FINITE SPACES AND APPLICATIONS TO THE EULER CHARACTERISTIC

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Abstract. The aim of this paper is to introduce finite spaces and their simplicial complexes. Next, we give an application to combinatorics in the form of a relation between the Euler characteristic and the Möbius function. We begin by giving an overview of finite topological spaces. We introduce beat points and weak homotopy equivalences. Then, we show that finite spaces are weak homotopy equivalent to their associated simplicial complexes. Lastly, we discuss the Euler characteristic of finite spaces. We introduce the Möbius function of posets to show its relation to the Euler characteristic.

Contents

1. Finite topological spaces and preordered sets

We begin with an exposition of finite topological spaces as an analogy to finite preordered sets.

Definition 1.1. A *finite topological space* is a topological space with only a finite number of points.

Definition 1.2. A *finite preordered set* is a set along with a binary relation "$\leq$" that compares elements and designates when one element is greater than another. For a preordered set P and elements $a, b, c \in P$, the relation follows two properties:

- $a \leq a$,
- If $a \leq b$ and $b \leq c$ then $a \leq c$.

We now show that these two concepts are in fact the same. Given a finite topological space X, first note that the intersection of arbitrary open sets is open, as there are only a finite number of open sets to begin with. Thus for each $x \in X$, define the minimal open set U_x to be the intersection of all open sets in X containing x. The set of these U_x forms a basis for the space X; call this basis the *minimal basis*. This is called the minimal basis because any other basis B for X must contain the minimal basis: if U_x is not an element of B, then it must be the union of elements of B. But if that were the case, then one of those elements must contain x, and thus that element must be U_x as it cannot contain anything outside of U_x. Therefore, the minimal basis is a subset of B.

Now, we define a preorder on a finite topological space X as $x \leq y$ if $x \in U_y$. Now if X were a finite preordered set, we can define a topology on X as the topology with a basis of sets $\{y \in X \mid y \leq x\}$ for all $x \in X$. According to this definition, if $y \leq x$, then y is contained in every basis element that also contains x. Thus y is contained in their intersection, and thus y is contained in U_x. On the other hand, if in a finite topological space $y \in U_x$, then $y \leq x$. Thus, according to these two definitions, $y \leq x$ if and only if $y \in U_x$, which implies that these two definitions are mutually inverses.Thus finite topological spaces and finite preordered sets are in a one-to-one correspondence.

A common axiom used in topology is the T_0 separation axiom.

Definition 1.3. A space satisfies the T_0 *axiom* if no two points have an exactly identical set of open neighborhoods.

Likewise, a common extension of the idea of a preordered set is to that of a partially ordered set.

Definition 1.4. A *partially ordered set*, also known as a *poset*, is a preordered set under the restriction that if $x \leq y$ and $y \leq x$, then $x = y$.

Remark 1.5. If X is a T_0 finite topological space with $x, y \in X$, then $x \in U_y$ and $y \in U_x$ implies that x and y are the same point. Translating this over to preordered set terminology, $x \leq y$ and $y \leq x$ implies $x = y$. But this is exactly the requirement for a preordered set to be a poset. Thus posets and T_0 topological spaces are in one-to-one correspondence as well.

This correspondence between finite spaces and preordered sets is interesting, but what is really important about it is that the correspondence extends to maps of these spaces.

Definition 1.6. A map $f : X \to Z$ between preordered sets is *order preserving* if $x \leq y$ implies that $f(x) \leq f(y)$ for all x and y in X.

We now show that these maps correspond exactly to continuous maps between finite topological spaces.

Lemma 1.7. *A function f between finite spaces is continuous if and only if it is order preserving.*

Proof. Consider a function $f : X \to Y$ between finite spaces. Let x and y be elements of X with $x \leq y$. If f is continuous, then $f^{-1}(U_{f(y)})$ is open. But $y \in f^{-1}(U_{f(y)})$, and since $x \leq y$, it follows that $x \in f^{-1}(U_{f(y)})$. Thus $f(x) \in f^{-1}(U_{f(y)})$, so $f(x) \leq f(y)$, and thus f is order preserving.

