GAUSS-BONNET FOR DISCRETE SURFACES

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Abstract. Gauss-Bonnet is a deep result in differential geometry that illustrates a fundamental relationship between the curvature of a surface and its Euler characteristic. In this paper I introduce and examine properties of discrete surfaces in effort to prove a discrete Gauss-Bonnet analog. I preface this proof with an exploration of how the Euler characteristic and Gaussian curvature, two key players in Gauss-Bonnet, function in the smooth and discrete settings. After developing these ideas, I ultimately demonstrate the existence of Gauss-Bonnet for both discrete surfaces with and without boundary. I conclude by computing examples that compare the results of Discrete Gauss-Bonnet against the original formulation of Gauss-Bonnet for smooth surfaces.

Contents

1. Introduction

Computer graphics and solid modeling make extensive use of discrete surfaces. Defining surface normals and curvatures on discrete surfaces in a manner analogous to smooth surfaces allows programmers to draw on the existing knowledge of differential geometry to streamline various geometry processing tasks. In this paper I present a definition of Gaussian curvature for discrete surfaces that yields a discrete analog of the Gauss-Bonnet theorem. Unless otherwise noted, I am following and expanding on topics and exercises in Crane's *Digital Geometry Processing with Discrete Exterior Calculus* [1]. Though these ideas and theorems have received complete treatment long prior, the proofs and examples that follow are my own.

Definition 1.1. For the purposes of this paper, a **discrete surface** refers to any surface that is piecewise linear, formed by gluing together polygons along their edges. The term **discrete surface** is synonymous with **polygonal mesh**, which appears more frequently in computer science literature.

Definition 1.2. A **simplicial surface** is a simplicial complex of dimension 2.

Example 1.3. Squares, triangles, cubes, and trigonal pyramids are all examples of discrete surfaces. Triangles and trigonal pyramids are also examples of simplicial surfaces.

Definition 1.4. The **Gauss map** $N : X \to S^2$ continuously maps a surface $X \subset \mathbb{R}^3$ to the unit sphere, such that for all $p \in X$, $N(p)$ is the normal vector to X at p [2]

Definition 1.5. The **shape operator** is the directional derivative of the Gauss map [2].

Definition 1.6. The **principal curvatures**, k_1 and k_2, are the eigenvalues of the shape operator at a given point. They measure the maximum and minimum bending of the surface at the given point [2].

Definition 1.7. Gaussian curvature of a surface $X \subset \mathbb{R}^3$ at a point $p \in X$ is the determinant of the shape operator of X at p [2]. Intuitively, this is the local explosion factor for areas under the Gauss map. As the determinant of the shape operator is equivalent to the product of its eigenvalues, Gaussian curvature is also equivalent to the product of the shape operator eigenvalues, k_1 and k_2, the principal curvatures.

Example 1.8. Compute the Gaussian curvature of a point on a sphere of radius R.

Solution. The Gauss map that maps a point p on a sphere of radius R to the unit sphere is

$$N(p) = \frac{1}{R}(p)$$

Lets consider the exterior normal to be positive, and the interior normal to be negative. Then the Shape operator is

$$S(p) = \frac{1}{R^2}(p)$$

and

$$S = \frac{1}{R^2}\mathbb{I}$$

where $\mathbb{I}$ is the identity matrix. Then the determinant and Gaussian curvature at p is $\frac{1}{R^2}$.

This traditional definition of Gaussian curvature doesn't immediately translate to discrete surfaces. Problems arise in the formulation of the Gauss map of a discrete surface, namely the assignment of a normal vector to a vertex. This assignment is non-trivial. While common practice, adding up the neighboring face normals for the vertex normal can be inconsistent, yielding different normals for different tessellations of the same geometry.

Furthermore, there is no consensus on the existence of a best method for weighting the neighboring face normals accordingly, whether by incident angle or face area. Here lies the necessity for understanding Gaussian curvature specific to discrete surfaces.

