# EXPLORING THE TOPOLOGY OF SPACES OF POLYNOMIALS VIA VECTOR BUNDLE THEORY

RANDALL R. VAN WHY

Abstract. This paper is partially expository but contains some original work. The results of this paper come out of joint work with Nir Gadish and Peter Haine. We give an exposition of a beautiful correspondence between configuration spaces, polynomials, and braid groups. This correspondence is used to show the existence of a real vector bundle $\xi_n$ over the space of monic, square-free, degree $n$, polynomials $Poly_n(\mathbb{C})$. We then outline a trivialization atlas for $\xi_n$ and use it to discuss $Poly_n(\mathbb{C})$ in the case $n = 2$.

Contents



## 1. Background

It would not take much to convince the reader that polynomials have been important objects of study for mathematicians for many years. It comes as a surprise that there is still much that we do not know about polynomials. This paper is the result of a project that aimed at using vector bundle theory to explore the topology of certain spaces of polynomials. This, on the surface, seems like a very strange idea. What do vector bundles have to do with polynomials? In order to see the connection, we need to explore a bit of background. The following is a retelling of a beautiful story that links certain classes of polynomials to other important mathematical structures.

This paper is concerned with the space of monic, square-free, degree $n$, polynomials with complex coefficients which we will call $Poly_n(\mathbb{C})$. The square-free condition in conjunction with the Fundamental Theorem of Algebra tells us that that polynomials in $Poly_n(\mathbb{C})$ factor into $n$ *distinct* complex roots. A polynomial

*Date*: November 11, 2015.

being square free is not enough to guarentee that these $n$ distinct roots are unique to the polynomial. Non-monic polynomials can have the same roots as other polynomials (e.g. $x^2 - 1$ and $2x^2 - 2$). However, two monic polynomials have the same roots if and only if they are equal.

These spaces of polynomials are intimately connected to certain types of configuration spaces.

**Definition 1.1.** Let $M$ be a connected manifold. The **ordered configuration space** of $n$ elements is the subspace of $M^n$ given by

$$Conf_n(\mathbb{C}) := \{(z_1, ..., z_n) : z_i \in M, z_i \neq z_j \ for \ i \neq j\}$$

Configuration spaces have found applications in many contexts inside and outside of mathematics including robotics and particle physics. The ordered configuration space is used when we want to distinguish two points in a configuration from each other. If one does not wish to distinguish the points in the configuration, one can simply factor out the action of the symmetric group.

**Construction 1.2.** The symmetric group on $n$ elements, $\Sigma_n$, acts on $Conf_n(M)$ as follows

$$\sigma(z_1, ..., z_n) = (z_{\sigma^{-1}(1)}, ..., z_{\sigma^{-1}(n)}), \forall \sigma \in \Sigma_n$$

We define the **unordered configuration space**, $UConf_n(M)$, as the orbit space $Conf_n(M)/\Sigma_n$ of this action. We write $UConf_n(M) = \{Z \subset \mathbb{C} : |Z| = n\}$.

Now we will stop to think about $Poly_n(\mathbb{C})$. What is an element $p(x) \in Poly_n(\mathbb{C})$ but an unordered list of $n$ complex roots? This data is all we need to characterize the polynomial. Thus, immediately, an interesting bridge between worlds appears.

**Theorem 1.3.** *$UConf_n\mathbb{C} \cong Poly_n(\mathbb{C})$ in the category of topological spaces*

*Proof.* Given a set of distinct unordered points $\{z_1, ..., z_n\} \in UConf_n(\mathbb{C})$ we construct the polynomial $p(x) = \Pi_{i=1}^n (x - z_i)$. This polynomial is obviously monic and square-free by construction. Moreover, $\deg p(x) = n$ and thus $p(x) \in Poly_n(\mathbb{C})$. Now, take $f(x) \in Poly_n(\mathbb{C})$. Since $\mathbb{C}$ is algebraically closed, we may assume $f(x) = \Pi_{i=1}^n (x - z_i)$. We let $z = \{z_i\}$ the set of roots of $f(x)$. Since $f(x)$ is square free, each $z_i$ is distinct and thus $z \in UConf_n(\mathbb{C})$. Using the above processes, we can construct a continuous bijective function: $g : Poly_n(\mathbb{C}) \to UConf_n(\mathbb{C})$ with continuous inverse $g^{-1} : UConf_n(\mathbb{C}) \to Poly_n(\mathbb{C})$ making $g$ a homeomorphism. □

