

INTRODUCTION TO FOURIER ANALYSIS

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ABSTRACT. In this paper we introduce basic concepts of Fourier analysis, develop basic theory of it, and provide the solution of Basel problem as its application. We also include a proof of the uniqueness of trigonometric series.

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1. INTRODUCTION

The development and study of Fourier series is motivated by the desire to model general functions as sums of simple trigonometric functions. The ability to model functions in a simple manner is tremendously valuable to science and technology, as it allows for the understanding and mathematical manipulation of the original function in terms of the simpler decomposition.

Jean-Baptiste Joseph Fourier, after whom the field is named, worked extensively with trigonometric series and used the Fourier series in order to solve the heat equation in physics. The ability of Fourier analysis to model complicated functions has led to its application in partial differential equations, quantum physics, thermodynamics, signal processing, and many other fields.

We begin our study of Fourier analysis with some definitions:

Definition 1.1. If f is an integrable function defined on the interval $[a, b]$ of length L , the n -th **Fourier coefficient** of f is defined by

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x / L} dx$$

Definition 1.2. The **Fourier series** of f is formally defined by

$$\mathcal{F}(f(x)) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x / L}$$

Definition 1.3. The N -th **Fourier partial sum** of f is defined by

$$S_N(f(x)) = \sum_{n=-N}^N \hat{f}(n)e^{2\pi inx/L}$$

Due to Euler's formula ($e^{ix} = \cos x + i \sin x$), this definition of Fourier series expresses a function f in terms of a summation of sine and cosine functions. This fact leads to our first intuition about Fourier series: like the trigonometric functions it is composed of, the Fourier series of a function is periodic in nature. As we will see later, a function need only satisfy a mild condition in order for its Fourier series to converge to the function itself.

2. CONVERGENCE

In order to justify the use of Fourier series to model functions and explore the various application of Fourier analysis, we must first investigate whether the Fourier series is, indeed, a good approximation of the original function. Until now, we have only stated the definition of Fourier series, and demonstrated that it is a summation of sine and cosine waves dependent on the original function, but have not proven that the series converges to its original function.

Before embarking on the subtle task of proving convergence, we must first understand what the formula provided in the definition conceptually entails. The idea of convolutions is central to understanding Fourier series, and will also serve a pivotal role in more advanced Fourier analysis.

Definition 2.1. Given two functions f and g integrable on $[a, b]$, their **convolution** $f * g$ on $[a, b]$ is defined as

$$(f * g)(x) = \int_a^b f(y)g(x - y)dy$$

Conceptually, convoluting two functions gives the area of their overlap as one is shifted, or translated, over the other. The operation is, loosely speaking, a way of determining the “blend” or “weighted average” of the two functions.

Some of the essential properties of convolutions are stated here:

Proposition 2.2. *Let c be any complex number and let f , g , and h be functions integrable on the interval of convolution. Then we have:*

- (1) $f * g = g * f$
- (2) $f * (g * h) = (f * g) * h$
- (3) $f * (g + h) = (f * g) + (f * h)$
- (4) $c(f * g) = (cf) * g$
- (5) $(f * g)$ is continuous

The first few properties are easily checked using the definition of convolution; the operation preserves commutativity, associativity, distributivity, and scalar associativity. Property (5) illuminates a key aspect of the convolution: it is more “regular” than the original functions. Whereas f and g need only be integrable, their convolution $f * g$ is continuous.

Given the concept of convolution, we can rearrange terms in the Fourier partial sum to more deeply study Fourier series. We will be using 2π -periodic functions defined on the interval $[-\pi, \pi]$ henceforth when discussing the properties of Fourier series, for the sake of simplicity and standardization. Since L is set to 2π , the exponent in the Fourier coefficient ($2\pi inx/L$) reduces to (inx) . We have:

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^N \hat{f}(n)e^{inx} \\ &= \sum_{n=-N}^N \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)e^{-iny} dy \right) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^N e^{in(x-y)} \right) dy \end{aligned}$$

After some rearranging of terms and exchanging the integral and the summation, the Fourier partial sum reveals itself as a convolution of f and the trigonometric sum $\sum_{n=-N}^N e^{inx}$. This trigonometric sum, critically important in studying Fourier series, is called the Dirichlet kernel.

Definition 2.3. The N -th **Dirichlet kernel** is defined to be

$$D_N(x) = \sum_{n=-N}^N e^{inx}$$

Therefore, the Fourier partial sum can be represented as a convolution of its original function and the Dirichlet kernel:

$$S_N(f)(x) = (f * D_N)(x)$$

The Dirichlet kernel has certain essential properties that will be necessary in the investigation of the convergence of Fourier series.