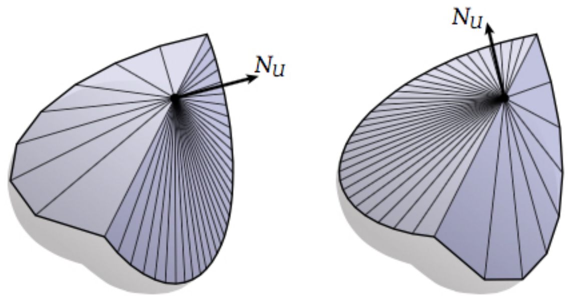

FIGURE 1
Image Source: *[1] Keenan Crane. Digital Geometry Processing with Discrete Exterior Calculus. http://www.cs.columbia.edu/ keenan/Projects/DGPDEC/paper.pdf*

2. Euler Characteristic

The Gauss-Bonnet Theorem expresses a relation between the the topology of a surface and its Gaussian curvature. The topology of a surface is expressed through its Euler characteristic. The Euler characteristic χ is a topologically invariant property of a surface. Accordingly, the Euler characteristic is a number that can describe something about the underlying shape or structure of a surface.

Definition 2.1. A **topological disk** is any image of the unit disk under homeomorphism. Similarly, a **topological sphere** is any image of the unit sphere under homeomorphism.

Initially, the Euler characteristic was classically defined for polyhedra with vertices V, edges E and faces F as

$$\chi = V - E + F$$

Definition 2.2. A **polygonal disk** is a topological disk constructed solely of simple polygons. Similarly, a **polyhedron** is a topological sphere constructed solely of simple polygons.

Proposition 2.3. ***Euler's Polyhedron Formula***. *For any polygonal disk with V vertices, E edges, and F faces, the following relationship holds*

$$V - E + F = 1$$

Proof. Consider an arbitrary polygonal disk D composed of n polygons. Suppose $n = 1$. Then D is a polygon which by definition has 1 face and an equal number of vertices and edges. Thus $F = 1$ and $V = E$. This implies that

$$V - E + F = 0 + 1 = 1$$

Now suppose the relation holds when $n = N$. Let D be composed of $N + 1$ polygons. Removing any exterior edge from D and identifying the vertices of the removed edge can either reduce the number of faces F by one, as in Figure 2, or leave it unchanged, as in Figure 3.

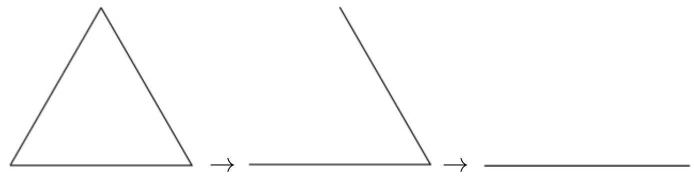

FIGURE 2. Example of removing an edge and reducing the number of faces by one

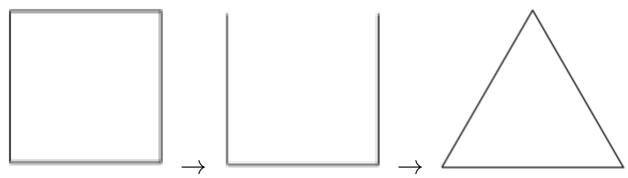

FIGURE 3. Removing an edge and leaving the number of faces unchanged

The case where F is reduced by one occurs when one edge of a triangle is removed. Call the resulting polygonal disk D' with vertices V', edges E', and faces F'.

$$\begin{aligned} V' - E' + F' &= (V - 1) - (E - 2) + (F - 1) \\ &= V - 1 - E + 2 + F - 1 \\ &= V - E + F \\ &= 1 \end{aligned}$$

by the inductive hypothesis.

The case where F is left unchanged occurs when an edge is removed from a polygon with more than 3 edges. Call the resulting polygonal disk D' with vertices V', edges E', and faces F'. As

$$\begin{aligned} V' - E' + F' &= (V - 1) - (E - 1) + F \\ &= V - E + F \end{aligned}$$

removing an edge in this case leaves the value of $V - E + F$ unchanged.

So in removing edges until the number of faces is reduced from $N + 1$ to N, the Euler characteristic is preserved. Then by adding edges in the reverse process, the Euler characteristic when $n = N$ is preserved when $n = N + 1$. Thus by induction, for any polygonal disk, $V - E + F = 1$. □

Corollary 2.4. *For any polyhedron* $V - E + F = 2$.

