POLYNOMIALS WITH SPECIFIED ROOT MULTIPLICITIES

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ABSTRACT. Fix a partition λ of $n > 0$. The space of monic polynomials of degree n whose root multiplicities partition n by λ forms an algebraic variety. We give a method for computing its class in the Grothendieck ring of varieties and use the Grothendieck-Lefschetz Trace Formula to count the number of such polynomials over a finite field.

CONTENTS

0. INTRODUCTION

A monic polynomial $f(z)$ of degree $n > 0$ with coefficients in a field K and $m \leq n$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_m$ in an algebraic closure $\bar{K}$ can be factored as

$$(z-\alpha_1)^{\lambda_1}(z-\alpha_2)^{\lambda_2}\cdots(z-\alpha_m)^{\lambda_m} \tag{0.1}$$

for some positive integers λ_i which partition n. We say λ_i is the *multiplicity* of the root α_i.

Definition. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n, define $Poly_n(\lambda)$ as the set of such $f(z)$ yielding λ as above.

An important special case is when $\lambda = 1 + 1 + \ldots + 1$. $Poly_n(1 + 1 + \ldots + 1)$ is the space of monic square-free polynomials, and we write this simply as $Poly_n$.

There may be repetitions among the multiplicities λ_i. Let $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_r}$ be the set of distinct multiplicities obtained by forgetting these repetitions. Write τ_j for number of times λ_{i_j} appears in λ, i.e. the number of distinct roots of $f(z)$ which are assigned multiplicity λ_{i_j}. The partition $\tau_1 + \tau_2 + \ldots + \tau_r = m$, denoted τ, may also be obtained as follows. Assuming that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$, rewrite this as

$$\lambda_1 = \lambda_2 \ldots = \lambda_{i_1} < \lambda_{i_1+1} = \lambda_{i_1+2} = \ldots = \lambda_{i_2} < \lambda_{i_2+1} = \ldots\ldots = \lambda_{i_r}$$

where $0 = i_0 < i_1 < i_2 < \ldots < i_r = m$. Then $\tau_j = i_j - i_{j-1}$.

A key observation is that the partition τ is more essential to the structure of $Poly_n(\lambda)$ than λ, since an element of $Poly_n(\lambda)$ is uniquely characterized by a set of m distinct roots in $\bar{K}$ partitioned, according to multiplicity, into r labelled subsets with sizes given by τ. For example, $Poly_5(1+2+2)$ is in bijection with $Poly_{11}(3+4+4)$. The relevant feature here is that a polynomial in either space has three distinct roots, two of which are labelled by a certain multiplicity and a third labelled by another, different multiplicity. The particular multiplicities chosen are irrelevant. In §1 we make this observation more precise, showing in Proposition 1.4 that τ determines the isomorphism class of $Poly_n(\lambda)$ as an algebraic variety.

Acknowledgements. The author thanks Kevin Casto, Weiyan Chen, Ben O'Connor, Benson Farb, Nir Gadish, Francois Greer, Ian Le, and Tom Nevins for their helpful comments.

1. $Poly_n(\lambda)$ as a variety

Any monic polynomial over K of degree n can be identified with a point of K^n with coordinates given by its coefficients, so we can view affine n-space over K as the *coefficient* space of such polynomials, and write this as $\mathbb{A}^n_{\text{coeff}}$.

1.1. **Proposition.** *(1) $Poly_n(\lambda)$ is a locally closed subvariety of $\mathbb{A}^n_{coeff}$*
(2) $Poly_n := Poly_n(1+1+\ldots+1)$ is an open subvariety of $\mathbb{A}^n_{coeff}$

Proof. Affine n-space $\mathbb{A}^n_{\text{root}} = \operatorname{Spec} K[x_1, x_2, \ldots, x_n]$, thought of as the ordered space of roots of degree n polynomials, maps to $\mathbb{A}^n_{\text{coeff}}$ via the morphism of schemes $\pi : \mathbb{A}^n_{\text{root}} \to \mathbb{A}^n_{\text{coeff}}$ given by the elementary symmetric polynomials. π is the quotient map by the action of S_n on $\mathbb{A}^n_{\text{root}}$ and is surjective on closed points.

Let L_E be the S_n-orbit of the intersection of hyperplanes in $\mathbb{A}^n_{\text{root}}$ defined by the equations

$$\left\{x_i = x_j : (i,j) \in E\right\} \tag{1.1}$$

where $E \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ is the set

$$E := \left\{(i,j) : \sum_{\ell=1}^{k} \lambda_\ell < i, j \leq \sum_{\ell=1}^{k+1} \lambda_\ell,\ 1 \leq k \leq m\right\}$$

which constrains the coordinates of $\mathbb{A}^n_{\text{root}}$ to m collections of λ_k equal coordinates. The K-points of the image $\pi(L_E)$ of L_E contains the set $Poly_n(\lambda)$ along with all polynomials with root multiplicities partitioning n more coarsely than λ.

