(V) = 1$. Due to Proposition 6.1, we have $R(u,v) = 1$.

On the other hand, assume $R(u,v) = 1$. As shown in Proposition 6.1, there exists a γ_R-function h of H such that $h(u) = 0,\ h(v) = 2$ or $h(v) = 0,\ h(u) = 2$. Assume, without loss of generality, that $h(u) = 0$ and $h(v) = 2$, then we have a RDF f for G defined as $f(u) = 1$ and $f(w) = h(w)$ for all other $w \in V, w \neq u$. Note that $f(V) = \gamma_R(H) + 1$. Hence $f(V) = \gamma_R(H) + 1 = \gamma_R(G) - R(u,v) + 1 = \gamma_R(G)$. By definition, f is a γ_R-function of G with $f(u) = 1$ and $f(v) = 2$. □

7. Summary

In this expository paper we have systematically introduce the basic set-up of Roman domination in graphs. The existing research tends to focus on special properties of this idea and this paper serves to build a foundation for understanding various advanced problems. In the process we provide a lot of proofs omitted in original papers, link some of the ideas in those paper together and extend them.

Acknowledgments. It is a pleasure to thank my mentor, Jonathan Sheperd, for his support, continuous guidance, meticulous suggestions and for his inexhaustible patience during the whole REU. I would also like to thank Professor Peter May for organizing this wonderful REU program which gives me the opportunity to appreciate various advanced topics in modern mathematics.

References

[1] I. Stewart, Defend the Roman Empire!, Sci. Amer. 281 (6) (1999) 136139.
[2] C.S. ReVelle, K.E. Rosing, Defendens imperium romanum: a classical problem in military strategy, Amer. Math. Monthly 107 (7) (2000), 585594.
[3] E.J. Cockayne, P.M. Dreyer Jr., S.M. Hedetniemi, et al., On Roman domination in graphs, Discrete Math. 278 (2004), 1122.
[4] X. Fu, Y. Yang, B. Jiang, Roman domination in regular graphs, Discrete Math. 309 (2009) 1528-1537
[5] E.W. Chambers, W. Kinnersley, N. Prince, D.B. West, Extremal problems for Roman domination, SIMA Journal on Discrete Mathematics, 23(3)(2009), 1575-1586.
[6] N. Jafari Rad, L. Volkmann, Changing and unchanging the Roman domination number of a graph, Util. Math. 89 (2012), 7995.