Proof. Now consider a polyhedron P with vertices, edges, and faces V, E, and F. Delete one face from P and call the resulting discrete surface P' with vertices, edges, and face V', E', and F'. P' is a polygonal disk where $V' - E' + F' = 1$. Adding that face back in,

$$\begin{aligned} V - E + F &= V' - E' + F' + 1 \\ &= 1 + 1 = 2 \end{aligned}$$

as desired. □

Definition 2.5. The **genus** of a surface is the maximum number of cuttings along non-intersecting closed simple curves that can be made while the surface is still connected. Intuitively, this is the number of "handles" in a surface.

The Euler characteristic of closed orientable surfaces is related to the genus g, another topological invariant. Specifically,

$$\chi = 2 - 2g.$$

As polyhedrons are of genus 0, this relationship agrees with the results of Euler's Polyhedron Formula.

3. Gauss-Bonnet

I now define the remaining elements in the Gauss-Bonnet formula and state its classical theorem.

Definition 3.1. The **boundary** of a surface M, is the set of points in M not belonging to the interior of M [3].

Definition 3.2. The **boundary of a discrete surface** is the set of edges that are contained in only one face.

Definition 3.3. Consider the unit tangent vector T of a curve C parametrized with respect to arc length at a point p. The **geodesic curvature** of C at p is the algebraic value of the covariant derivative of T at p [4]. Intuitively, geodesic curvature measures how far a curve is from being a geodesic.

Definition 3.4. The **exterior angle** at the junction of two piecewise differentiable curves is the angular difference between the tangent vectors at the junction [4].

Definition 3.5. A subset $S \subset \mathbb{R}^n$ is a **regular surface** if for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^n$ and a map $x : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^n$ such that x is differentiable, x is a homeomorphism, and for each $q \in U$ the differential $dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one [4].

Theorem 3.6. ***Gauss Bonnet****. Let R be a regular oriented surface, with Gaussian curvature K, and ∂R with geodesic curvature k_g. Let $C_1 \ldots C_n$ be the closed, simple, piecewise, regular curves which form ∂R. Let $\alpha_{ext_i} \ldots \alpha_{ext_p}$ be the set of exterior angles of the curves $C_i \ldots C_n$. Then,*

$$\int_R K dA + \sum_{i=1}^{n} \int_{C_i} k_g ds + \sum_{i=1}^{p} \alpha_{ext_i} = 2\pi\chi(T)[4]$$

Proof. Refer to Do Carmo's proof of the Global Gauss-Bonnet Theorem [4]. □

4. Consequences of Gauss-Bonnet

One interesting consequence of Gauss-Bonnet is an equation for the area of spherical triangles.

Proposition 4.1. *For a sphere with radius R and a spherical triangle with interior angles α_1, α_2 and α_3, the area of the spherical triangle $A = R^2(\alpha_1 + \alpha_2 + \alpha_3 - \pi)$.*

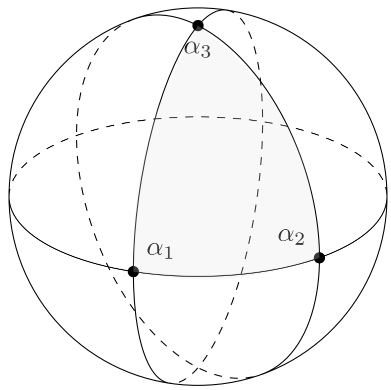

FIGURE 4. A spherical triangle with interior angles $\alpha_1, \alpha_2, \alpha_3$.