Now let f be order preserving. Since the set of U_x's constitutes a basis, we only need to show that $f^{-1}(U_z)$ is open for all $z \in Z$. Suppose $y \in f^{-1}(U_z)$. As $x \leq y$, $f(x) \leq f(y) \leq z$. Thus $x \in f^{-1}(U_z)$. Thus f is continuous as U_z is a union of minimal basis elements with maximums in the set $f^{-1}(z)$. □

2. Connectedness and Homotopies of finite spaces

Before this section, note that throughout this paper we will make use of the term *map* between topological spaces generally to refer to continuous maps.

Definition 2.1. A path between points x and y in X is a map $\alpha : I \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Proposition 2.2. *If $x, y \in X$ for X a finite space, and $x \leq y$, then there is a path connecting x to y.*

Proof. We prove this by constructing a path α. Define α to be the function such that

$$\alpha(t) = \begin{cases} x & \text{if } 0 \leq t < 1 \\ y & \text{if } t = 1. \end{cases}$$

Now we show that α is a continuous. For all open subsets $U \subset X$, $\alpha^{-1}(U)$ is either $\emptyset$, $[0, 1)$, or $[0, 1]$ since if U contains y then U also contains x. Thus α is continuous and thus is a path. □

Proposition 2.3. *Suppose $x, y \in X$ for X a finite space. If there is a path connecting x to y, then there exists a sequence of points $x_1, x_2, ..., x_n$ with $x = x_1$ and $y = x_n$ such that x_i is comparable to x_{i+1}.*

Proof. Let X be a finite space and let $x \in X$. Let

$$S = \{y \in X \mid \exists\ x_1, x_2, ..., x_n \text{ with } x = x_1, y = x_n \text{ and } x_i \text{ is comparable to } x_{i+1}\}.$$

But then S is open as if $z \in S$, then so is U_z. On the other hand, if $z \notin S$ then neither is U_z. Thus $X \setminus S$ is open, and S is closed. But since S is both open and closed, and X is connected, it must be the case that $S = X$. □

Now we examine homotopies of maps using the mapping space.

Definition 2.4. Let X and Y be topological spaces. The *mapping space* Y^X denotes the set of maps from X to Y. Y^X can be considered a topological space by using the *compact-open topology*, which is the topology having a subbasis of sets of the form

$$W(C, U) = \{f \mid f(C) \subset U\}$$

for all C compact in X and U open in Y.

Definition 2.5. For X and Y finite spaces and $f, g \in Y^X$, the *pointwise order* on Y^X is defined by$f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Proposition 2.6. *For X and Y finite spaces, the intersection of all open sets in Y^X containing a map g is $\{f \in Y^X \mid f \leq g\}$.*

Proof. Let V denote the intersection of all open sets containing g. We know that V is open as X and Y are finite. Let $f \leq g$. g is an element of V, thus $f \in V$. Conversely, if f is in V then fix an $x \in X$. g is in $W(\{x\}, U_{g(x)})$, thus $f \in W(\{x\}, U_{g(x)})$. Thus $f(x) \in U_{g(x)}$ and therefore $f(x) \leq g(x)$. □

Proposition 2.6 can be restated as saying that the pointwise ordering coincides with the preordering associated with the compact-open topology.

Throughout algebraic topology, one of the most important ideas is that of a homotopy relation.

Definition 2.7. Two maps $f, g : X \to Y$ between topological spaces X and Y are *homotopic* if there exists a map $G : X \times I \to Y$ such that for all $x \in X$, $G(x, 0) = f(x)$ and $G(x, 1) = g(x)$. If two maps f and g are homotopic, we write $f \simeq g$.

Definition 2.8. Two spaces X and Y are said to be *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We write $X \simeq Y$.

We use this opportunity to introduce some language that will be useful later.

Definition 2.9. A *strong deformation retract* of a space X is a subspace A such that there exists a map $F : X \times I \to X$ with $F(x, 0) = x$, $F(x, 1) \in A$, and $F(a, t) = a$ for all $x \in X$, $a \in A$, and $t \in I$.