To cut $\pi(L_E)$ down to size, we remove polynomials with root multiplicities too coarse as follows. For each subset $\hat{E} \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ strictly containing E, let $L_{\hat{E}}$ be the intersection of hyperplanes defined as is L_E by the set of equations (1.1) above, but with further equations imposed by pairs of indices in $\hat{E}$.

Then

$$\pi(L_E) - \bigcup_{\hat{E} \supsetneq E} \pi(L_{\hat{E}}) \tag{1.2}$$

has K-points $Poly_n(\lambda)$ and is a locally closed subvariety of $\mathbb{A}^n_{\text{coeff}}$ because π is a finite morphism (hence is closed) and all L_E and $L_{\hat{E}}$ are closed in $\mathbb{A}^n_{\text{root}}$, thus proving (1). For (2), note that if $\lambda = 1 + 1 + \ldots + 1$ then the equations given by E impose no conditions on $\mathbb{A}^n_{\text{coeff}}$ so that $\pi(L_E) = \mathbb{A}^n_{\text{coeff}}$ and the subvariety given by (1.2) above is open.

□

Definition. $Conf_m$, the *ordered space of m distinct roots*, denotes the K-subvariety of affine m-space $\mathbb{A}^m_{\mathrm{root}}$ given by $x_i \neq x_j$ for $i \neq j$.

When $K = \mathbb{C}$, $Conf_m$ is the configuration space of m labeled points in the complex plane, hence our notation.

The product of symmetric groups $S_{\tau_1} \times S_{\tau_2} \times \ldots \times S_{\tau_r}$ acts on $Conf_m$ by permuting coordinates. More precisely, $Conf_m = \operatorname{Spec} S^{-1}K[x_1, x_2, \ldots, x_m]$ where $S^{-1}K[x_1, x_2, \ldots, x_m]$ denotes the localization of $K[x_1, x_2, \ldots, x_m]$ at the multiplicative subset S generated by all $x_i - x_j$ with $i < j$. $\prod_{j=1}^r S_{\tau_j}$ acts on $K[x_1, x_2, \ldots, x_m]$ via the action of S_{τ_j} on

$$\{x_{i_{j-1}+1}, x_{i_{j-1}+2}, \ldots, x_{i_j}\}$$

given by

$$\sigma_j(x_{i_{j-1}+k}) = x_{i_{j-1}+\sigma_j{}^{-1}(k)}$$

for $\sigma_j \in S_{\tau_j}$. Equivalently, this is the action obtained by viewing S_τ inside the symmetric group S_m and restricting the action

$$\sigma(x_j) = x_{\sigma^{-1}(j)}$$

Notation. Write S_τ for the product of symmetric groups $\prod_{j=1}^r S_{\tau_j}$.

Since S_τ maps S to itself, the action of S_τ on $K[x_1, x_2, \ldots, x_m]$ induces an action on the localization $S^{-1}K[x_1, x_2, \ldots, x_m]$, hence on $Conf_m$. The scheme-theoretic quotient by this action is the affine K-variety

$$Conf_m/S_\tau = \operatorname{Spec}\,(S^{-1}K[x_1, x_2, \ldots, x_m])^{S_\tau}$$

where $(S^{-1}K[x_1, x_2, \ldots, x_m])^{S_\tau}$ denotes the subring of $S^{-1}K[x_1, x_2, \ldots, x_m]$ invariant under S_τ.

We will show that $Conf_m/S_\tau \cong Poly_n(\lambda)$ as K-varieties if K is perfect. Over $\mathbb{C}$, this variety is a configuration space of m distinct points in the complex plane which have been 'colored' according to their multiplicities. It will turn out that this description is more useful for proving results about $Poly_n(\lambda)$ than that given in Proposition 1.1 above.

Notation. Write A for the invariant subring $K[x_1, x_2, \ldots, x_m]^{S_\tau}$, and B for $(S^{-1}K[x_1, x_2, \ldots, x_m])^{S_\tau}$ so that $Conf_m/S_\tau = \operatorname{Spec} B$.

For $j = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, \tau_j$, let ψ^i_j be the coefficient of z^{i-1} of the polynomial

$$\prod_{k=1}^{\tau_j}(z - x_{i_{j-1}+k})$$

in z. ψ^i_j is a symmetric polynomial in $x_{i_{j-1}+1}, x_{i_{j-1}+2}, \ldots, x_{i_j}$, hence belongs in A.