Proof. Let T be a spherical triangle with edges E_i on a sphere of radius R. Then

$$\int_T KdA + \sum_{i=1}^{3} \int_{E_i} k_g ds + \sum_{i=1}^{3} \alpha_{ext_i} = 2\pi\chi(T)$$

Because T is piecewise bounded by geodesics C_i which have a geodesic curvature of 0,

$$\sum_{i=1}^{3} \int_{E_i} k_g ds = 0$$

By Definition 3.4,

$$\sum_{i=1}^{3} \alpha_{ext_i} = \sum_{i=1}^{3} \pi - \alpha_i$$

T is a topological disk, so $\chi(T) = 1$ and $2\pi\chi(T) = 2\pi$. Gauss-Bonnet thus yields the relation

$$\int_T KdA + \sum_{i=1}^{3} \pi - \alpha_i = 2\pi$$

This implies

$$KA + 3\pi - (\alpha_1 + \alpha_2 + \alpha_3) = 2\pi$$

Rearranging,

$$\begin{aligned} A &= \frac{1}{K}(\alpha_1 + \alpha_2 + \alpha_3 + 2\pi - 3\pi) \\ &= \frac{1}{K}(\alpha_1 + \alpha_2 + \alpha_3 + \pi) \end{aligned}$$

As demonstrated in Exercise 1.8, the gaussian curvature K at any point on a sphere is $\frac{1}{R^2}$. Plugging in for K yields the desired formula. □

Remark 4.2. Proposition 4.1 can also be proved without the use of Gauss-Bonnet.

Proof. (Proposition 4.1 Alternate) Referring to Figure 1, let A_1 be the wedge-shaped subset of the sphere swept out by angle α_1. Define A_2 and A_3 similarly. Looking at Figure 1, it is evident that

$$2A_1 + 2A_2 + 2A_3 = 4\pi R^2 + 4A.$$

A_1, A_2, A_3 each have an area of $2\theta R^2$, where θ is the angle being swept out. This implies that

$$\begin{aligned} 4\pi R^2 + 4A &= 2A_1 + 2A_2 + 2A_3 \\ &= 4\alpha_1 R^2 + 4\alpha_2 R^2 + 4\alpha_3 R^2 \end{aligned}$$

Dividing through by 4,

$$\pi R^2 + A = R^2(\alpha_1 + \alpha_2 + \alpha_3)$$

yielding

$$A = R^2(\alpha_1 + \alpha_2 + \alpha_3 - \pi)$$

□

Proposition 4.3. *The area of a polygon with consecutive interior angles* $\beta_1, \ldots, \beta_n$ *on the unit sphere is*

$$A = (2-n)\pi + \sum_{i=1}^{n} \beta_i$$

Proof. Let P be a polygon with n vertices labeled clockwise $v_1 \ldots v_n$ with angles $\beta_1 \ldots \beta_n$ as in Figure 5.

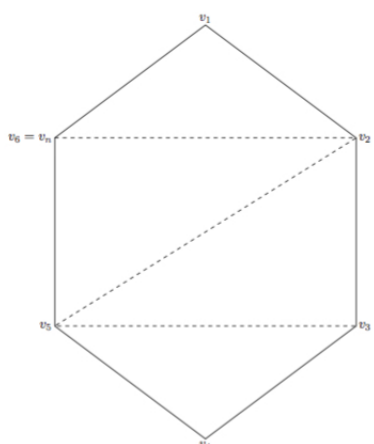

FIGURE 5

Now, we can triangulate the polygon by adding the following edges:

$$\overline{v_2 v_n}, \overline{v_2 v_{n-1}}, \overline{v_3 v_{n-1}}, \overline{v_3 v_{n-2}}, \ldots, \overline{v_{\lceil \frac{n}{2} \rceil -1} v_{\lceil \frac{n}{2} \rceil +1}}.$$

There are $n-2$ edges and $n-2$ triangles. The sum of all the angles in the triangles is equal to $\sum_i^n \beta_i$. By the Proposition 4.1, the area of all these triangles is

$$\sum_i^n \beta_i - (n-2)\pi.$$

Rearranging yields the desired formula. □

5. Discrete Gaussian Curvature

For discrete surfaces, gaussian curvature is defined on the vertices.

Definition 5.1. Discrete Gaussian curvature at a vertex v is the area on the unit sphere bounded by a spherical polygon whose vertices are the unit normals of the faces around v. See Figure 6.