Now we know that we can discuss unordered point configurations in the complex plane and certain types of polynomials as one and the same. Visualizing a polynomial as an unordered configuration of its roots in the complex plane gives us a convenient way to discuss the shape of $Poly_n(\mathbb{C})$.

1.1. **The Artin Braid Group $B_n$.** In a world seemingly separate from Polynomials and configuration spaces we have the Artin Braid Group on $n$ strands. This group, denoted $B_n$ can be intuitively defined with a few picutres. Imagine taking two strands and crossing one over the other. We then lock the endpoints of the strands in place. The result might look something like this:

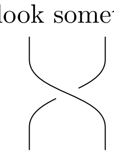
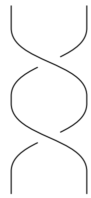


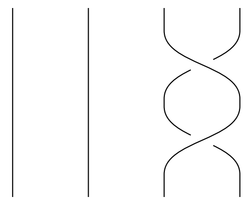
Figure 1: A “braid” or an element $\sigma \in B_2$.

Indeed, $B_n$ has the structure of a group. We compose, two braids on $n$ strands by gluing the ends of the strands of one braid to the top of the strands of the other.



The result of gluing $\sigma$ above to itself

Finally, we want to say that two braids are equivalent if we cam deform one into the other without permuting the endpoints.



Two equivalent braids.
We can "comb" the right braid until it looks like the left.

The above description of $B_n$ is intuitive but not rigorous. We would like a formalism that would allow us to unambiguously define $B_n$ in a way that still fits our intuition. There are many ways to do this, but the most illuminating formalism for our purposes is its realization as particle dances which we borrow from [7].

**Definition 1.4.** An $n$-particle dance is a trajectory of complex numbers beginning and ending at integer points $\{1, ..., n\}$ i.e. a function $\beta : [0, 1] \to \mathbb{C}^n$:

$$\beta(t) = (\beta_1(t), ..., \beta_n(t)), \beta_i(t) \neq \beta_j(t) \ for\ i \neq j$$

with $\beta(0) = (1, ..., n)$ and $\beta(1) = (\sigma^{-1}(1), ..., \sigma^{-1}(n))$ for some unique $\sigma \in \Sigma_n$. $\sigma$ is called the permutation of the particle dance.

With a bit of thought, one can convince themselves that particle dances trace out braid-like objects in the space $[0, 1] \times \mathbb{C}$. Using this formalism we can define our group operation: composition of particle dances.

*Remark* 1.5. Given two particle dances $\alpha(t)$ and $\beta(t)$, we can construct another particle dance $\gamma(t)$ given by:

$$\gamma(t) = \begin{cases} \alpha(2t) & for\ 0 \leq t < 1/2 \\ \beta(2t-1) & for\ 1/2 \leq t \leq 1 \end{cases}$$

We write $\gamma = \alpha * \beta$

Now to formalize equivalent braids. Obviously similar trajectories will result in similar particle dances. We need a bit of language from homotopy theory to discuss this similarity.

**Definition 1.6.** A homotopy between two continuous maps of spaces $f, g : X \to Y$ is a continuous function $h : I \times X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$. If such a function exists, we say $f$ is homotopic to $g$ and write $f \simeq g$.

*Remark* 1.7. $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$. The equivalence classes of this relation are called homotopy classes.

Now we have a simple definition.

**Definition 1.8.** A braid $b$ is a representative of a homotopy class of particle dances. Thus two braids are equivalent if their corresponding particle dances belong to the same homotopy class.

We can now make a simple observation.