1.2. **Lemma.** *A is generated as a K-algebra by the collection of ψ^i_j.*

Proof. In fact the result is true if K is only a commutative ring; upon this hypothesis we induct on r. When $r = 1$, A is just $K[x_1, x_2, \ldots, x_{i_1}]^{S_{i_1}}$ and the result follows from the well known theorem on symmetric polynomials.

If $r > 1$, we have

$$A = A'[x_{i_1+1}, x_{i_1+2}, \ldots, x_m]^{\prod_{j>1} S_{\tau_j}}$$

where $A' = K[x_1, x_2, \ldots, x_{\tau_1}]^{S_{\tau_1}}$. The lemma follows from the inductive hypothesis. $\square$

1.3. **Lemma.** *The map $(S \cap A)^{-1}A \hookrightarrow B$ induced by the injection $A \hookrightarrow B$ is an isomorphism. In particular, every element of B can be written as a ratio of elements of A.*

Proof. Write an element of B in the form g/s where $g \in K[x_1, x_2, \ldots, x_m])$ and $s \in S$ with

$$s = \prod_{i<j} (x_i - x_j)^{e_{ij}}$$

for some $e_{ij} \geq 0$ and

$$\left(\frac{g}{s}\right)^{\sigma} = \frac{g}{s}$$

for all $\sigma \in S_\tau$.

For any transposition $(ij) \in S_\tau$, we have

$$s^{(ij)} = (-1)^{e_{ij}} s$$

and thus

$$\frac{g}{s} = \left(\frac{g}{s}\right)^{(ij)} = \frac{g^{(ij)}}{s^{(ij)}} = (-1)^{e_{ij}} \frac{g^{(ij)}}{s}$$
$$\Rightarrow g^{(ij)} = (-1)^{e_{ij}} g$$

Therefore sg and s^2 are fixed by all transpositions in S_τ. So $sg \in A$ and $s^2 \in A \cap S$, and $g/s = \frac{sg}{s^2}$ lies in the image of $(S \cap A)^{-1}A$. □

1.4. **Proposition.** *If K is perfect then $Conf_m/S_\tau$ and $Poly_n(\lambda)$ are isomorphic as varieties over K.*

Proof. We show only that there is a morphism

$$\iota : Conf_m/S_\tau \to Poly_n(\lambda)$$

of K-varieties which is bijective on K-points. An inverse can be constructed by mapping from the space of roots of $Poly_n(\lambda)$.

For $i = 1, 2, \ldots, n$ let ϕ_i be the polynomial in $x_1, x_2, \ldots, x_m$ given by the coefficient of z^{i-1} of the polynomial

$$\prod_{j=1}^{m} (z - x_j)^{\lambda_j}$$

in z so that

$$z^n + \phi_n z^{n-1} + \phi_{n-1} z^{n-2} + \ldots + \phi_2 z + \phi_1 = \prod_{j=1}^{m} (z - x_j)^{\lambda_j} = \prod_{j=1}^{r} \Big(\prod_{k=1}^{\tau_j} (z - x_{i_{j-1}+k}) \Big)^{\lambda_{i_j}}$$

$$= \prod_{j=1}^{r} (z^{\tau_j} + \psi_j^{\tau_j} z^{\tau_j - 1} + \ldots + \psi_j^2 z + \psi_j^1)^{\lambda_{i_j}} \tag{1.3}$$

Write $\iota^\sharp : K[y_1, y_2, \ldots, y_n] \to B$ for the map given by

$$y_i \mapsto \phi_i$$

and let $\iota : Conf_m/S_\tau \to \mathbb{A}^n$ be the induced map on spectra. A K-point of $Conf_m/S_\tau$ is a map κ of K-algebras

$$\kappa : B \to K$$

The pullback $\kappa \circ \iota^\sharp$ of κ by $\iota^\sharp$ is the map $y_i \mapsto \kappa(\phi_i)$ which can be identified with the polynomial

$$f(z) := z^n + \kappa(\phi_n)z^{n-1} + \kappa(\phi_{n-1})z^{n-2} + \ldots + \kappa(\phi_2)z + \kappa(\phi_1)$$

We show that the polynomial $\prod_{j=1}^r f_j$ where

$$f_j = z^{\tau_j} + \kappa(\psi_j^{\tau_j})z^{\tau_j - 1} + \ldots + \kappa(\psi_j^2)z + \kappa(\psi_j^1)$$

is square-free, which by equation (1.3) implies that $f(z) \in Poly_n(\lambda)$ and hence that ι maps K-points into $Poly_n(\lambda)$.