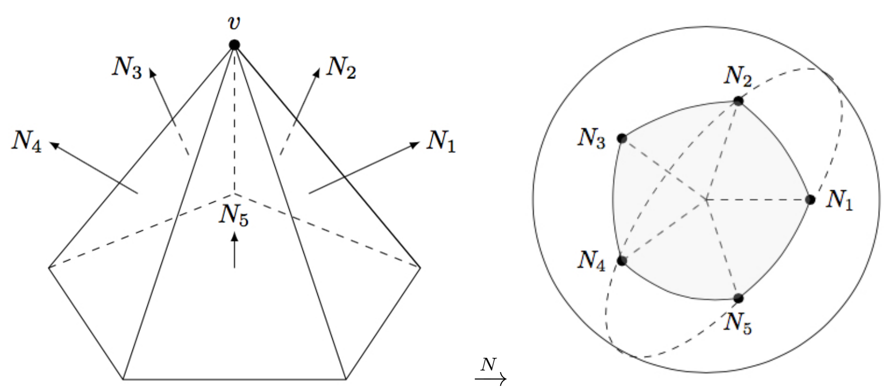

Figure 6

Definition 5.2. The **angle defect**, $d(v)$ of a vertex v is the amount by which the angles incident to v fail to add up to 2π. When F_v is the set of faces containing v and α_f is the interior angle of the face f at v

$$d(v) := 2\pi - \sum_{f \in F_v} \alpha_f$$

Exercise 5.3. Show that the discrete Gaussian curvature at a vertex v is equal to its angle defect.

Solution. Using Figure 6 as a model, consider where N_5 intersects the sphere, forming a vertex of a spherical polygon. On the polyhedron, N_5 is the normal vector to face 5. Face 5 shares an edge with face 1, labeled with N_1, and an edge with face 4, labeled with N_4. Call these edges E_{51} and E_{54}. Using the naming conventions of Definition 5.2, call the interior angle of face 5 at v α_5. For clarity, see Figure 7. Return to the image of the spherical polygon where N_5 is a vertex, Figure 8. Consider a tangent vector to the sphere at N_5 pointing towards N_4. Call this vector T_{54}. T_{54} is orthogonal to N_5. If we consider the vector T_{54} in the polyhedron, as illustrated in Figure 7, T_{54} lies in face 5. T_{54} is in the span of N_5 and N_4. Because E_{54} is orthogonal to both N_5 and N_4, and T_{54} is in the span of N_5 and N_4, T_{54} is orthogonal to E_{54}. By similarly defining T_{51}, using an identical argument, T_{51} is orthogonal to E_{51} and of course N_5. E_{54}, E_{51}, T_{51} and T_{54} can now form a quadrilateral with clearly defined angles. The angle between E_{54} and E_{51} is α_5, and recall that $T_{54} \perp E_{54}$ and $T_{51} \perp E_{51}$. Thus the angle between T_{54} and T_{51} is

$$2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \alpha_5 = \pi - \alpha_5.$$

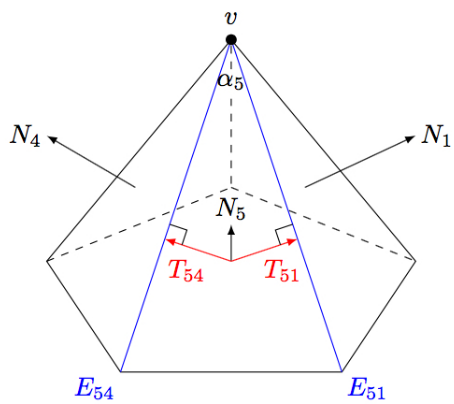

Figure 7

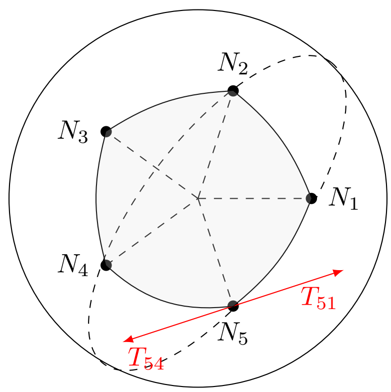

Figure 8

This is precisely the interior angle at N_5 on the spherical polygon. So in general, on the map defining discrete Gaussian curvature, an angle of α_f at a vertex v on a polyhedron transforms to an angle of $\pi - \alpha_f$ at the vertex N_f on a spherical polygon. By Exercise 4.3. the area of a polygon with interior angles $\beta_1 \ldots \beta_n$ on the unit sphere is

$$A = (2-n)\pi + \sum_{i=1}^{n} \beta_i$$

Thus discrete Gaussian curvature at a vertex v is equal to

$$\begin{aligned}(2-n)\pi + \sum_{f \in F_v} (\pi - \alpha_f) &= 2\pi - n\pi + n\pi - \sum_{f \in F_v} \alpha_f \\ &= 2\pi - \sum_{f \in F_v} \alpha_f \\ &= d(v)\end{aligned}$$

