EILENBERG-MACLANE SPACES AS A LINK BETWEEN COHOMOLOGY AND HOMOTOPY

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ABSTRACT. This paper gives an exposition of an established theorem in algebraic topology: there exists an isomorphism between reduced cohomology groups of a given CW complex and basepoint-preserving homotopy classes of maps from that CW complex to a suitable Eilenberg-MacLane space. In particular, we show an isomorphism $H^n(X; G) \cong \langle X, K(G, n)\rangle$.

CONTENTS

1. INTRODUCTION

The relationship between cohomology and homotopy—both fundamental concepts—is a natural object of interest in algebraic topology. Eilenberg-MacLane spaces provide one fruitful description of this relationship. An Eilenberg-MacLane space K(G,n) has the property that $\pi_n(K(G, n)) \cong G$ and every other homotopy group is trivial. Verifying the axioms for cohomology for $\langle X, K(G, n)\rangle$, the set of basepoint-preserving homotopy classes of maps from a CW complex X to the Eilenberg-MacLane space K(G,n), allows, with a little extra work, the demonstration of an isomorphism $H^n(X; G) \cong \langle X, K(G, n)\rangle$. The more precise statement of the theorem is as follows.

Theorem 1.1. *For all CW complexes X and $n > 0$, there are natural bijections $T : \langle X, K(G, n)\rangle \to H^n(X; G)$ with G any abelian group. If H^n is a reduced cohomology theory, the restriction on n may be omitted.*

The approach to the above demonstration will largely follow the one given in Hatcher's *Algebraic Topology* with adjustments and rearrangements made when necessary to clarify the central theorem as a standalone topic rather than a subordinate element in a larger text. The succeeding section will lay out the axioms for cohomology, which will be followed by a section checking these axioms for $\langle X, K(G, n)\rangle$. The final section deals with the coefficient group and wraps up the proof of the theorem.

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2. The Axioms for Cohomology

A reduced cohomology theory of CW complexes (X, A) is a contravariant functor $\tilde{h}^n$ to abelian groups with natural coboundary homomorphisms $\delta : \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A)$ satisfying three axioms. The axioms uniquely determine the cohomology theory.

- **Homotopy Axiom:** Given $f, g : X \to Y$, if $f \simeq g$, then $f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$.
- **Wedge Sum Axiom:** For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the map $\prod_\alpha i_\alpha : \tilde{h}^n(X) \to \prod_\alpha \tilde{h}^n(X_\alpha)$ is an isomorphism for each n.
- **Long Exact Sequence Axiom:** For each CW pair (X, A) there exists a long exact sequence

$$... \xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \xrightarrow{q^*} ...$$

3. Verifying the Axioms

The first task is to verify that $\langle X, K(G, n)\rangle$ satisfies the three axioms for cohomology outlined in Section 2. To that end, a couple of definitions are useful.

Definition 3.1. An **Ω-spectrum** is a sequence of CW complexes $\{K_n\}$ with weak homotopy equivalences $K_n \to \Omega K_{n+1}$ for all n.

Definition 3.2. A **cofibration sequence** of a CW pair (X, A) is a sequence of the following form.

$$A \hookrightarrow X \to X/A \to \Sigma A \hookrightarrow \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \hookrightarrow \Sigma^2 X \to ...$$

Armed with these definitions, we proceed to the main object of this section.

Theorem 3.3. *Given an Ω-spectrum $\{K_n\}$, the functors $X \mapsto h^n(X) = \langle X, K_n\rangle$ for integer n define a reduced cohomology theory on the category of pointed CW complexes and basepoint-preserving maps.*

Proof. First consider the homotopy axiom. A basepoint-preserving map $f : X \to Y$ induces a map $f^* : \langle Y, K_n\rangle \to \langle X, K_n\rangle$ which sends a map $Y \to K_n$ to a map $X \xrightarrow{f} Y \to K_n$. This f^* depends only on the homotopy class of f; moreover, regarding K_n as ΩK_{n+1} and using the concatenation of loops as a group operation makes it apparent that f^* is a homomorphism. This confirms the homotopy axiom.

Verification of the wedge sum axiom is equally brief. The axiom holds since a map $\bigvee_\alpha X_\alpha \to K_n$ is just a collection of basepoint-preserving maps $X_\alpha \to K_n$.

Moving to the final axiom, let (X, A) be a CW pair and CZ denote the cone on a CW complex Z. Then the successive addition of mapping cones produces the following sequence of inclusions.

$$A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA)$$

Making use of homotopy equivalences yields a more instructive sequence—the cofibration sequence defined above.

$$A \hookrightarrow X \to X/A \to \Sigma A \hookrightarrow \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \hookrightarrow \Sigma^2 X \to ...$$

Fixing a space K and defining maps by composition with this sequence gives a new sequence of basepoint-preserving homotopy classes of maps.

$$\langle A, K\rangle \leftarrow \langle X, K\rangle \leftarrow \langle X/A, K\rangle \leftarrow \langle \Sigma A, K\rangle \leftarrow \langle \Sigma X, K\rangle \leftarrow ...$$

This sequence is exact. Since the method of successively adding cones determines a term by its two predecessors, a proof of the exactness of the whole sequence reduces to a proof of the exactness of $\langle A, K\rangle \leftarrow \langle X, K\rangle \leftarrow \langle X \cup CA, K\rangle$. In this case, consider a map $X \to K$. That this map goes to zero in $\langle A, K\rangle$ means that its restriction to A is nullhomotopic, which is the same as its extending to a map $X \cup CA \to K$.Thus, exactness is established. Therefore, if $K = K(G, n)$ for some G and n, then applying the relation $\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle$ to the entire sequence gives the desired long exact sequence. □

4. Wrapping up

With the results of Section 3, the proof of Theorem 1.1 can be completed. This section follows the discussion in Frankland [**2**, p.8].

Proof. (Of Theorem 1.1) By Theorem 3.3 and the uniqueness of cohomology theories under the axioms of Section 2, for arbitrary CW complex X and for abelian group G, we have the following.

$$\langle X, K(G, n)\rangle \cong \tilde{H}^n(X; \langle S^0, K(G, 0)\rangle)$$

By the definition of an Eilenberg-MacLane space we know the following.

$$\langle S^0, K(G, 0)\rangle = \pi_0 K(G, 0) \cong G$$

Hence, $\langle X, K(G, n)\rangle \cong H^n(X; G)$, which is the assertion of Theorem 1.1. □

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References

[1] Allen Hatcher. Algebraic Topology. http://www.math.cornell.edu/ hatcher/AT/AT.pdf.
[2] Martin Frankland. Eilenberg-Maclane Spaces and Cohomology. http://www.math.uiuc.edu/˜franklan/Math527_0329.pdf.