Let Δ be the discriminant of degree m. For $k = 1, 2, \ldots, m$ write Ψ_k for the symmetric polynomial in $x_1, x_2, \ldots, x_m$ given by the coefficient of z^{k-1} of the polynomial

$$\prod_{j=1}^{m}(z - x_j)$$

in z so that

$$\Delta(\Psi_k) = \prod_{1 \le i < j \le m}(x_i - x_j)^2 \tag{1.4}$$

Then

$$\prod_{j=1}^{r} f_j = z^m + \kappa(\Psi_m)z^{m-1} + \ldots + \kappa(\Psi_2)z + \kappa(\Psi_1)$$

and

$$\Delta\big(\kappa(\Psi_k)\big) = \kappa\big(\Delta(\Psi_k)\big) \neq 0$$

since $\Delta(\Psi_k) \in S$ by equation (1.4) above and κ does not vanish on the localized set S. Thus $\prod_{j=1}^r f_j$ is square-free and $\iota(\kappa) \in Poly_n(\lambda)$.

Any $f(z) \in Poly_n(\lambda)$ can be factored as

$$f(z) = \prod_{j=1}^{r} {f_j}^{\lambda_{i_j}} \tag{1.5}$$

where each f_j is the unique monic, square-free polynomial of degree τ_j whose roots are exactly the distinct roots of f with multiplicity λ_{i_j}. A priori, f_j is only contained in $\bar{K}[z]$, but in fact it has coefficients in K: if $\sigma \in Gal(\bar{K}/K)$ then

$$f(z) = f(z)^\sigma = \prod_{j=1}^{r} \left(f_j^\sigma\right)^{\lambda_{i_j}}$$

so that $f_j = f_j^\sigma$ and $f_j \in K[z]$.

Since the f_j are uniquely determined by $f(z)$, the values of κ on ψ_j^i are uniquely determined by its values on ϕ_i due to equation (1.3) above. By Lemmas 1.2 and 1.3, κ is uniquely determined by its values on ψ_j^i. Furthermore, any K-point of $f(z) \in Poly_n(\lambda)$ arises as the pull-back under ι of κ mapping ψ_j^i to the coefficient of z^{i-1} in f_j. It follows that ι is injective on K-points and surjects onto $Poly_n(\lambda)$. $\square$

Convention. In light of Proposition 1.4 we will assume K is perfect for the remainder of this paper.

1.5. **Proposition.** *$Poly_n(\lambda)$ embeds as an open subvariety, written $Poly_\tau$, of the product of square-free polynomial spaces*

$$Poly_{\tau_1} \times Poly_{\tau_2} \times \ldots \times Poly_{\tau_r} \tag{1.6}$$

Proof. Any $f(z) \in Poly_n(\lambda)$ may be factored uniquely as in equation (1.5) above to obtain polynomials $f_1, f_2, \ldots, f_r$. The map $Poly_n(\lambda) \to Poly_{\tau_1} \times Poly_{\tau_2} \times \ldots \times Poly_{\tau_r}$ given by

$$f(z) \mapsto (f_1, f_2, \ldots, f_r)$$

is an embedding whose image consists of all $(f_1, f_2, \ldots, f_r)$ such that f_i and f_j don't share roots for $i \neq j$, and is injective because $f(z)$ may be recovered by $f(z) = \prod_{j=1}^r {f_j}^{\lambda_{i_j}}$. The image is open because it is the complement of the union of hypersurfaces in $\prod Poly_{\tau_j}$ given by the resultant locus on each pair of factors $Poly_{\tau_j}$.

$\square$

1.5.1. **Corollary.** *Over a finite field $\mathbb{F}_q$ of order q,*

$$|Poly_n(\lambda)/_{\mathbb{F}_q}| = \sum_{g \in Poly_m/_{\mathbb{F}_q}} |\nu^{-1}(g)|$$

where $\nu : Poly_\tau \to Poly_m$ is the multiplication map

$$(f_j) \mapsto \prod f_j$$

and $|\nu^{-1}(g)|$ is the size of the fiber over a square-free polynomial g, i.e. the number of ordered lists of factors $(f_1, f_2, \ldots, f_r)$ of g with degrees specified by τ multiplying to g.

2. Preliminaries

We recall two notions from algebraic geometry which will be used to count $Poly_n(\lambda)$ over a finite field.

The Grothendieck-Lefschetz Trace Formula. For a smooth projective variety X over $\mathbb{F}_q$ the number $|X(\mathbb{F}_q)|$ of $\mathbb{F}_q$-points is given by the formula

$$|X(\mathbb{F}_q)| = \sum_{i \geq 0} (-1)^i \operatorname{Trace}(\mathrm{Frob_q}^* \, | H^i_{\acute{e}t}(\overline{X}; \mathbb{Q}_\ell)) \tag{2.1}$$

where ℓ is a prime different from the characteristic of $\mathbb{F}_q$, $\overline{X}$ is the $\overline{\mathbb{F}}_q$-variety $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, and $\mathrm{Frob_q}^* \, | H^i_{\acute{e}t}(\overline{X}; \mathbb{Q}_\ell)$ is the map on the i^{th} ℓ-adic cohomology group induced by the geometric Frobenius morphism $\mathrm{Frob_q} : \overline{X} \to \overline{X}$.