In other words, let $i : A \hookrightarrow X$ denote the inclusion. F is a homotopy between id_A and a map $i \circ r$ for some map $r : X \to A$. If A is a deformation retract of X, then clearly X and A are homotopy equivalent.

Definition 2.10. A space is *contractible* if there exists a homotopy from the identity map on that space to a constant map.

Now we return to the context of finite spaces.

Proposition 2.11. *For finite spaces X and Y, the homotopies $h : X \times I \to Y$ are in bijective correspondence with the paths $j : I \to Y^X$.*

Proof. This correspondence is given by $h(x, t) = j(t)(x)$. □

Proposition 2.11 implies that homotopy classes of maps between finite spaces are equivalent to path components in the mapping space.

Proposition 2.12. *Let f and g be maps between two finite topological spaces X and Y. Then $f \simeq g$ if and only if there exist maps $f_1, ..., f_n$ with $f_1 = f$, $f_n = g$, and such that $f_1 \le f_2$, $f_2 \ge f_3$,$f_3 \le f_4$, $f_4 \ge f_5$, and so on.*

Proof. First suppose that there is a string of maps $f_1, ..., f_n$ that satisfy the conditions above. Since for all i, either $f_i \le f_{i+1}$ or $f_{i+1} \le f_i$, we have a path α connecting f_i with f_{i+1} in Y^X given by

$$\alpha(t) = \begin{cases} \min(f_i, f_{i+1}) & \text{if } 0 \le t < 1 \\ \max(f_i, f_{i+1}) & \text{if } t = 1 \end{cases}$$

Since this is order-preserving, it is continuous under the compact-open topology. Since paths in Y^X correspond to homotopies, $f_i \simeq f_{i+1}$ for all i. Inductively, $f \simeq g$.

If $f \simeq g$, then by proposition 2.11 we know that there is a path between f and g in Y^X. Let Z be the set of points that are on this path. Thus Z is a connected subspace of Y^X. But then by proposition 2.3, all maps including g are connected to f by exactly a sequence $f_1, ..., f_n$ such that $f_1 = f$, $f_n = g$, and such that $f_1 \le f_2$, $f_2 \ge f_3$,$f_3 \le f_4$, and so on. □

So far, we have mainly been dealing with finite spaces in general. However, algebraic topology usually only concerns itself with identifying spaces up to homotopy, and in the case of finite spaces it turns out that every space is homotopy equivalent to a T_0 finite space. We show this below.

Lemma 2.13. *Every finite space X is homotopy equivalent to a finite T_0 space X_0.*

Proof. If X is a finite space, let X_0 denote the space $X/\sim$ where $x \sim y$ if $x \leq y$ and $y \leq x$. Immediately we see that X_0 is a T_0 space as if $x \leq y$ and $y \leq x$ in X_0, then $x = y$ by the definition of X_0 and quotient space. Thus, X_0 as a preordered set satisfies antisymmetry, which is the requirement for a preordered set to be a poset. As we showed in remark 1.5, X_0 being a poset is exactly the same as the space X_0 being T_0.

All that remains to be shown is that X_0 is indeed homotopy equivalent to X. Since X_0 is a quotient of X, there is a quotient map $q : X \to X_0$. Let p be the function such that $q \circ p = \mathrm{id}_{X_0}$. Thus $p \circ q$ is order preserving, and furthermore, $p \circ q \leq \mathrm{id}_X$ as it only sends elements to themselves or to something less than or equal to themselves. Thus p is a homotopy inverse of q, and $X \simeq X_0$. □

This implies that from now on we only need concern ourselves with T_0 finite spaces up to homotopy.

3. Beat points and cores

Definition 3.1. A *down beat point* x of a finite T_0 space is a point such that there exists a y such that $y \leq x$, and if $z \leq x$ then $z \leq y$. An *up beat point* x of a finite T_0 space is a point such that there exists a y such that $y \geq x$, and if $z \geq x$ then $z \geq y$.

Definition 3.2. A *beat point* is a point that is either an up beat point or a down beat point.

Intuitively, a beat point is a point with a point directly above or below it, that is to say, there is nothing between the two points.