Proposition 2.4.

$$D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})}$$

Proof. Let $r = e^{ix}$. Recall that the sum of a geometric series $\sum_{n=-N}^N r^n$ is given by $\frac{1-r^{N+1}}{1-r}$.

$$\begin{aligned} D_N &= \sum_{n=0}^N r^n + \sum_{n=-N}^{-1} r^n = \frac{1-r^{N+1}}{1-r} + \frac{r^{-N}-1}{1-r} \\ &= \frac{r^{-N-1/2}-r^{N+1/2}}{r^{-1/2}-r^{1/2}} = \frac{e^{-(N+1/2)ix}-e^{(N+1/2)ix}}{e^{-ix/2}-e^{ix/2}} \\ &= \frac{-2i \sin((N+1/2)x)}{-2i \sin(x/2)} = \frac{\sin((N+1/2)x)}{\sin(x/2)} \end{aligned}$$

□

Given that many other kernel functions used in convolutions exist (e.g. the **Poisson kernel** $P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}$), it is useful to define certain sets of properties that cause the kernel to be “good.”

Definition 2.5. A family of functions $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is a family of **good kernels** if:

- (a) For all $n \in \mathbb{N}$, $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1$.
- (b) There exists $M > 0$ such that for all $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} |K_n(x)| dx \leq M$.
- (c) For all $\delta > 0$, $\int_{\delta \leq |x| \leq \pi} |K_n(x)| dx \rightarrow 0$ as $n \rightarrow \infty$

Functions on the circle and 2π -periodic functions on \mathbb{R} are naturally connected. Since any point on the circle can be represented as $e^{i\theta}$, we may define a 2π -periodic function f by its counterpart F on the circle: $f(\theta) = F(e^{i\theta})$. The importance of good kernels arises from their use in convolutions with functions.

Proposition 2.6. Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels, and f be an integrable function on the circle. Then, whenever f is continuous at x ,

$$\lim_{n \rightarrow \infty} (f * K_n)(x) = f(x)$$

If the Dirichlet kernel was a good kernel, the problem of Fourier series convergence would be easily solved with the proof of this sole proposition. However, the Dirichlet kernel does not satisfy all of the requirements in Definition 2.5. The definition of the Dirichlet kernel as a sum of exponentials quickly shows us that Property (a) holds.

However, the “goodness” of D_N falls apart at Property (b). While the mean value of D_N is 1, the integral of its absolute value is very large, due to the fact that $D_N(x)$ oscillates rapidly between positive and negative values. More precisely,

$$\int_{-\pi}^{\pi} |D_N(x)| dx \geq c \log n, \text{ as } N \rightarrow \infty$$

All of this is to demonstrate that the problem of Fourier convergence is more subtle than expected at first glance, due to the Dirichlet kernel’s extreme behavior that prevents it from being a good kernel.

Convergence of Fourier series can be understood in several ways (e.g. mean-square convergence, Abel summability, etc.). Our proof will focus on the pointwise convergence of a Fourier series to its original function. The Riemann-Lebesgue Lemma is central to the proof of pointwise convergence.

Lemma 2.7. Riemann-Lebesgue Lemma: If f is an arbitrary Riemann-integrable function on compact interval $I = [a, b]$, then:

$$\lim_{n \rightarrow \infty} \int_I f(x) e^{-inx} dx = 0$$

Proof. Suppose first that f is a piecewise constant function on a compact interval $I = [a, b]$. This means that I can be subdivided into finitely many subintervals

$I_k = [a_k, b_k]$ such that f is constant on each subinterval. Then, f can be written as a summation of the constant functions:

$$\text{For all } x \in (a_k, b_k), f(x) = \sum_{k=1}^N c_k g_k(x)$$

where $g_k(x) = 1$ in I_k and $g_k(x) = 0$ outside of I_k .

We then get:

$$\int_I f(x) e^{-inx} dx = \sum_{k=1}^N \int_I c_k g_k(x) e^{-inx} dx = \sum_{k=1}^N c_k \int_{I_k} e^{-inx} dx = \sum_{k=1}^N c_k |I_k| \frac{e^{-inx}}{-in}$$

This is a sum of finitely many terms, all of which approach 0 as n approaches infinity.