**Proposition 1.9.** *Two equivalent particle dances, and thus braids, have the same permutation.*

**Construction 1.10.** The braid group on $n$ strands, denoted $B_n$, is the group of braids under group operation $*$ (particle dance composition).

*Remark* 1.11. $B_n$ satisfies all the properties of a group.

1.2. **Connecting $B_n$ to Polynomials.** As we stated before, this paper is about polynomials. We will now justify the exposition of braids by reconnecting them to $Poly_n(\mathbb{C})$. This is done via the theory of classifying spaces which we will now outline.

**Definition 1.12.** A topological group $G$ is a group that is also a topological space. In addition, we require that the group operation $* : G \times G \to G$ and inversion $()^{-1} : G \to G$ are continuous functions.

*Remark* 1.13. Every group is a topological group when given the discrete topology.

We can associate to each topological group $G$ a topological space called the "classifying space" of $G$ which we will denote by $BG$. A classical construction of the classifying space of $G$ is Milnor's construction first outlined in [9]. For our purposes, the most important part of the theory of classifying spaces is that they are functorial. Given a group homomorphism $\phi$ between two topological groups $G$ and $H$, we may associate a continuous function $B\phi : BG \to BH$ between their classifying spaces. We will use functoriality in Section 2.

If $G$ is given the discrete topology, a classifying space $BG$ is an example of an important object in algebraic topology, a $K(G, 1)$ space. These spaces are completely characterized by their homotopy type.

**Definition 1.14.** The $n$th homotopy group of a path-connected space $X$, $\pi_n(X, x_0)$, is the group of homotopy classes of maps $g : [0,1]^n \to X$ such that $g(\partial[0,1]^n) = x_0 \in X$ with group operation $*$ defined by:

$$(f * g)(t) = \begin{cases} f(2t_1, t_2, ..., t_n) & for\ 0 \leq t_1 < 1/2 \\ g(2t_1 - 1, t_2, ..., t_n) & for\ 1/2 \leq t_1 \leq 1 \end{cases}$$

A space is said to be weakly contractable if all of its homotopy groups are trivial.

*Remark* 1.15. $\pi_1(X, x_0)$ is the group of homotopy classes of loops beginning and ending at $x_0$.

**Definition 1.16.** A $K(G, 1)$-space X is a path-connected space with $\pi_1(X) = G$ and $\pi_i(X) = 0, \forall i > 1$.

Indeed, the importance of these spaces is apparent with the following construction.

**Construction 1.17.** (Mod 2 Group Cohomology) We define the group cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ as

$$H^*(G;\mathbb{Z}/2\mathbb{Z}) := H^*(K(G,1);\mathbb{Z}/2\mathbb{Z})$$

Because group cohomology of $G$ is defined as the cohomology of $K(G,1)$ spaces, the study of these spaces is motivated by the study of their corresponding groups. With the following theorem, we will connect the world of braids to the world of polynomials.

**Theorem 1.18.** *$UConf_n(\mathbb{C})$ is a classifying space for $B_n$ and thus a $K(B_n,1)$ space.*

There are many proofs of this statement which go beyond the scope of this paper. We would like to point the reader to [1] for one example of a proof. The reader however can see this intuitively. It's easy to see that $UConf_n(\mathbb{C})$ is path connected. Thus our choice of $x_0$ is arbitrary. If we choose $x_0 = \{1,...,n\} \in UConf_n(\mathbb{C})$ we can see that a loop begining and ending at $x_0$ is, by definition, a particle dance. Thus it is easy to see that $\pi_1(UConf_n(\mathbb{C})) \cong \pi_1(Poly_n(\mathbb{C})) = B_n$.

V.I. Arnol'd demonstrated and used this theorem in [2] in order to study the Cohomology of $B_n$ with integer coefficients. This theorem was also used to study polynomials in [3] via Theorem 1.3. This paper builds on some of the work of D.B. Fuks in order to study polynomials from the perspective of vector bundle theory.