Definition 3.3. A *minimal finite space* is a space without beat points. A *core* of a finite T_0 space X is a minimal finite space that is a deformation retract of X.

Proposition 3.4. *Let X be a finite T_0 space and let $x \in X$ be a beat point of X. Then $X \setminus \{x\}$ is a deformation retract of X.*

Proof. Let x be a down beat point of X. Let y be as in definition 3.1. The retraction $r : X \to X \setminus \{x\}$ is given by $r(x) = y$, and otherwise r is the identity. This is an order-preserving map. Additionally the inclusion $i : X \setminus \{x\} \to X$ implies that $i \circ r \leq \mathrm{id}_X$. Thus $i \circ r \simeq \mathrm{id}_X$, and thus r is deformation retraction. A similar proof works for up beat points. □

It immediately follows from the previous result that every finite T_0 space has a core as one can remove beat points until none are left. Since the removal of beat points constitutes a deformation retract, then one would be left with a minimal deformation retract which is a core.

Proposition 3.5. *If X is a minimal finite T_0 space and $f : X \to X$ is homotopic to id_X, then $f = \mathrm{id}_X$.*

Proof. First suppose $f \geq \mathrm{id}$. Then $f(x) \geq x$ for all $x \in X$. We proceed with induction. For the base case: if x is a maximal point of X, then necessarily $f(x) = x$. For the inductive step: fix a y and suppose $f(x) = x$ for all $x \geq y$. Thus for all $x \geq y$, $x = f(x) \geq f(y) \geq y$. But if $f(y) \neq y$, then y is a beat point by definition. This contradicts the minimality of X, so $y = f(y)$ for all y inductively. A similar argument works to show that if $f \leq \mathrm{id}$, then f is the identity. Since we have already shown that any map homotopic to the identity has a sequence of comparable maps between it and the identity, by induction they must all be the identity map, and thus any map homotopic to the identity must be the identity. □

Corollary 3.6. *A homotopy equivalence between minimal finite spaces is a homeomorphism.*

Proof. Let X and Y be minimal finite spaces, and let f be a homotopy equivalence between them. Let g be the homotopy inverse of f, and thus $f \circ g \simeq \mathrm{id}$. By proposition 3.4, $f \circ g = \mathrm{id}$ and similarly $g \circ f = \mathrm{id}$. Thus X and Y are homeomorphic. □

Since two cores of a space are homotopy equivalent to the original space, and they are both minimal, this implies that cores are unique. Additionally, it follows that two finite spaces are homotopic if and only if they have homeomorphic cores.

4. Weak homotopy equivalences and Simplicial Complexes

Definition 4.1. A continuous map $f : X \to Y$ between topological spaces is called a *weak homotopy equivalence* if $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection and for all $n \geq 1$, $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for every base point x_0.

Homotopy equivalences are all weak homotopy equivalences, and in many cases weak homotopy equivalences are homotopy equivalences. Whitehead has shown that weak homotopy equivalences between CW-complexes are in fact homotopy equivalences. However, this is not true for general spaces. Weak homotopy equivalences are very important in the study of finite spaces, and while we will not go too deeply into this, we will see some of their uses.

Now we will state a key theorem about weak homotopy equivalences without proof. This is a theorem that allows maps to be identified as weak homotopy equivalences if they are locally weak homotopy equivalences, in a sense. This is a theorem from general algebraic topology, its proof can be found at [2, corollary 4K.2]. It will be useful at the end of this section to prove that a specific map is a weak homotopy equivalence.

Definition 4.2. A *basis like open cover* U of a space X is an open cover such that for each $U_1, U_2 \in U$ and $x \in U_1 \cap U_2$, there exists $U_3 \in U$ such that $x \in U_3 \subset U_1 \cap U_2$.

Theorem 4.3. *Let $f : X \to Y$ be a continuous map between spaces X and Y. If there exists a basis like open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$,*

$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$

is a weak homotopy equivalence, then f is a weak homotopy equivalence.

Now we introduce the concept of a simplicial complex that is associated to each finite T_0 space X.