The trace formula (2.1) holds for non-projective X if we replace $H^i_{\acute{e}t}(\overline{X}; \mathbb{Q}_\ell)$ by compactly supported étale cohomology groups $H^i_{\acute{e}t,c}(\overline{X}; \mathbb{Q}_\ell)$ (cf. [2]), so that

$$|X(\mathbb{F}_q)| = \sum_{i \geq 0} (-1)^i \operatorname{Trace}(\mathrm{Frob_q}^* \, | H^i_{\acute{e}t,c}(\overline{X}; \mathbb{Q}_\ell)) \tag{2.2}$$

Furthermore, if X is smooth, then Poincaré duality for étale cohomology ([4], Theorem 24.1) gives an isomorphism of Galois representations

$$H^i_{\acute{e}t,c}(\overline{X}; \mathbb{Q}_\ell) \cong H^{2 \dim X - i}_{\acute{e}t}(\overline{X}; \mathbb{Q}_\ell(-\dim X))^\vee$$

where, for an integer e, $\mathbb{Q}_\ell(e)$ denotes the 1-dimensional Galois representation over $\mathbb{Q}_\ell$ on which $\mathrm{Frob_q}$ acts with weight q^{-e}. Plugging this into formula (2.2), we obtain

$$|X(\mathbb{F}_q)| = q^{\dim X} \sum_{i\geq 0} (-1)^i \operatorname{Trace}(\mathrm{Frob_q}^* \,| H^i_{ét}(\overline{X}; \mathbb{Q}_\ell)^\vee) \tag{2.3}$$

The Grothendieck ring of varieties.

Definition. The Grothendieck ring $K_0(Var_K)$ of varieties over K is, as a group, the quotient of the free abelian group on isomorphism classes $[X]$ of K-varieties X by the relation

$$[X] = [X - Z] + [Z]$$

for a closed subvariety $Z \subset X$. Multiplication is given by

$$[X][Y] = [X \times_K Y]$$

We denote the class $[\mathbb{A}^1]$ of the affine line by $\mathbb{L}$.

In §3 we give an algorithm to compute the class of $Poly_n(\lambda)$ in the Grothendieck ring. The following lemma implies that the algorithm, in particular, counts $Poly_n(\lambda)$ over a finite field:

2.1. **Lemma.** *The map $K_0(Var_{\mathbb{F}_q}) \to \mathbb{Z}$ which sends the class $[X]$ of an $\mathbb{F}_q$-variety X to the number of $\mathbb{F}_q$-points of X is a ring homomorphism.* □

We will also use the following crucial fact about $K_0(Var_K)$:

2.2. **Lemma.** *If X is a disjoint union of locally closed subvarieties $X_1, X_2, \ldots X_e$ then*

$$[X] = [X_1] + [X_2] + \ldots + [X_e]$$

Proof. See the proof of Lemma 2.2 in [1] □

3. Calculations in the Grothendieck ring

As we will see in Theorem 3.2, the class of $Poly_n(\lambda)$ in the Grothendieck ring is a polynomial in classes of square-free spaces $[Poly_d]$, so we first show that $[Poly_d]$ has a simple formula in terms of $\mathbb{L}$:

3.1. **Proposition.** $[Poly_n] = \mathbb{L}^n - \mathbb{L}^{n-1}$ *in the Grothendieck ring $K_0(Var_K)$ if $n > 1$*

Proof. We induct on n. For ℓ a nonnegative integer such that $n - 2\ell \geq 0$ let R_ℓ be the subset of $\mathbb{A}^n_{\text{coeff}}$ containing all polynomials which can be written in the form ab^2, where a and b are polynomials over K of degree $n-2\ell$ and ℓ respectively. Then we have the filtration

$$\emptyset = R_{d+1} \subset R_d \subset R_{d-1} \subset \ldots \subset R_1 \subset R_0 = \mathbb{A}^n$$

where $d = \lfloor n/2 \rfloor$.

The closed subvariety of $\mathbb{A}^n_{\text{root}}$ given by the S_n-orbit of the intersection of hyperplanes

$$x_j = x_{j+1}$$

for $j = 1, 3, 5, \ldots, 2\ell - 1$ is mapped onto R_ℓ by the quotient

$$\mathbb{A}^n_{\text{root}} \xrightarrow{S_n} \mathbb{A}^n_{\text{coeff}}$$

which is a closed map, so R_ℓ is closed.