## 2. Preliminaries and Motivation

We require a bit of the language from the theory of vector bundles in order to discuss how we can use the information described in the background to study polynomials. We begin by defining a vector bundle.

**Definition 2.1.** A **real vector bundle** is a 4-tuple $(F,E,B,\pi)$ where $\pi : E \to B$ is a continuous map of spaces, $F_b := \pi^{-1}(b) \cong F$, and $F$ has the structure of a real $n$-dimensional vector space. In addition, for every $x$, there exists an open neighborhood $U$ of $x$ and a homeomorphism $\phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$, called a local trivialization, such that $\phi|_b : \{b\} \times \mathbb{R}^n \to \pi^{-1}(b)$ is an isomorphism of vector spaces. $F$ is called the **fibre**, $E$ the **total space**, $B$ the **base space**, and $\pi$ the **bundle map**.

For convenience, we will refer to vector bundles by their bundle maps. We will primarily be concerned with the construction of new vector bundles from old ones. The most important way of doing so, for our purposes, is via pullback bundles.

**Construction 2.2.** Suppose we have a vector bundle $\xi : E \to B$ and a map $f : B' \to B$. We define $E' = \{(b,e) \in B' \times E : f(b) = \xi(e)\}$. Then there exists a new bundle $f^*\xi : E' \to B'$ (called the **pullback bundle** of $f$) and map $\pi_2 : E' \to E$ defined by $f^*\xi(b,e) = b$ and $\pi_2(b,e) = e$, respectively. We have the following commutative diagram.

$$\begin{array}{ccc} E' & \xrightarrow{\pi_2} & E \\ \downarrow{\scriptstyle f^*\xi} & & \downarrow{\scriptstyle \xi} \\ B' & \xrightarrow{f} & B \end{array}$$

Thus the existence of a vector bundle over a space $B$ and a map of spaces $B' \to B$ is sufficient to construct a vector bundle over $B'$. We want to construct a vector bundle over $Poly_n(\mathbb{C})$ and see what it can tell us about its topology. We first need an existing vector bundle before we can even think about pullbacks. For reasons that will become clear later, the canonical bundle over the Grassmanians is precisely the bundle we need.

**Definition 2.3.** The **Grassmanian** $Gr(r, V)$ is the space of $r$-dimensional subspaces of $V$. For Euclidean spaces, we let $Gr(r, n) := Gr(r, \mathbb{R}^n)$. We may think of $Gr(r, n)$ as $r$-planes through the origin in Euclidean space $\mathbb{R}^n$ with $n = r + k$ for some $k$.

The reader should think of the Grassmanians as a generalization of projective space.

**Example 2.4.** If $n < \infty$ then $Gr(1, n + 1) \cong \mathbb{RP}^n$

**Construction 2.5.** Let $\mathbb{R}^\infty$ denote the vector space of real sequences that are eventually 0. We define $Gr(r, \infty) := Gr(r, \mathbb{R}^\infty)$ which we can think of as $r$-planes through the origin in $\mathbb{R}^\infty$.

This gives us all that we need to define a simple $n$-dimensional vector bundle.

**Construction 2.6.** Let $E = \{(V, x) \in Gr(n, \infty) \times \mathbb{R}^\infty : x \in V\}$. We define $\gamma_n : E \to Gr(n, \infty)$ by $\gamma_n(V, x) = V$. Then $\gamma_n$ is a real vector bundle over $Gr(n, \infty)$. We call this the **canonical $\mathbb{R}^n$ bundle**.

Now that we have our vector bundle over $Gr(n, \infty)$, we just need a map $Poly_n(\mathbb{C}) \to Gr(n, \infty)$ in order to produce a pullback bundle. The following theorem will help us find it.

**Theorem 2.7.** *Let $O(n)$ be the orthogonal group (i.e. $GL(n)$ matrices $Q$ such that $QQ^T = Q^TQ = I$). Then $Gr(n, \infty)$ is a classifying space $BO(n)$.*

Now for the crescendo. We have a sequence of group homomorphisms.

$$B_n \to \Sigma_n \to O(n)$$

The map $B_n \to \Sigma_n$ is the map that sends a braid to its permutation (in other words, it "forgets" braiding).