Definition 4.4. A *simplicial complex* K consists of a set V_K of vertices and a set S_K of finite and nonempty subsets of V_K called the simplices. It is also required that all single points of V_K are simplices, and all subsets of simplices are simplices.

Often we just refer to something as being in K if it is in one of these two sets, and context will make it clear which one. The *dimension* of a simplex is one less than the number of vertices it contains.

Definition 4.5. Let $\sigma = \{v_1, ..., v_n\}$ be a simplex in K. Then the *closed simplex* $\overline{\sigma}$ is the set of convex combinations $\sum_{i=0}^{n} \alpha_i v_i$ for $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. This is given the metric topology with metric

$$d(\sum_{i=0}^{n} \alpha_i v_i, \sum_{i=0}^{n} \beta_i v_i) = \sqrt{\sum_{i=0}^{n} (\alpha_i - \beta_i)^2}.$$

Associated to each simplicial complex, there is a useful notion of its geometric realization.

Definition 4.6. The *geometric realization* $|K|$ of a simplex K, consists of the set of all convex combinations $\sum_{v \in V_K} \alpha_v v$ such that $\{v \mid \alpha_v > 0\} \in S_K$. Note that each $\overline{\sigma}$ is a subset of $|K|$. $|K|$ is given the topology such that $U \in |K|$ is open if $U \cap \overline{\sigma}$ is open in $\overline{\sigma}$ for each $\sigma \in S_K$, otherwise known as the final topology.

In the case that K is a finite simplex, the topology on $|K|$ coincides with the metric topology under the metric

$$d(\sum_{v \in V_K} \alpha_v v, \sum_{v \in V_K} \beta_v v) = \sqrt{\sum_{v \in V_K} (\alpha_v - \beta_v)^2}.$$

Now let X be a finite T_0 space. The simplicial complex associated with X is denoted $\mathcal{K}(X)$, and it is the simplicial complex whose simplices are the nonempty chains in the poset X. If $f : X \to Y$ is a map between topological spaces, then the associated simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined to be $\mathcal{K}(f)(x) = f(x)$. Categorically, $\mathcal{K}(X)$ is a functor from finite T_0 spaces to simplicial complexes.

The geometric realization of a simplicial complex associated to a poset $|\mathcal{K}(X)|$ consists of points $k = \sum_{i=1}^{n} t_i x_i$ with $x_1 < x_2 < ... < x_n$ being a chain in X, and with $\sum_{i=1}^{n} t_i = 1$ and $t_i > 0$ for all i.

Definition 4.7. The *support* of a point k in the geometric realization of a simplicial complex of a finite poset is the set $\{x_1, x_2, ..., x_n\}$.

Now we come to a map which is useful to showing relations between a poset and its associated simplicial complex.

Definition 4.8. The $\mathcal{K}$-McCord map of a finite T_0 space X is $\mu_X : |\mathcal{K}(X)| \to X$ such that $\mu_X(k) = \min(\text{support}(k))$.

Lastly we prove a useful theorem showing a deep relation between a poset and its associated simplicial complex.

Theorem 4.9. *The $\mathcal{K}$-McCord map for a finite T_0 space X is a weak homotopy equivalence between X and $|\mathcal{K}(X)|$.*

Proof. The idea of this proof is to first show that μ_X is continuous. Then we apply theorem 4.3 on the minimal basis of X by showing that for all $x \in X$ both U_x and $\mu_X^{-1}(U_x)$ are contractible.

Let X be a finite space and fix an $x \in X$. Since the minimal basis is a basis and since x was chosen arbitrarily, to show that μ_X is continuous, it suffices to show that $\mu_X^{-1}(U_x)$ is open. In order to do this we first assign $L = \mathcal{K}(X \setminus U_x)$. Now we will show that

$$\mu_X^{-1}(U_x) = |\mathcal{K}(X)| \setminus |L|.$$

First, let $\alpha \in \mu_X^{-1}(U_x)$. Let $v = \min(\text{support}(\alpha))$. Thus, $v \in U_x$. By definition of μ_X, $\alpha \in \mathcal{K}(K)$. But since $v \in U_x$, $\alpha \notin |L|$ as the support of anything in $|L|$ cannot contain anything in U_x.