Writing Z_ℓ for the locally closed subvariety $R_\ell - R_{\ell+1}$, we have

$$[\mathbb{A}^n] = [Z_0] + [Z_1] + [Z_2] + \ldots + [Z_d]$$

The morphism of varieties $Poly_{n-2\ell} \times \mathbb{A}^\ell \to Z_\ell$ which maps a squarefree a of degree $n - 2\ell$ and b of degree ℓ by

$$(a, b) \mapsto ab^2$$

is in fact an isomorphism; one can construct its inverse by mapping from the space of roots of Z_ℓ. Thus $Z_0 \cong Poly_n$ and

$$[Z_d] = [Poly_{n-2d} \times \mathbb{A}^d] = \mathbb{L}^{n-d}$$

and by the inductive hypothesis

$$[Z_\ell] = [Poly_{n-2\ell} \times \mathbb{A}^\ell] = \big(\mathbb{L}^{n-2\ell} - \mathbb{L}^{n-2\ell-1}\big)\mathbb{L}^\ell = \mathbb{L}^{n-\ell} - \mathbb{L}^{n-\ell-1}$$

for $0 < \ell < d$ and so

$$\begin{aligned}[Poly_n] = [Z_0] =& [\mathbb{A}^n] - \Big([Z_1] + [Z_2] + \ldots + [Z_d]\Big) = \mathbb{L}^n - \sum_{\ell=1}^{d-1} \big(\mathbb{L}^{n-\ell} - \mathbb{L}^{n-\ell-1}\big) - [Z_d] \\ =& \mathbb{L}^n - \mathbb{L}^{n-1} + \mathbb{L}^{n-d} - [Z_d] = \mathbb{L}^n - \mathbb{L}^{n-1}\end{aligned}$$

□

3.2. **Theorem.** *There is a recursive formula, given by equation (3.3) below, for the class* $[Poly_n(\lambda)] = [Poly_\tau]$ *of* $Poly_n(\lambda)$ *in the Grothendieck ring which reduces to*

$$[Poly_n(\lambda)] = [Poly_\tau] = \prod [Poly_{\tau_j}] + \sum_{|\Lambda| < |\tau|} a_\Lambda \big(\prod_j [Poly_{\Lambda_j}]\big) \tag{3.1}$$

for some $a_\Lambda \in \mathbb{Z}$*, where the sum on the righthand side is taken over all partitions* Λ *of integers smaller than* $|\tau| = \sum \tau_j$*.*

Proof. We induct on $m = |\tau|$, the case $m = 1$ being trivially true, so assume $m > 1$ and suppose $(f_1, f_2, \ldots, f_r) \in \prod Poly_{\tau_j}$. For each nonempty subset $S \subset \{1, 2, \ldots, r\}$, the set of $\alpha \in \bar{K}$ such that

1. $f_j(\alpha) = 0$ for $j \in S$
2. $f_j(\alpha) \neq 0$ for $j \notin S$

is finite and Galois-invariant, hence is precisely the set of roots of a unique monic square-free polynomial h_S over K.

Alternatively, h_S is the monic polynomial of largest degree dividing precisely those f_k with $k \in S$. The roots of h_S are the roots shared by all $f_k \in S$ and no other f_j.

Thus $(f_1, f_2, \ldots, f_r)$ induces a function, which we write as F, from the set $\mathcal{P}(r)$ of nonempty subsets of $\{1, 2, \ldots, r\}$ to non-negative integers $\mathbb{Z}_{\geq 0}$ given by

$$F(S) = \deg(h_S)$$

which satisfies, for all j,

$$\sum_{S \ni j} F(S) = \tau_j \tag{3.2}$$

where the sum is taken over S.

Roughly speaking, F keeps track of the manner in which the f_j are sharing roots. The equality (3.2) says that each of the τ_j roots of f_j is shared via a unique $S \in \mathcal{P}(r)$.

Notation. We write

(1) $\mathcal{F}$ for the set of all $F : \mathcal{P}(r) \to \mathbb{Z}_{\geq 0}$ satisfying (3.2).
(2) Z_F for the subset of $\prod Poly_{\tau_j}$ containing all $(f_1, f_2, \ldots, f_r)$ inducing F.
(3) $F_0 \in \mathcal{F}$ for the map $F_0(\{j\}) = \tau_j$ which is 0 elsewhere.

Note that (f_j) induces F_0 if and only if no roots are shared among (f_j), so that $Z_{F_0} = Poly_\tau$.

Z_F is locally closed in $\prod Poly_{\tau_j}$ since it is the intersection of images, and complements thereof, under the quotient map

$$\prod Conf_{\tau_j} \xrightarrow{S_\tau} \prod Poly_{\tau_j}$$

of finitely many closed subsets of $\prod Conf_{\tau_j}$ given by intersections of hyperplanes (cf. proof of Proposition 1.1).