The map $\Sigma_n \to O(n)$ is the regular representation of $\Sigma_n$ in $O(n)$ from representation theory. For $\sigma \in \Sigma_n$, this is just the matrix that results from permuting the columns of the identity matrix according to $\sigma$.

The functoriality of classifying spaces $BG$ ensures that this chain of homomorphisms induces a chain of continuous maps.

$$BB_n \to B\Sigma_n \to BO(n)$$

As has been already established, $BB_n = Poly_n(\mathbb{C})$ and $BO(n) = Gr(n, \infty)$. Thus we have a map

$$Poly_n(\mathbb{C}) \to Gr(n, \infty)$$

This, in conjunction with the existence of $\gamma_n$, induces a vector bundle on $Poly_n(\mathbb{C})$. We call this vector bundle $\xi_n$.

## 3. CW Complex Structure of $Poly_n(\mathbb{C})^*$

We want to produce a trivialization atlas for $\xi_n$. To do this, we will turn to a paper by D.B. Fuks. In [4] Fuks gives $Poly_n(\mathbb{C})^*$, the one point compactification of $Poly_n(\mathbb{C})$, the structure of a CW-Complex for purposes of computing the cohomology of the $B_n$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients. This is done by passing to Poincare Isomorphism $H^*(B_n;\mathbb{Z}/2\mathbb{Z}) := H^*(Poly_n(\mathbb{C});\mathbb{Z}/2\mathbb{Z}) \cong H_*(Poly_n(\mathbb{C})^*;\mathbb{Z}/2\mathbb{Z})$. We will use the ideas put forth in the cellular decomposition for purposes of trivializing our vector bundle. We elaborate Fuks' idea for the cells as follows:

**Definition 3.1.** A $k$-tuple of natural numbers $\lambda = (m_1, .., m_k)$ is called an **composition of** $n$ if $\sum_{i=1}^{k} m_i = n$. We write $\lambda \vdash n$ and say "lambda composes n".

**Construction 3.2.** Let $\lambda = (m_1, ..., m_k) \vdash n$. An $(n+k)$-cell of $Poly_n(\mathbb{C})^*$, $e(m_1, .., m_k)$, is the subset of $Poly_n(\mathbb{C})$ which consists of points $\{z_1, ..., z_n\} \in Poly_n(\mathbb{C})$ such that the points $z_1, ..., z_n$ lie on $k$ pairwise distinct vertical lines in the plane. Moreover we require that $m_1$ of the points lie on the leftmost line, $m_2$ to lie on the second leftmost line, and so on. We may also write $e(\lambda)$.

The following illustration will help the reader visualize the requirement in Construction 3.2:

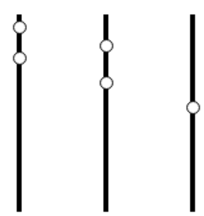


Figure 1. A point in $e(2,2,1) \subset Poly_5(\mathbb{C})$.
Lines added for reference.

The proof that the sets $e(\lambda)$ are cells in the sense of CW-Complexes as well as the boundary and attaching maps of the decomposition is outlined in [4]. The definition of the cells is all that we will need for trivializing $\xi_n$

## 4. Local Trivializations of $\xi_n$

We will use Fuks' intuitive cellular decomposition to produce a trivialization atlas for $\xi_n$. As far as we are aware, a specific trivialization atlas for this bundle does not exist in the literature. This atlas will allow us to analyze the vector bundle in more intimate detail as well as do some computations. We define our atlas as follows:

**Definition 4.1.** Let $\varepsilon > 0$ be a real number. An open strip of radius $\varepsilon$ centered at $z$ is a subset of $\mathbb{C}$ defined by

$$V(z, \varepsilon) = \{w \in \mathbb{C} : |Re(w) - Re(z)| < \varepsilon\}$$

**Construction 4.2.** Let $m_1, ..., m_k$ be a composition of $n$. We define $U(m_1, ..., m_k)$ as the subset of $UConf_n(\mathbb{C})$ such that there exists $k$ non-intersecting open strips $\{V_i\}$ such that $m_i$ of the points lie in strip $V_i$ ordered from left to right. "Left to right" should be understood to mean ascending order of the real part of the center of the strips. In addition, we require that on each strip, no two points share the same imaginary part.