6. Discrete Gauss-Bonnet

In the case of compact surfaces without boundary, the $\int_{\partial M} k_g ds$ term in Gauss Bonnet theorem vanishes. Recall that for discrete surfaces, the boundary is the set of edges contained in only one face. Accordingly, polyhedrons have no boundary, whereas polygonal disks do.

Theorem 6.1. ***Gauss Bonnet Theorem for discrete surfaces without boundary***. *Consider a discrete surface, S, with finitely many vertices V, edges E and faces F and no boundary.*

$$\sum_{v \in V} d(v) = 2\pi\chi(S)$$

where χ is the Euler characteristic of the surface, and $d(v)$ is the angle defect.

Proof. For ease of computation, tesselate the discrete surface S so that it is simplicial. By Exercise 5.3,

$$\begin{aligned} d(v) &= (2-n)\pi + \sum_{i=1}^{n} \beta_i \\ &= 2\pi - n\pi + \sum_{i=1}^{n} \beta_i \end{aligned}$$

for the vertex's corresponding spherical polygon. This implies that

$$\sum_{v \in V} d(v) = \sum_{v \in V} 2\pi - \sum_{v \in V} n(v)\pi + \sum_{v \in V} \sum_{i=1}^{n} \beta_i$$

for said polygon. The first term

$$\sum_{v \in V} 2\pi = 2\pi V.$$

In the second term

$$\sum_{v \in V} n(v)\pi = \pi \sum_{v \in V} n(v),$$

$n(v)$ is the number of interior angles in the spherical polygon associated with v. This number is equivalent to the number of edges coming out of v. As each edge is associated with two vertices,

$$\begin{aligned} \sum_{v \in V} n(v)\pi &= \pi \sum_{v \in V} n(v) \\ &= 2\pi E \end{aligned}$$

As demonstrated in Exercise 5.3, the third term

$$\sum_{v \in V} \sum_{i=1}^{i=n} \beta_i$$

can be re-written in terms of the angles incident to the vertex v such that

$$\sum_{v \in V} \sum_{i=1}^{i=n} \beta_i = \sum_{v \in V} \sum_{F | v \in F} (\pi - \alpha_f)$$

Switching the sums so that the angles are summed according to the face they are in instead of the vertex they are incident yields

$$\begin{aligned}\sum_{v\in V}\sum_{i=1}^{i=n}\beta_i &= \sum_{v\in V}\sum_{F|v\in F}(\pi-\alpha_f)\\ &= \sum_{F}\sum_{v\in F}(\pi-\alpha_f)\end{aligned}$$

Because the surface is simplicial, summing $(\pi-\alpha_f)$ over each face yields

$$\begin{aligned}3\pi-(\alpha_{F_{v_1}}+\alpha_{F_{v_2}})+\alpha_{F_{v_3}} &= 3\pi-\pi\\ &= 2\pi\end{aligned}$$

The third term is now

$$\begin{aligned}\sum_{v\in V}\sum_{i=1}^{i=n}\beta_i &= \sum_{v\in V}\sum_{F|v\in F}(\pi-\alpha_f)\\ &= \sum_{F|v\in F}\sum_{v\in V}(\pi-\alpha_f)\\ &= \sum_{F}2\pi\\ &= 2\pi F\end{aligned}$$

Thus

$$\begin{aligned}\sum_{v\in V}d(v) &= \sum_{v\in V}2\pi-\sum_{v\in V}n(v)\pi+\sum_{v\in V}\sum_{i=1}^{n}\beta_i\\ &= 2\pi V-2\pi E+2\pi F\\ &= 2\pi\chi(S)\end{aligned}$$

□

This parallels the smooth Gauss-Bonnet theorem for surfaces without boundary, where

$$\sum_{i=1}^{n}\int_{C_i}k_g ds+\sum_{i=1}^{p}\alpha_{ext_i}=0$$

and

$$\int_M K dA = 2\pi\chi(M)$$

The following examples compare the results of the smooth and discrete Gauss-Bonnet theorems for surfaces without boundary.