Now, let f be an arbitrary Riemann-integrable function on compact I . By the definition of Riemann-integrability, the integral of f can be approximated by rectangles (i.e. the integral of a piecewise constant function). In other words, for any $\epsilon > 0$ there exists a piecewise constant function g such that:

$$\int_I |f(x) - g(x)| dx < \epsilon$$

Let g be a function as described above.

$$\begin{aligned} \left| \int_I f(x) e^{-inx} dx \right| &= \left| \int_I (f(x) - g(x)) e^{-inx} dx + \int_I g(x) e^{-inx} dx \right| \\ &\leq \int_I |f(x) - g(x)| |e^{-inx}| dx + \left| \int_I g(x) e^{-inx} dx \right| \\ &\leq \int_I |f(x) - g(x)| dx + \left| \int_I g(x) e^{-inx} dx \right| < \epsilon + 0 \end{aligned}$$

as $n \rightarrow \infty$. □

Proposition 2.8. *If f is bounded on $[a, b]$, $c \in [a, b]$, and for any $\delta > 0$, f is integrable on $[a, c - \delta] \cup [c + \delta, b]$, then f is integrable on $[a, b]$.*

Proof. Since f is bounded, there exists a number M such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $4\delta M \leq \epsilon/3$. Since f is integrable on $[a, c - \delta] \cup [c + \delta, b]$, we can choose P_1 and P_2 to be partitions of $[a, c - \delta] \cup [c + \delta, b]$ such that $U(P_i, f) - L(P_i, f) < \epsilon/3$, $i = 1, 2$. Let $P = P_1 \cup (c - \delta, 0] \cup (0, c + \delta) \cup P_2$. Then $U(P, f) - L(P, f) < \epsilon/3 + 2\delta M + 2\delta M + \epsilon/3 \leq \epsilon$. Thus, f is integrable on the whole interval $[a, b]$. □

With the powerful Riemann-Lebesgue lemma, the dirichlet kernel, and Proposition 2.8, the pointwise convergence of Fourier series is readily investigated.

Theorem 2.9. *Let f be an integrable function on $[-\pi, \pi]$ differentiable at a point x_0 . Then $S_N(f)(x_0) \rightarrow f(x_0)$ as $N \rightarrow \infty$.*

Proof. Let $x_0 \in [-\pi, \pi]$ be such that f is differentiable at x_0 . Let $F(t)$ be the function defined by:

$$F(t) = \begin{cases} \frac{f(x_0-t) - f(t)}{t} & t \neq 0, t \in (-\pi, \pi) \\ -f'(x_0) & t = 0 \end{cases}$$

Due to the differentiability at x_0 and integrability of f , we know that F is bounded on $[-\pi, \pi]$ and integrable on $[-\pi, -\delta] \cup [\delta, \pi]$ for every $\delta \in (0, \pi)$. By Proposition 2.8, F is integrable on $[-\pi, \pi]$.

We know from our analysis of the Dirichlet kernel that $S_N(f)(x_0) = (f * D_N)(x_0)$, $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$, and $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N dx = 1$. Then,

$$\begin{aligned} S_N f(x_0) - f(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0 - t) D_N(t) dt - f(x_0) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x_0 - t) - f(x_0)] D_N(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) t D_N(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) \frac{t}{\sin(t/2)} \sin((N+1/2)t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) \frac{t}{\sin(t/2)} [\sin(Nt) \cos(t/2) + \cos(Nt) \sin(t/2)] dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) \frac{t \cos(t/2)}{\sin(t/2)} \sin(Nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) t \cos(Nt) dt \end{aligned}$$

Let $Q(t) = \frac{F(t)t \cos(t/2)}{\sin(t/2)}$ and let $R(t) = F(t)t$. We know that Q and R are continuous since $F(t)$, t , $\cos(t/2)$, and $\frac{t}{\sin(t/2)}$ are all continuous in the interval $[-\pi, \pi]$. Then, we can apply the Riemman-Lebesgue lemma to Q and R to finish the proof:

$$S_N f(x_0) - f(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(t) \sin(Nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} R(t) \cos(Nt) dt \rightarrow 0 \text{ as } N \rightarrow \infty$$

□

3. UNIQUENESS

In previous sections we used Fourier series to express periodic functions as trigonometric series. A natural question is whether this expression is unique. In this section, we are going to prove the Cantor-Riemann theorem which answers the above question affirmatively.

Several concepts and lemmas must be addressed before the uniqueness proof may be initiated. The first and most essential of these is the Cantor-Lebesgue Theorem.