The original inspiration for the definition of these open sets came from the idea of "thickening" the lower dimensional cells in Fuks' decomposition. This idea is echoed by the following remarks

*Remark* 4.3. $e(m_1, ..., m_k) \subseteq U(m_1, ..., m_k)$. In particular, $e(1, ..., 1) = U(1, ..., 1)$.

*Remark* 4.4. $U(m_1, ..., m_k)$ strong deformation retracts onto $e(m_1, ..., m_k)$.

*Proof.* Given $Z \in U(m_1, ..., m_k)$, let $Z_k \subset Z$ be the $m_i$ points in the $i$th strip ordered left to right. Let $r_i = \sum_{z \in Z_i} Re(z)/m_i$. Define for each $Z_i$, $f_t : Z_i \to \mathbb{C}$ by $f_t(z) = (Re(z)(1-t)+r_i t)+Im(z)i$. Note that $f_0(z) = z$ and $f_1(z) = r_i+Im(z)$. We also note that for every $t$, $\{f_t(z_i)\} \in U(m_1, ..., m_k)$. It is clear that $f_t$ is continuous. Now we define $F : U(m_1, ..., m_k) \times I \to U(m_1, ..., m_k)$ by $F(\{z_i\}, t) = \{f_t(z_i)\}$. Evidently, this map is a strong deformation retract. □

Before we begin discussing the trivializations and transition maps in the general case, the reader may find it useful to have a motivating example to fall back on.

**Example 4.5.** When $n = 2$, we have two open sets $U(1, 1)$ and $U(2)$. Our trivializing map $\phi_{1,1} : U(1, 1) \times \mathbb{R}^2 \to \xi_2^{-1}(U(1, 1))$ is defined by

$$\phi_{1,1}(\{z_1, z_2\}, (r_1, r_2)) = \{(z_1, r_1), (z_2, r_2)\}$$

where $Re(z_1) < Re(z_2)$. We define $\phi_2 : U(2) \times \mathbb{R}^2 \to \xi_2^{-1}(U(2))$ analogously with $Im(z_1) < Im(z_2)$.

Let $p = \{z_1, z_2\} \in U(1, 1) \cap U(2)$ such that $Re(z_1) < Re(z_2)$. Then there is a unique permutation $\sigma$ such that $Im(z_{\sigma^{-1}(1)}) < Im(z_{\sigma^{-1}(2)})$. Thus, the transition map $\phi_2 \circ \phi_{1,1}^{-1} : U(1, 1) \cap U(2) \to GL(2)$ at the point $p$ is given by the regular representation of $\sigma$. Similarly, $\phi_{1,1} \circ \phi_2^{-1}(p)$ is the regular representation of $\sigma^{-1}$.

*Remark* 4.6. The transition functions in Example 4.5 are continuous. To see why, note that any point $x \in U(1, 1) \cap U(2)$ belongs to one of two path-components. The transition functions are constant on each of these path-components.

The idea of the case $n = 2$ can be generalized to higher values of $n$. To do this, we must generalize our local orderings to work on more general open sets.

**Definition 4.7.** Let $(m_1, .., m_k) \vdash n$ and let $\{V_i\}$ be the strips as in Construction 4.2. Let $Z \in U(m_1, ..., m_k)$. The **stripwise order** ($\prec$) of the points $z_i \in Z$ is the order defined as follows:

(1) If $z_i, z_j \in V_k$ and $Im(z_i) < Im(z_j)$ then $z_i \prec z_j$
(2) If $z_i \in V_k$ and $z_j \in V_l$ for some $l > k$, then $z_i \prec z_j$

All points within a strip are ordered by their imaginary parts. Additionally, points in one strip are viewed as being less than points in the strips to its right.