Example 6.2. Let's compare a sphere and a simple, albeit crude, discrete approximation of a sphere, a cube. Both a sphere and a cube are of genus 0, with $\chi=2$.

The sphere, S, has no boundary. The area of a sphere is $4\pi R^2$ and by Exercise 1.8, the Gaussian curvature of a sphere is $\frac{1}{R^2}$. Thus for a sphere

$$\begin{aligned}\int_s K dA &= \frac{1}{R^2} 4\pi R^2 \\ &= 4\pi \\ &= 2\pi\chi(S)\end{aligned}$$

as expected.

For the cube, C, each vertex has the same angle defect. This means

$$\begin{aligned}\sum_{v\in V} d(v) &= \sum_{v\in V}(2\pi - \sum_{f\in F_v} \alpha_f) \\ &= \sum_{v\in V}(2\pi - (\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2})) \\ &= \sum_{v\in V} \frac{\pi}{2} \\ &= 8\frac{\pi}{2} \\ &= 4\pi \\ &= 2\pi\chi(C)\end{aligned}$$

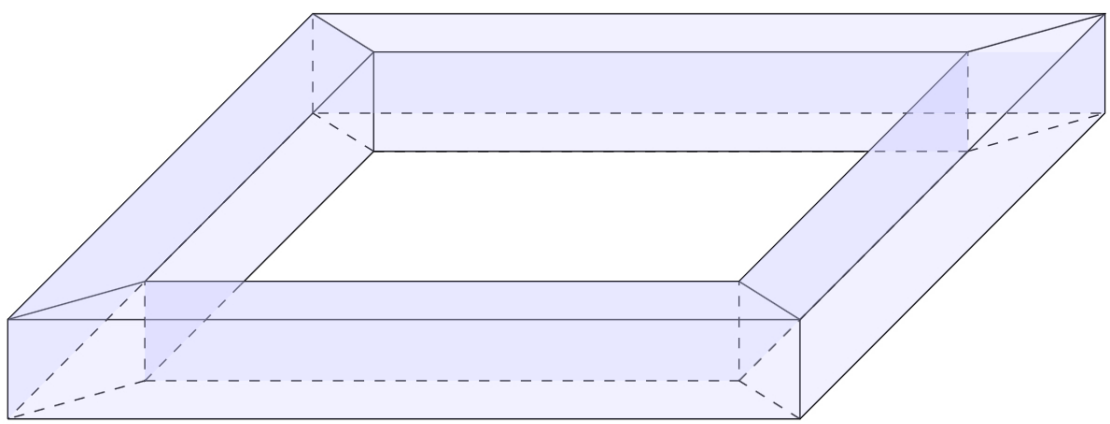

FIGURE 9. Discrete torus

Example 6.3. Now lets confirm the results of the discrete Gauss Bonnet theorem for a surface of genus = 1, a torus T (See Figure 9). The vertices on the interior have the same angle defect. The angle defect of one of these interior vertices, v_i is

$$\begin{aligned}d(v_i) &= 2\pi - \frac{3\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} \\ &= -\frac{\pi}{2}\end{aligned}$$

Similarly, the vertices on the exterior are identical, and the angle defect of one of these exterior vertices, v_e is

$$\begin{aligned} d(v_e) &= 2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{v \in V} d(v) &= 8 * \frac{\pi}{2} + 8 * -\frac{\pi}{2} \\ &= 0 \\ &= 2\pi\chi(T) \end{aligned}$$

as expected.

Theorem 6.4. ***Discrete Gauss Bonnet for surfaces with boundary****. For a discrete surface S with boundary,*

$$\sum_{v \in S_{int}} d(v) + \sum_{v \in \partial S} \alpha_{ext}(v) = 2\pi\chi(S)$$

where is the exterior angle at v by as

$$\alpha_{ext} = \pi - \sum_i \alpha_i$$

for α_i incident to v .