Z_F is isomorphic to $Poly_{\tau(F)}$ (cf. Proposition 1.5), where $\tau(F)$ is the partition

$$\sum_{S \in \mathcal{P}(r)} F(S)$$

via the correspondence

$$(h_S)_{S\in\mathcal{P}(r)} \leftrightarrow (\prod_{S \ni j} h_S)_{j=1}^r$$

which 'extracts and collapses' common divisors among (f_j) to obtain polynomials (h_S) which do not share roots. Since $\prod Poly_{\tau_j}$ is a disjoint union of the Z_F, it follows from Lemma 2.2 that

$$[\prod Poly_{\tau_j}] = \sum_{F\in\mathcal{F}} [Z_F] = \sum_{F\in\mathcal{F}} [Poly_{\tau(F)}]$$

Solving for $[Poly_n(\lambda)]$, we obtain

$$[Poly_n(\lambda)] = [Poly_\tau] = [Z_{F_0}] = [\prod Poly_{\tau_j}] - \sum_{\substack{F\in\mathcal{F}\\ F\neq F_0}} [Poly_{\tau(F)}] \tag{3.3}$$

while $|\tau(F)| < |\tau|$ for all $F \neq F_0$, so by the inductive hypothesis each $[Poly_{\tau(F)}]$ satisfies the formula 3.1 above, which, plugged into equation 3.3, completes the proof. □

4. Trace Formula

The formula given in Proposition 3.1 and the counting homomorphism in Lemma 2.1 together imply that the number of monic, square-free polynomials of degree m over a finite field $\mathbb{F}_q$ of order q is $q^m - q^{m-1}$. In this section we obtain this count using the Grothendieck-Lefschetz Trace Formula, introduced in §2, and attempt to generalize the method to $Poly_n(\lambda)$ for arbitrary λ.

Let $\mathrm{Frob_q} : Conf_m/_{\overline{\mathbb{F}}_q} \to Conf_m/_{\overline{\mathbb{F}}_q}$ be the geometric Frobenius automorphism which raises coordinates to the q^{th} power. $\mathrm{Frob_q}$ acts on the i^{th} ℓ-adic cohomology group $H^i_{\acute{e}t}(Conf_m/_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ by multiplication by q^i ([6], Proposition 3.3). $Poly_m/_{\overline{\mathbb{F}}_q}$ is the quotient of $Conf_m/_{\overline{\mathbb{F}}_q}$ by S_m, so transfer gives an isomorphism of Galois representations

$$H^i_{\acute{e}t}(Poly_m/_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^i_{\acute{e}t}(Conf_m/_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)^{S_m}$$

i.e. the subspace of $H^i_{ét}(Conf_m/_{\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ fixed by the action of S_m, which is indeed a Galois representation because the actions of S_m and Frobenius commute.

$Poly_m/_{\mathbb{F}_q}$ is smooth and m-dimensional since it is an open subvariety of affine m-space, so the trace formula (2.3) gives

$$|Poly_m/_{\mathbb{F}_q}| = q^m \sum_{i\geq 0} (-q)^{-i} \dim H^i_{ét}(Conf_m/_{\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)^{S_m} \tag{4.1}$$

It therefore suffices to find the dimension of $H^i_{ét}(Conf_m/_{\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)^{S_m}$, which, by Artin's comparison theorem, is the same as the $\mathbb{C}$-dimension of the analogous singular cohomology group $H^i(Conf_m/_{\mathbb{C}};\mathbb{C})^{S_m}$. The singular cohomology ring $H^*(Conf_m/_{\mathbb{C}};\mathbb{C})$ and its structure as an S_m-representation is well understood and was studied by Arnol'd (see [5] and §2.2 in [7]), whose results show that

$$H^i(Conf_m/_{\mathbb{C}};\mathbb{C})^{S_m} \cong \begin{cases} \mathbb{C} & \text{if } i = 0, 1 \\ 0 & \text{else} \end{cases}$$

From equation (4.1) it follows that

$$|Poly_m/_{\mathbb{F}_q}| = q^m(1 - q^{-1}) = q^m - q^{m-1}$$

Now we count $Poly_n(\lambda)$, which, as an open subvariety of $\prod Poly_{\tau_j}$, is smooth and m-dimensional. As before, transfer gives an isomorphism

$$H^i_{ét}(Poly_n(\lambda)/_{\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \cong H^i_{ét}(Conf_m/_{\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)^{S_\tau}$$

of Galois representations and therefore

$$|Poly_n(\lambda)/_{\mathbb{F}_q}| = q^m \sum_{i\geq 0} (-q)^{-i} \dim H^i(Conf_m/_{\mathbb{C}};\mathbb{C})^{S_\tau} \tag{4.2}$$