*Remark* 4.8. The stripwise order always emits a *unique* chain $z_1 \prec ... \prec z_n$.

*Remark* 4.9. If $\lambda := (1, ..., 1) \vdash n$, the stripwise order of $Z \in U(\lambda)$ is identical to ordering by real parts. The stripwise order of $Z \in U(n)$ is identical to ordering by imaginary parts.

Now we can define our trivialization and transition maps in general. These will work analogously to $n = 2$.

**Construction 4.10.** Let $\lambda \vdash n$ and let $\prec_\lambda$ be the stripwise order of the points in $U(\lambda)$. We define our local trivialization $\phi_\lambda : U(\lambda) \times \mathbb{R}^n \to \pi^{-1}(U)$ by

$$\phi_\lambda(Z, (r_1, ..., r_n)) = \{(z_1, r_1), ..., (z_n, r_n)\},\ z_i \in Z$$

where $z_1 \prec_\lambda ... \prec_\lambda z_n$.

Let $\lambda_1 := (m_1, ..., m_k) \vdash n$ and $\lambda_2 := (l_1, ..., l_j) \vdash n$. Pick $z_i = p \in U(\lambda_1) \cap U(\lambda_2)$ with stripwise orderings $\prec_{\lambda_1}$ and $\prec_{\lambda_2}$. Then $\phi_{\lambda_2} \circ \phi_{\lambda_1}^{-1}(p)$ is the regular matrix representation given by the unique permutation $\sigma$ such that $z_1 \prec_{\lambda_1} ... \prec_{\lambda_1} z_n$ is taken to $z_{\sigma^{-1}(1)} \prec_{\lambda_2} ... \prec_{\lambda_2} z_{\sigma^{-1}(n)}$.

These open sets and trivializing maps give us a trivialization atlas for $\xi_n$. It becomes clear that this atlas is combinatorial in nature. This is due to the fact that trivializing the bundle over $Poly_n(\mathbb{C})$ forces us to pick and maintain an ordering of the roots on some open set. We emphasize that this *cannot* be done in a canonical way; Otherwise, there would be a canonical ordering of the roots of polynomial equations which is absurd! The benefit of using this trivialization is it has an easy to visualize description that, in principle, can be analyzed using a computer (see Appendix A).

We apply our atlas to the case $n = 2$ in the next section.

## 5. Examples and Computations

**Example 5.1.** $Poly_2(\mathbb{C})$. Using the above methods, we have a trivialization atlas $\{U(1,1), U(2)\}$. The intersection $U(1,1) \cap U(2)$ has two different path components which we label $P_1$ and $P_2$ corresponding to the path components containing $\{0, 1 \pm i\}$ respectively. We will call $P_1$ the "regular component" and $P_2$ the "twisted component". We notice that any local sections defined on $U(1,1)$ can be partially extended to $U(2)$ through the regular component. The obstruction preventing the extension of a local section to the rest of our space is the twisted component $P_2$.

**Example 5.2.** $Poly_2(\mathbb{C})$. The following argument uses techniques introduced in [8]. Let $\lambda_{Re} : \mathbb{C}^2 \to \mathbb{C}$ be defined by $\lambda_{Re}(z_1, z_2) = z_1 - z_2$ if $Re(z_1) > Re(z_2)$ and 0 otherwise. Define $\lambda_{Im} : \mathbb{C}^2 \to \mathbb{R}$ analagously for $Im(z_1) > Im(z_2)$. Consider $\psi_{1,1} : U(1,1) \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\psi_2 : U(2) \times \mathbb{R}^2 \to \mathbb{R}^2$ defined componentwise by

$$\psi_{1,1}^1[\{(z_1, r_1), (z_2, r_2)\}] = r_1\lambda_{Re}(z_1, z_2) + r_2\lambda_{Re}(z_2, z_1)$$

$$\psi_{1,1}^2[\{(z_1, r_1), (z_2, r_2)\}] = r_2\lambda_{Re}(z_1, z_2) + r_1\lambda_{Re}(z_2, z_1)$$

$$\psi_2^1[\{(z_1, r_1), (z_2, r_2)\}] = r_1\lambda_{Im}(z_1, z_2) + r_2\lambda_{Im}(z_2, z_1)$$

$$\psi_2^2[\{(z_1, r_1), (z_2, r_2)\}] = r_2\lambda_{Im}(z_1, z_2) + r_1\lambda_{Im}(z_2, z_1)$$