Theorem 3.1. Cantor-Lebesgue Theorem *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers. If*

$$\lim_{n \rightarrow \infty} a_n \cos x + b_n \sin x = 0$$

for every $x \in \mathbb{R}$, then $a_n, b_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let $p_n = \sqrt{a_n^2 + b_n^2}$ and choose θ_n such that $p_n \sin \theta_n = b_n$, $p_n \cos \theta_n = a_n$. Then $a_n \cos(nx) + b_n \sin(nx) = p_n \cos(nx - \theta_n) \rightarrow 0$ as $n \rightarrow \infty$. If we can prove that $p_n \rightarrow 0$ then, since \sin and \cos are bounded, a_n, b_n must approach 0.

Assume $p_n \not\rightarrow 0$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ and a number δ such that for any positive integer k , $p_{n_k} \geq \delta > 0$. We can further assume that

$\frac{n_{k+1}}{n_k} \geq 3$. Then we attempt to put a lower bound on the $\cos(nx - \theta)$ term for all n at at least one point. We define I_1 to be

$$\left[\frac{(\theta_{n_1} - \frac{\pi}{3})}{n_1}, \frac{(\theta_{n_1} + \frac{\pi}{3})}{n_1} \right]$$

Then we have: $\cos(n_1x - \theta_{n_1}) \geq \frac{1}{2}$, for all $x \in I_1$. It is known that $|I_1| = \frac{2\pi}{3n_1}$ and $\frac{n_2}{n_1} \geq 3$. As x ranges over I_1 , $(n_2x - \theta_{n_2})$ ranges over $(n_2I_1 - \theta_{n_2})$. We have: $|n_2I_1 - \theta_{n_2}| = n_2|I_1| = n_2(\frac{2\pi}{3n_1}) \geq 2\pi$. Just as we defined I_1 to be the interval for which $\cos(n_1x - \theta_{n_1}) \geq 1/2$, there exists an interval I_2 for which $\cos(n_2x - \theta_{n_2}) \geq 1/2$ for all $x \in I_2$. Since the range of $(n_2I_1 - \theta_{n_2})$ exceeds 2π , there exists $I_2 \subset I_1$ such that:

$$\cos(n_2x - \theta_{n_2}) \geq 1/2 \text{ for all } x \in I_2$$

$$|I_2| = \frac{2\pi}{3n_2}$$

Proceeding by induction, we can construct intervals $I_k, k \in \mathbb{Z}_{>0}$ such that $I_1 \supset I_2 \supset I_3 \supset \dots$ for which $\cos(n_kx - \theta_{n_k}) \geq 1/2$ for all k , for all $x \in I_k$. By Cantor's closed interval theorem, there exists $\xi \in \bigcap_{k=1}^{\infty} I_k$ such that $\cos(n_k\xi - \theta_{n_k}) \geq \frac{1}{2}$ for all k . Therefore $p_{n_k} \cos(n_k\xi - \theta_{n_k}) \geq \delta/2$ for all k . This contradicts the convergence of $p_n \cos(nx - \theta_n)$ to 0, since it will always be greater than $\delta/2$ at $x = \xi$. \square

One of the first limitations encountered in attempting to prove the uniqueness of Fourier series, is the need to analyze function's second derivatives without imposing the restriction of differentiability at each point. For this reason, we use a generalized version of the second derivative rather than differentiating at x .

Definition 3.2. The **Schwarz second derivative** of a function F is defined by:

$$DF(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2}$$

Remark 3.3. We can see that the Schwartz second derivative coincides with $f''(x)$ by using *L'Hôpital's Rule* for f with continuous $f''(x)$:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{f''(x+h) + f''(x-h)}{2} = f''(x) \end{aligned}$$

Lemma 3.4. A continuous function F on \mathbb{R} with zero Schwarz second derivative everywhere is a linear function.

Proof. Let $A(x) = x^2$. We prove that $F + \epsilon A$ is convex for every $\epsilon > 0$. Suppose it is not, then there exist $a < t < b$ and a linear function $L(x) = \mu x + \lambda$ such that $G(x) = F(x) + \epsilon A(x) - L(x)$ satisfies $G(a) = G(b) = 0$ and $G(t) > 0$. Choose $s \in (a, b)$ where G attains its maximum in $[a, b]$. Then $DG(s) \leq 0$. Since $DL(s) = 0$ and $DA(s) = 2\epsilon > 0$, we see that $DF(s) < 0$, a contradiction.