Next suppose $\alpha \in |\mathcal{K}(X)|$ but $\alpha \notin |L|$. Since $\alpha \notin |L|$, there must exist some $w \in \text{support}(\alpha)$ such that $w \in U_x$ as otherwise α would be in $|L|$. Again let $v = \min(\text{support}(\alpha))$, and thus $v \leq w \leq x$. Thus $\mu_X(\alpha) = v \in U_x$. Therefore we have shown $\mu_X^{-1}(U_x) = |\mathcal{K}(X)| \setminus |L|$. But U_x is open, so $X \setminus U_x$ is closed, and $|\mathcal{K}(X \setminus U_x)| = |L|$ is closed. Thus $\mu_X^{-1}(U_x)$ is open, and μ_X is continuous.

We now show that $|\mathcal{K}(X)| \setminus |L|$ strong deformation retracts onto $|K(U_x)|$. Once we've shown this, showing $|\mathcal{K}(U_x)|$ is contractible will imply that $|\mathcal{K}(X)| \setminus |L|$ is also contractible.

$|\mathcal{K}(U_x)|$ is a subset of $|\mathcal{K}(X)| \setminus |L|$ as if $x \in |\mathcal{K}(U_x)|$, then $x \in \mathcal{K}(X)$, and $x \notin |L|$. Let $\alpha \in |\mathcal{K}(X)| \setminus |L|$. The support of α then contains elements from $|\mathcal{K}(U_x)|$, and possibly elements from $|L|$. But since $\min(\text{support}(\alpha))$ is an element of $|\mathcal{K}(U_x)|$, at least one element of the support of α must be from $|\mathcal{K}(U_x)|$. Thus α can be written as a linear combination of an element in $|\mathcal{K}(U_x)|$ and an element in $|L|$. So

$$\alpha = \beta t + \gamma(1 - t)$$

for $\beta \in |\mathcal{K}(U_x)|$, $\gamma \in |L|$, and $0 < t \leq 1$. Now let $i : |\mathcal{K}(U_x)| \hookrightarrow |\mathcal{K}(X)| \setminus |L|$ be the inclusion map. Let $r : |\mathcal{K}(X)| \setminus |L| \to |\mathcal{K}(U_x)|$ be defined by $r(\alpha) = \beta$. For each $\sigma \in \mathcal{K}(X)$, $r|_{(|\mathcal{K}(U_x)|) \cap \overline{\sigma}} : |\mathcal{K}(U_x)|) \cap \overline{\sigma} \to \overline{\sigma}$ is continuous. Since $(|\mathcal{K}(X)| \setminus |L|)$ has the final topology with respect to each $\overline{\sigma}$, it follows that since r is continuous on each restriction, r is continuous overall.

Now we can construct a strong deformation retract H between $|\mathcal{K}(X)| \setminus |L|$ and $|\mathcal{K}(U_x)|$. Let $H : |\mathcal{K}(X)| \setminus |L| \times I \to |\mathcal{K}(X)| \setminus |L|$ be given by

$$H(\alpha, s) = (1 - s)\alpha + s(r(\alpha)).$$

Since the simplexes involved are finite, they have the metric topology as stated below definition 4.6. Since r is continuous, and H is a linear function, between $(1 - s)\alpha$ and $s(r(\alpha))$ it follows that H is continuous with respect to the referenced metric.

We check that H satisfies the conditions to be a strong deformation retract. $H(\alpha, 0) = \alpha$. $H(\alpha, 1) = r(\alpha) \in |\mathcal{K}(U_x)|$. And lastly, for all $\delta \in |\mathcal{K}(U_x)|$, $r(\delta) = \delta$. So $H(\delta, t) = (1 - t)\delta + t\delta = \delta$. Therefore $|\mathcal{K}(U_x)|$ is a strong deformation retract of $|\mathcal{K}(X)| \setminus |L|$.

We now turn toward the goal of showing U_x and $\mu_X^{-1}(U_x)$ are contractible. U_x has a maximum value, namely x. Let $c : U_x \to U_x$ be the constant map sending everything to x. Thus $\text{id}_{U_x} \leq c$, and thus U_x is contractible by proposition 2.12.