This then gives us a map $\psi : E(\xi_n) \to \mathbb{R}^4 \hookrightarrow \mathbb{R}^\infty$ defined by

$$\psi(Z) = (\psi_{1,1}^1(Z), \psi_2^1(Z), \psi_{1,1}^2(Z), \psi_2^2(Z))$$

This map is linear and injective on each fiber. We can then construct a vector bundle map $f : \xi_n \to \gamma_n$ defined by:

$$f(e) = (\psi(\pi^{-1}(\pi(e))), \psi(e))$$

By the theory introduced in [8] this map classifies the bundle $\xi_n$.

## Appendix A. An algorithm for producing stripwise orders.

Given a point $Z \in U(\lambda)$ for $\lambda = (m_1, ..., m_k) \vdash n$, we perform the following procedure.

(1) Order the points of $Z$ lexicographically with order of real part taking precedence over imaginary part. Store this ordered list $Z'$
(2) Partition $Z'$ from left to right corresponding to $(m_1, ..., m_k) \vdash n$. Store the resulting $k$ lists $Z_1, ..., Z_k$.
(3) Sort each $Z_i$ points by complex part and store the resulting lists as $Z'_1, ..., Z'_k$.
(4) Return the concatenation of the lists $Z'_1 Z'_2 ... Z'_k$. This list is the points of $Z$ sorted according to the stripwise order.

Thus we can compute the stripwise order of any point given a composition $\lambda$. In principle, we can compute the transition map between two open sets by finding the permutation that changes the list computed for one into the one computed for the other. This algorithm would allow us to compute all the transition maps for large $n$ with a computer. The algorithm's complexity grows at most quadratically with $n$.

**Acknowledgments.** I would like to thank Peter May for orchestrating the University of Chicago REU as well as for allowing students from outside of the school to participate. Your graciousness has brought me closer to my dream of mathematics research. I want to express great appreciation for the students and mentors of the REU and Benson Farb's group for making this a profoundly educational experience. Deep thanks to both Nir Gadish and Peter Haine for their collaboration on this project. Working with you two was very fulfilling and I have learned a lot from the both of you. I would like to thank Jesse Wolfson for the conversation, quotes, and reading suggestions. Finally, I would like to thank Benson Farb for being unreasonably influential and for introducing me to this beautiful topic. This experience was an absolute blessing.

## References

[1] Fred Cohen and Jonathan Pakianathan. Configuration Spaces and Braid Groups. http://www.math.rochester.edu/people/faculty/jonpak/newbraid.pdf http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.188.2699&rep=rep1&type=pdf
[2] V.I. Arnol'd. The Cohomology Ring of the Colored Braid Group. Mathematical Notes, 5(2):138140, 1969.
[3] V.I. Arnol'd On Some Topological Invariants of Algebraic Functions. Trans. Moscow Math. Soc., 21:3052, 1970.
[4] D.B. Fuks. The Mod 2 Cohomologies of The Braid Group. Funkts. Anal. Prilozhen., 4(2):46-59, 1970.
[5] J. P. May. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.
[6] A. Hatcher. Algebraic Topology. Cambridge University Press.
[7] D. Rolfsen, et al. Braids: Introductory Lectures on Braids, Configurations and Their Applications. World Scientific Publishing Company.
[8] John Milnor and James Stasheff. Characteristic Classes. Princeton University Press.
[9] John Milnor. Construction of Universal Bundles, II. Annals of Mathematics.