By Frobenius reciprocity,

$$\dim H^i(Conf_m/_{\mathbb{C}};\mathbb{C})^{S_\tau} = \Big\langle H^i(Conf_m/_{\mathbb{C}};\mathbb{C}),\ 1\Big\rangle_{S_\tau} \tag{4.3}$$

$$= \Big\langle H^i(Conf_m/_{\mathbb{C}};\mathbb{C}),\ \chi\Big\rangle_{S_m} \tag{4.4}$$

is an inner product of characters of representations of S_m, where χ denotes the character of the induced representation $\mathrm{Ind}_{S_\tau}^{S_m}(1)$ of S_m by the trivial representation of S_τ.

Plugging into (4.2), we have

$$|Poly_n(\lambda)/_{\mathbb{F}_q}| = q^m \sum_{i\geq 0} (-q)^i \Big\langle H^i(Conf_m/_{\mathbb{C}};\mathbb{C}),\ \chi\Big\rangle_{S_m} \tag{4.5}$$

Church, Ellenberg, and Farb prove in Theorem 3.7 of [6], using a version of Grothendieck-Lefschetz with twisted coefficients, the following formula for any class function χ on S_m:

$$q^m \sum_{i\geq 0} (-q)^i \Big\langle H^i(Conf_m/_{\mathbb{C}};\mathbb{C}),\ \chi\Big\rangle_{S_m} = \sum_{g\in Poly_m/_{\mathbb{F}_q}} \chi(\sigma_g)$$

where $\sigma_g \in S_m$ is a permutation on the roots of g induced by $\mathrm{Frob_q}$. Taking χ to be the character of $\mathrm{Ind}_{S_\tau}^{S_m}(1)$, we obtain:

4.1. **Proposition.**

$$|Poly_n(\lambda)/_{\mathbb{F}_q}| = \sum_{g \in Poly_m/_{\mathbb{F}_q}} \chi(\sigma_g)$$

□

Proposition 4.1 is Corollary 1.5.1 in disguise! For a square-free polynomial $g \in Poly_m/_{\mathbb{F}_q}$, write $g = g_1 g_2 \cdots g_s$ for its factorization into irreducibles. The partition $\mu =: (\mu_1, \mu_2, \ldots, \mu_s) \vdash m$ given by $\mu_i = \deg g_i$ is the cycle type of σ_g, and $\chi(\sigma_g)$ is the number of Young tabloids of type τ fixed by σ_g. Let $\sigma_g = \sigma_1 \sigma_2 \cdots \sigma_s$ be a cycle decomposition in which σ_i has order μ_i. A Young tabloid of type τ is fixed by σ_g if and only if each row of the tabloid is a union of entries of σ_i's. The number of such Young tabloids is the number of 'ways of refining' the partition τ to the partition μ, i.e. the number of lists $(\Sigma_1, \Sigma_2, \ldots, \Sigma_r)$ of r disjoint subsets partitioning the set $\{1, 2, \ldots, s\}$ such that

$$(\mu_k)_{k \in \Sigma_j}$$

is a partition of τ_j for $j = 1, 2, \ldots, r$. Each Σ_j determines a factor f_j of g of degree τ_j given by

$$f_j = \prod_{k \in \Sigma_j} g_k$$

and vice versa. So $\chi(\sigma_g)$ is the number of lists $(f_1, f_2, \ldots, f_r)$ of factors of g with $\deg f_j = \tau_j$. But this is exactly the size of the fiber over g of the multiplication map $Poly_\tau \to Poly_m$

$$(f_j) \mapsto \prod f_j$$

thus recovering Corollary 1.5.1.

Remark. Using equation (4.3), we may rewrite the formula (4.2) as

$$\begin{aligned} |Poly_n(\lambda)/_{\mathbb{F}_q}| &= q^m \sum_{i \geq 0} (-q)^i \Big\langle H^i(Conf_m/_{\mathbb{C}}; \mathbb{C}),\ 1 \Big\rangle_{S_\tau} \\ &= q^m \sum_{\gamma \in S_\tau} \sum_{i \geq 0} (-q)^i \operatorname{Trace}(\gamma, H^i(Conf_m/_{\mathbb{C}}; \mathbb{C})) \end{aligned}$$

Lehrer studied the polynomial

$$\sum_{i \geq 0} t^i \operatorname{Trace}(\gamma, H^i(Conf_m/_{\mathbb{C}}; \mathbb{C}))$$

in t, called the *Poincaré* series of γ, and gave an explicit formula (Theorem 5.5 in [8]), which, evaluated at $t = -q$, gives another count for $|Poly_n(\lambda)/_{\mathbb{F}_q}|$.

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