$|\mathcal{K}(U_x)|$ is contractable by the homotopy $J : |\mathcal{K}(U_x)| \times I \to |\mathcal{K}(U_x)|$ given by

$$J(y, t) = (1 - t)y + tx.$$

J is well-defined and continuous because for every simplex σ in $|\mathcal{K}(U_x)|$, $\sigma \cup \{x\}$ is also a simplex in $|\mathcal{K}(U_x)|$. Because $|\mathcal{K}(U_x)|$ is contractible, so must be $|\mathcal{K}(X)| \setminus |L|$.

Since we have shown that on the minimal basis of X, μ_X is continuous and takes contractible basis elements to contractible basis elements. This shows that μ_X is locally a weak homotopy equivalence. Thus by theorem 4.3, μ_X is a weak homotopy equivalence. □

5. The Euler Characteristic

The Euler characteristic is an important invariant throughout mathematics. If the homology of a topological space X is finitely generated and all the groups H_n are zero for some arbitrarily large n then the Euler characteristic is well defined. the Euler characteristic is given by

$$\chi(X) = \sum_{n\geq 0}(-1)^n \mathrm{rank}(H_n(X)).$$

For compact CW complexes, the euler characteristic is given by

$$\chi(X) = \sum_{n\geq 0}(-1)^n \alpha_n,$$

where α_n is the number of n-cells in X.

Now choose a finite T_0 space X. It is a result from general topology that weak homotopy equivalences induce isomorphisms in the homology groups of spaces. This can be found at [2, proposition 4.21]. By theorem 4.9, X is weak homotopy equivalent to the geometric realization of $\mathcal{K}(X)$. Because of this, we see that the Euler Characteristic of X is:

$$\chi(X) = \sum_{C\in\mathcal{C}(X)} (-1)^{|C|+1},$$

where $\mathcal{C}(X)$ is the set of chains in $\mathcal{K}(X)$, and $|C|$ is the number of points in a chain C.

This next theorem is a combinatorial approach to proving that the Euler characteristic of finite spaces is a homotopy invariant.

Theorem 5.1. *If X and Y are finite T_0 spaces and $X \simeq Y$, then $\chi(X) = \chi(Y)$.*

Proof. The main idea of this proof relies on the fact that cores of homotopic spaces are homeomorphic. Let X_c and Y_c be the cores of X and Y respectively. Since X_c is homeomorphic to Y_c, it follows that $\chi(X_c) = \chi(Y_c)$. Since cores of spaces are formed by removing beat points, it suffices to show that the Euler characteristic is invariant under removal of beat points. (If we can show that, then $\chi(X) = \chi(X_c) = \chi(Y_c) = \chi(Y)$ and we are done.)

Consider an arbitrary finite T_0 space P with beat point p. Since p is a beat point, there is a q such that if $x \in P$ is comparable with p, then x is comparable with p. Let $\mathcal{C}(P)$ be the set of chains of P. For every chain containing p and q, there is a chain identical except without q in it. Conversely, for every chain containing p and not containing q, there is a chain identical except with q inserted directly after p. This implies the set of chains containing q in bijection with the set of chains not containing q.

Now, consider

$$\chi(P) = \sum_{C \in \mathcal{C}(P)} (-1)^{|C|+1} = \sum_{\substack{C \in C(X) \\ C \ni p}} (-1)^{|C|+1} + \sum_{\substack{C \in C(X) \\ C \not\ni p}} (-1)^{|C|+1}$$

$$= \sum_{\substack{C \in C(X) \\ C \ni p}} (-1)^{|C|+1} + \chi(P \setminus \{p\})$$

$$\sum_{\substack{C \in C(X) \\ C \ni p}} (-1)^{|C|+1} = \sum_{\substack{C \in C(X) \\ C \ni p \\ C \ni q}} (-1)^{|C|+1} + \sum_{\substack{C \in C(X) \\ C \ni p \\ C \not\ni q}} (-1)^{|C|+1}.$$