As ϵ tends to 0, we see that F itself is convex. Similarly, we can prove that F is concave. Therefore, F is linear. \square

Theorem 3.5. *Cantor-Riemann Uniqueness Theorem*

Let $\{c_n\}_{n \in \mathbb{Z}}$ be complex numbers such that

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n e^{inx} = 0$$

for all $x \in \mathbb{R}$, then $c_n = 0$ for all $n \in \mathbb{Z}$.

Proof. Since $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges everywhere, $c_n e^{inx} + c_{-n} e^{-inx} \rightarrow 0$ for all x as $n \rightarrow \infty$. By the Cantor-Lebesgue Theorem, $|c_n| + |c_{-n}| \rightarrow 0$ as $n \rightarrow \infty$. Let F be the formal integral:

$$F = \int \int \sum_{n=-N}^N c_n e^{inx} = c_0 \frac{x^2}{2} + \sum_{n \neq 0} \frac{c_n}{(in)^2} e^{inx}$$

Using the Weierstrass M-test on the following expression

$$\left| \frac{c_n e^{inx}}{(in)^2} + \frac{c_{-n} e^{-inx}}{(-in)^2} \right| \leq \frac{\sup(|c_n| + |c_{-n}|)}{n^2}$$

we deduce that $F(x) - c_0 \frac{x^2}{2}$ is a continuous function and F is the uniform limit of its partial sums. Applying the Schwarz second derivative, first to e^{ix} and then to F , we find:

$$D(e^{ix}) = \frac{e^{i(x+h)} - 2e^{ix} + e^{i(x-h)}}{h^2} = -e^{ix} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2$$

$$DF(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} = \lim_{h \rightarrow 0} \left[c_0 + \sum_{n \neq 0} c_n e^{inx} \left(\frac{\sin \frac{nh}{2}}{\frac{nh}{2}} \right)^2 \right]$$

Since $\lim_{h \rightarrow 0} \left(\frac{\sin \frac{nh}{2}}{\frac{nh}{2}} \right) = 1$ and $c_0 + \sum_{n \neq 0} \frac{c_n}{(in)^2} e^{inx} = 0$,

$$DF(x) = \lim_{h \rightarrow 0} \left[c_0 + \sum_{n \neq 0} \frac{c_n}{(in)^2} e^{inx} \left(\frac{\sin \frac{nh}{2}}{\frac{nh}{2}} \right)^2 \right] = 0$$

Since $DF(x) = 0$, the Schwarz lemma tells us that F is linear, so for some α, β :

$$F(x) = c_0 \frac{x^2}{2} + \sum_{n \neq 0} \frac{c_n}{(in)^2} e^{inx} = \alpha x + \beta$$

$$\sum_{n \neq 0} \frac{c_n}{(in)^2} e^{inx} = -c_0 \frac{x^2}{2} + \alpha x + \beta$$

Since $\sum_{n \neq 0} \frac{c_n}{(in)^2} e^{inx}$ is bounded in x , we have $c_0 = 0 = \alpha$. We now know that $s_N(x)$ converges uniformly to 0, where $s_N(x)$ is defined by:

$$s_N(x) = -\beta + \sum_{0 < |n| \leq N} \frac{c_n}{(in)^2} e^{inx}$$

However, for each $n \neq 0$ we can also express c_n in terms of s_N :

$$c_n = \frac{(in)^2}{2\pi} \int_0^{2\pi} s_N(x) e^{-inx} dx$$

Since $s_N(x)$ converges uniformly to 0, we have

$$c_n = \lim_{N \rightarrow \infty} \frac{(in)^2}{2\pi} \int_0^{2\pi} s_N(x) e^{-inx} dx = 0$$

□

Riemann's idea of examining the double integral of the trigonometric series and Cantor's insight that the static value c_n could be analyzed and determined using a sequence of functions s_N were seminal advances in mathematical understanding of trigonometric series and beyond.

4. PARSEVAL'S IDENTITY AND THE BASEL PROBLEM

The Basel problem, relevant to both mathematical analysis and number theory, was first introduced in the 17th century when Pietro Mengoli asked for the exact sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The problem puzzled mathematicians for many years before Euler finally offered a solution ($\frac{\pi^2}{6}$) in 1735, but it took him even longer to develop a rigorous proof. Using Fourier Analysis, however, the problem's treatment becomes simple.

The solution to the Basel problem will require a property of Fourier series called Parseval's Identity, which will first require examination of Fourier series convergence on the L^2 norm.