Proof. Take a second copy of S and glue it to the first along the boundary. Then, the exterior angle, α_{ext} of a boundary vertex v is simply half the discrete gaussian curvature of v on $2S$. This implies that

$$\begin{aligned} \sum_{v \in 2S} d(v) &= 2(\sum_{v \in S_{int}} d(v) + \sum_{v \in \partial S} \alpha_{ext}(v)) \\ &= 2\pi\chi(2S) \end{aligned}$$

Dividing through, this implies

$$\sum_{v \in S} d(v) + \sum_{v \in \partial S} \alpha_{ext}(v) = \pi\chi(2S)$$

As the boundary of S, ∂S, is piecewise linear, $\chi(\partial S) = 0$. Then

$$\begin{aligned} \chi(2S) &= 2\chi(S) - \chi(\partial S) \\ &= 2\chi(S) \end{aligned}$$

Thus

$$\begin{aligned} \sum_{v \in S} d(v) + \sum_{v \in \partial S} \alpha_{ext}(v) &= \pi\chi(2S) \\ &= 2\pi\chi(S) \end{aligned}$$

as desired. □

Example 6.5. Let's compare a spherical cap on a sphere of radius 1 and a cube with one face deleted. Both are topological disks, which by Proposition 2.3, have Euler characteristic 1.

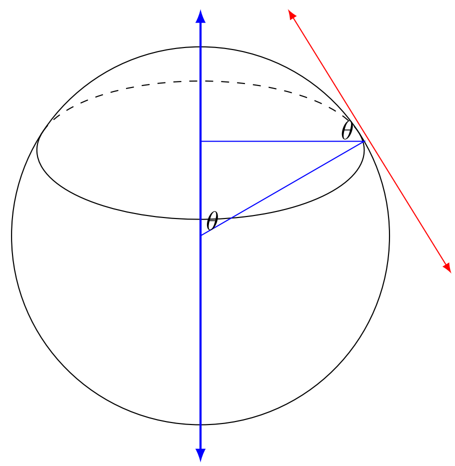

FIGURE 10. Spherical cap on a sphere of radius 1

Referring to Figure 10, because the radius of the sphere is 1, the base radius of the spherical cap S is $\sin\theta$. Then the area of

$$\begin{aligned} S &= \int_0^\theta 2\pi \sin\theta d\theta \\ &= 2\pi(1-\cos\theta). \end{aligned}$$

As the Gaussian curvature

$$K = \frac{1}{R^2} = 1,$$

$$\begin{aligned} \int_S K dA &= \int_S dA \\ &= 2\pi(1-\cos\theta). \end{aligned}$$

The boundary ∂S is the circumference of the base of the cap, $2\pi\sin\theta$. From the tangent vector on Figure 10, it is evident that the geodesic curvature k_g of ∂S is $\frac{\cos\theta}{\sin\theta}$. This implies that,

$$\begin{aligned} \sum_{i=1}^{n} \int_{\partial C_i} k_g ds + \sum_{i=1}^{p} \alpha_{ext_i} &= \int_{\partial S} k_g ds \\ &= \frac{\cos\theta}{\sin\theta} * 2\pi\sin\theta \\ &= 2\pi\cos\theta \end{aligned}$$

Thus,

$$\begin{aligned} \int_S K dA + \int_{\partial S} k_g ds &= 2\pi(1-\cos\theta) + 2\pi\cos\theta \\ &= 2\pi \\ &= 2\pi\chi(S) \end{aligned}$$

For the cube with one face deleted, C, there are 8 interior vertices with

$$d(v_{int}) = 2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}$$

C also has 4 vertices on the boundary with

$$\alpha_{ext} = \pi - (\frac{\pi}{2} + \frac{\pi}{2}) = 0.$$

Thus,

$$\begin{aligned}\sum_{v \in S_{int}} d(v) + \sum_{v \in \partial S} \alpha_{ext}(v) &= 4 * d(v_{int}) + 4 * 0 \\ &= 4 * \frac{\pi}{2} \\ &= 2\pi \\ &= 2\pi\chi(C)\end{aligned}$$

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