As $\{C \in \mathcal{C}(X) \mid p \in C, q \notin C\}$ is in bijection with $\{C \in \mathcal{C}(X) \mid p, q \in C\}$,

$$\sum_{\substack{C \in C(X) \\ C \ni p \\ C \not\ni q}} (-1)^{|C|+1} = \sum_{\substack{C \in C(X) \\ C \ni p \\ C \ni q}} (-1)^{(|C|+1)+1}.$$

(The only difference is adding in the q directly after p, so the cardinality increases by 1.) Thus

$$\sum_{\substack{C \in C(X) \\ C \ni p \\ C \ni q}} (-1)^{|C|+1} + \sum_{\substack{C \in C(X) \\ C \ni p \\ C \not\ni q}} (-1)^{|C|+1} = \sum_{\substack{C \in C(X) \\ C \ni p \\ C \ni q}} (-1)^{|C|+1} + \sum_{\substack{C \in C(X) \\ C \ni p \\ C \ni q}} (-1)^{|C|} = 0.$$

Thus $\chi(P) = \chi(P \setminus \{p\})$, and therefore $\chi(X) = \chi(Y)$. □

6. The Möbius Function

The Möbius function is an important function that branches across a couple fields including combinatorics and number theory. We will end this paper by defining it, and showing it's connection to the Euler characteristic, a topological property.

Definition 6.1. Let P be a finite poset. The *incidence algebra* $\mathcal{U}(P)$ on P is the set of complex valued functions on $P \times P$ such that $f(x, y) = 0$ unless $x \leq y$.

Addition on these functions is defined pointwise, multiplication is defined to be

$$(fg)(x, y) = \sum_{\substack{z \in P \\ x \leq z \leq y}} f(x, z) g(z, y).$$

The identity element of the incidence algebra is

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Another important function is the zeta function, which is

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function μ is defined to be the inverse of the zeta function. We can find a basic recurrence for the Möbius function from the relation $\mu\zeta = \delta$. For $x \neq y$, we have

$$0 = \delta(x,y) = (\mu\zeta)(x,y) = \sum_{x \leq z \leq y} \mu(x,z)\zeta(z,y) = \sum_{x \leq z \leq y} \mu(x,z).$$

So

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z).$$

Since this recurrence is only for $x \neq y$, we must add the condition that $\mu(x,x) = 1$.

Now we compute the Möbius function more explicitly and show how it is directly related to the Euler characteristic. By Taylor series expansion,

$$\zeta^{-1} = (\delta + \zeta - \delta)^{-1} = \sum_{k \geq 0} (-1)^k (\zeta - \delta)^k.$$

Define the *length* of a chain to be one less than the number of points in it.

Lemma 6.2. *In a poset P with $x, y \in P$, $(\zeta - \delta)^k(x,y)$ represents the number of chains of length k between x and y.*

Proof. We see that $(\zeta - \delta)(x,y) = 1$ if $x < y$, and 0 otherwise. This is exactly the amount of length 1 chains that there are between x and y. But then $(\zeta - \delta)^k(x,y)$ counts the number of chains of length k, as there is one counted for every unbroken k length sequence of 1-chains between x and y. □

Thus we get Hall's theorem.

Theorem 6.3.

$$\mu(x,y) = C_0 - C_1 + C_2 - C_3 + ...,$$

where C_i is the number of chains of length i going from x to y.

Now we define $\hat{P} = P \cup \{0,1\}$. By this we mean the poset $\hat{P}$ is identical to P but with maximum and minimum elements adjoined to it. By Hall's theorem, along with the characterization of the Euler characteristic of finite posets from before, we get the following theorem.

Theorem 6.4. *With respect to the incidence algebra on $\hat{P}$ of a poset P,*

$$\mu(0,1) = \chi(P) - 1.$$

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References

[1] Barmak. *Algebraic Topology of Finite Topological Spaces and Applications*. Lecture Notes in Mathematics, 2011.
[2] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.
[3] R. Stanley I. Gessel. *Algebraic Enumeration*, volume Handbook of Combinatorics, vol. 2. 1995.
[4] Katie Lyons. Posets and the möbius function. Online.
[5] May. *Finite Spaces and Larger Contexts*. 2015.