Definition 4.1. The **Hilbert inner product** of two functions f and g on the interval $[-\pi, \pi]$ (where g^* denotes the complex conjugate of g such that if $g(c) = a + bi$ then $g^*(c) = a - bi$) is defined by:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)g(x)^* dx$$

Definition 4.2. The L^2 norm of a function f is defined by:

$$\|f\| = \sqrt{\langle f, f^* \rangle}$$

Definition 4.3. Two functions f, g on $[-\pi, \pi]$ are **orthogonal** (denoted by $f \perp g$) if $\langle f, g \rangle = 0$.

Proposition 4.4. Pythagoras Theorem If f and g are orthogonal, then

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2$$

Lemma 4.5. Best L^2 Approximation Amongst all the functions of the form $P_N = \sum_{n=1}^N b_n e_n$, where $e_n = e^{inx}$, the n -th Fourier partial sum $S_N f$ best approximates f in the L^2 norm. More precisely:

$$\|f - P_N\| \geq \|f - S_N f\|$$

Proof. Let $E_N f = f - S_N f$. Notice that the family e_n is orthonormal and that $\hat{f} = \langle f, e \rangle$. Consequently:

$$\begin{aligned} \langle E_N f, e_n \rangle &= 0 \text{ for all } n \leq N \\ \langle (S_N f - P_N), E_N f \rangle &= 0 \end{aligned}$$

We can write:

$$f - P_N = f - S_N f + S_N f - P_N = E_N f + (S_N f - P_N)$$

The Pythagoras theorem then gives us:

$$\|f - S_N f\| = \|E_N f\| \leq \|f - P_N\|$$

□

Theorem 4.6. Mean-square convergence *If f is a square-integrable function, then $\|f - S_N f\| \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. Fix $\epsilon > 0$. We begin by approximating f using a continuous function g that satisfies:

$$\begin{aligned} \sup |g(x)| &\leq \sup |f(x)| = B \text{ on } [-\pi, \pi] \\ \int_{-\pi}^{\pi} |f(x) - g(x)| dx &< \epsilon^2 \end{aligned}$$

We then find:

$$\begin{aligned} \|f - g\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - g(x)|^2 dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - g(x)| |f(x) - g(x)| dx \\ &\leq \frac{2B}{2\pi} \int_{-\pi}^{\pi} |f(x) - g(x)| dx \\ &\leq C\epsilon^2 \end{aligned}$$

Since g is continuous, we may approximate it by a trigonometric polynomial P :

$$\|g - P\| < \epsilon$$

Then,

$$\|f - P\| < C'\epsilon$$

We conclude with the Best Approximation Lemma:

$$\|f - S_N f\| \leq \|f - P\| < C'\epsilon$$

□

Lemma 4.7. *For an integrable function f on $[-\pi, \pi]$, we have*

$$\langle f - S_N f, S_N f \rangle = 0$$

Proof. The lemma follows from the statement in our proof of the Best Approximation Lemma that $\langle E_N f, e_n \rangle = 0$. □

Theorem 4.8. Parseval's Identity *For a periodic square-integrable function f on the circle with convergent Fourier series, Fourier coefficient \hat{f} , and complex conjugate f^* :*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x))^2 dx$$

Proof. By the above lemma and Pythagoras theorem,

$$\|f\|^2 = \|f - S_N f\|^2 + \|S_N f\|^2$$

for every $N \geq 0$. By theorem 4.4, we see that $\|f\|^2 = \lim_{N \rightarrow \infty} \|S_N f\|^2$. Since $\{e^{inx}\}_{n \in \mathbb{Z}}$ is an orthonormal set, we have

$$\|S_N f\|^2 = \sum_{n=-N}^N |\hat{f}(n)|^2$$

Let $N \rightarrow \infty$, we get Parseval's identity

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

□

Theorem 4.9. Solution to Basel Problem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof. Let $f(x) = x$. Since f is continuous, differentiable, integrable, and real, we apply Parseval's identity to get:

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx$$

Taking the Fourier coefficient of $f(x) = x$ and integrating by parts:

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx = \frac{1}{2\pi} \left[\frac{ix e^{-inx}}{n} + \frac{e^{-inx}}{n^2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[\frac{x}{n} (i \cos(nx) + \sin(nx)) + \frac{1}{n^2} (\cos(nx) - i \sin(nx)) \right]_{-\pi}^{\pi} \\ &= \frac{\cos(n\pi)i}{n} = \frac{(-1)^n}{n} i \end{aligned}$$

Then we find

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{4\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{6}$$

□

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